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SOME NEW RESULTS OF PYTHAGOREAN FUZZY SOFT TOPOLOGICAL SPACES

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Abstract. This paper aims to originate new notion that is pythagorean fuzzy soft topological space and investigate some important properties of pythagorean fuzzy soft topological spaces. Furthermore the notions of pythagorean fuzzy soft interior, pythagorean fuzzy soft closure, pythagorean fuzzy soft image and pre-image, pythagorean fuzzy soft continuity are presented.

Keywords: Pythagorean fuzzy soft set, pythagorean fuzzy soft topological spaces, pythagorean fuzzy soft interior, pythagorean fuzzy soft closure, pythagorean fuzzy soft mapping, pythagorean fuzzy soft continuity.

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1. Introduction

In the literature, there are many theories which are used to the exploration of uncertainties and many complicated models in engineering, economics, medical sciences etc. Among these theories, the most popular theories are fuzzy set theory [29], intuitionistic fuzzy set theory [2], soft set theory [14], rough set theory [19]. One of these theories, soft set theory was originate by Molodtsov [14], which is different from the other existing theories due to its parameterization tools. Soft set theory is free from the fuzzy set theory, rough set theory, inherent complications and uncertain knowledge easy. Research on soft set has been very active and many important results habe been achieved in the theoretical aspect. Maji et al. [13] introduced various algebraic operation in soft set theory. Ali et al. [1] further improved the existing literature and presented some new operation on soft set theory. Shabir and Naz [23] presented soft topological spaces. They also defined some notions of soft sets on soft topological spaces. The combined structure of soft set theory and fuzzy set theory was presented bu Maji et al. [11] to introduce the concept of fuzzy soft set. Tanay and Kandemir [25] initiated the notion of fuzzy soft topology using the fuzzy soft sets and Chang [5] gave some basic notions of its. Intuitionistic fuzzy set (IFS), introduced by Atanassov [2], consist of two functions which are the generalization of fuzzy set and like a fuzzy set. These two functions also map on unit closed interval. Coker [6],

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introduced the notion of intuitionistic fuzzy topology and studied some analogue versions of classical topology such as continuity and compactness. Maji et al. extended soft sets to intuitionistic fuzzy soft sets which were discussed in Maji et al. [12]. Furthermore, some authors studied the concept of intuitionistic fuzzy soft topological spaces [4, 9, 18, 28].

Yager [26], introduced Pythagorean fuzzy set characterized by a membership degree and a non-membership degree that satisfies the case in which the square sum of its membership degree and a non-membership degree is less than or equal to one. Yager and Abbasov [27], gave an example to illustrate this situation: An expert giving his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his non-membership is $\frac{1}{2}$. Since the sum of two values is bigger than 1, they are not avaliable for intuitionistic fuzzy set but it feasible for pythagorean fuzzy set. Obviously, pythagorean fuzzy set is more effective than intuitionistic fuzzy set [22]. Some different author studied on pythagorean fuzzy set theory [3, 8, 15, 16, 24, 30]. In 2019, Olgun et al. [17] introduced pythagorean fuzzy topological space and studied continuity, some importan properties on pythagorean fuzzy topological spaces.

Peng et al. [21], defined the pythagorean fuzzy soft set theory and studied its important properties. Guleria and Bajaj [7], investigated pythagorean fuzzy soft matrix and its diverse feasible types. M. Kirisci [10] defined new type of pythagorean fuzzy soft set and proposed to the solution of decision-making problem. By viewing existing literature, it is clear that, there does not axist any concepts of Pythagorean fuzzy soft topological spaces.

In this paper, we introduce the notion of pythagorean fuzzy soft topological spaces. We also define pythagorean fuzzy soft closure, pythagorean fuzzy soft interior and some important properties on pythagorean fuzzy soft topological spaces. Finally, pythagorean fuzzy soft continuous mapping is investigated and some interesting results are derived which may be of value for further research.

2. Preliminaries

Definition 2.1. [29] *Let X be an universe. A fuzzy set F in X, $F = \{(x, \mu_F(x)) : x \in X\}$, where $\mu_F : X \to [0, 1]$ is the membership function of the fuzzy set F; $\mu_F(x) \in [0, 1]$ is the membership of $x \in X$ in f. The set of all fuzzy sets over X will be denoted by $FS(X)$.*

Definition 2.2. [2] *An intuitionistic fuzzy set F in X is $F = \{(x, \mu_F(x), v_F(x)) : x \in X\}$, where $\mu_F : X \to [0, 1]$, $v_F : X \to [0, 1]$ with the condition $0 \leq \mu_F(x) + v_F(x) \leq 1$, $\forall x \in X$. The numbers $\mu_F, v_F \in [0, 1]$ denote the degree of membership and non-membership of x to F, respectively. The set of all intuitionistic fuzzy sets over X will be denoted by $IFS(X)$.*

Definition 2.3. [14] *Let E be a set of parameters and X be the universal set. A pair (F, E) is called a soft set over X, where F is a mapping $F : E \to \mathcal{P}(X)$. In other words, the soft set is a parameterized family of subsets of the set X.*

Definition 2.4. [11] *Let E be a set of parameters and X be the universal set. A pair (F, E) is called a fuzzy soft set over X, If $F : E \to FS(X)$ is a mapping from E into set of all fuzzy sets in X, where $FS(X)$ is set of all fuzzy subset of X.*

Definition 2.5. [12] *Let X be an initial universe E be a set of parameters. A pair (F, E) is called an intuitionistic fuzzy soft set over X, where F is a mapping given by, $F : E \to IFS(X)$.*

In general, for every $e \in E$, $F(e)$ is an intuitionistic fuzzy set of X and it is called intuitionistic fuzzy value set of parameter e. Clearly, $F(e)$ can be written as a intuitionistic fuzzy set such that $F(e) = \{(x, \mu_F(x), v_F(x)) : x \in X\}$

Definition 2.6. [26] *Let X be a universe of discourse. A pythagorean fuzzy set (PFS) in X is given by, $P = \{(x, \mu_P(x), v_P(x)) : x \in X\}$ where, $\mu_P : X \to [0,1]$ denotes the degree of membership and $v_p : X \to [0,1]$ denotes the degree of nonmembership of the element $x \in X$ to the set P with the condition that $0 \leq (\mu_P(x))^2 + (v_P(x))^2 \leq 1$.*

Theorem 2.1. [27] *The set of Pythagorean membership grades is greater than the set of intuitionistic membership grades.*

Proof. I. Every point (a, b) that is an intuitionistic membership grade is also a Pythagorean membership grade. We first observe that any $a, b \in [0,1]$ then $a^2 \leq a$ and $b^2 \leq b$ from this we observe that if $a + b \leq 1$ then $a^2 + b^2 \leq 1$.

II. There are Pythagorean membership grades that not intuitionistic membership grades. Consider now the point $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$. Thus, this is a Pythagorean membership grade. However, since $\frac{\sqrt{3}}{2} = \frac{1.72}{2} = 0.866$ then $0.5 + 0.866 > 1$. This is not an intuitionistic membership grade. □

Definition 2.7. [20, 26] *Let $P_1 = \{(x, \mu_{P_1}(x), v_{P_1}(x)) : x \in X\}$ and $P_2 = \{(x, \mu_{P_2}(x), v_{P_2}(x)) : x \in X\}$ be two pythagorean fuzzy sets. Then their operations are defined as follows;*

(1) $P_1 \cup P_2 = \{(x, \max\{\mu_{P_1}(x), \mu_{P_2}(x)\}, \min\{v_{P_1}(x), v_{P_2}(x)\})\}$
(2) $P_1 \cap P_2 = \{(x, \min\{\mu_{P_1}(x), \mu_{P_2}(x)\}, \max\{v_{P_1}(x), v_{P_2}(x)\})\}$
(3) $P^c = \{(x, v_P(x), \mu_P(x)) : x \in X\}$
(4) $P_1^c \cup P_2^c = (P_1 \cap P_2)^c$
(5) $P_1^c \cap P_2^c = (P_1 \cup P_2)^c$

Definition 2.8. [17] *Let $X \neq \emptyset$ be a set and let τ be a family of Pythagorean fuzzy subsets of X. If*

(1) $1_X, 0_X \in \tau$,
(2) *for any $P_1, P_2 \in \tau$, we have $P_1 \cap P_2 \in \tau$,*
(3) *for any $\{P_i\}_{i \in I} \subset \tau$, we have $\underset{i \in I}{\cup} P_i \in \tau$*

where I is an arbitrary index set then τ is called a Pythagorean fuzzy topology on X.

Remark 2.1. *According to study* [17], *As any fuzzy subset or intuitionistic fuzzy subset of a set can be considered as a Pythagorean fuzzy subset, we observe that any fuzzy topological space or intuitionistic fuzzy topological space is a Pythagorean fuzzy topological space as well. On the other hand, it is obvious that a Pythagorean fuzzy topological space needs not to be a fuzzy topological space or intuitionistic fuzzy topological space. Even an open Pythagorean fuzzy subset may be neither a fuzzy subset nor an intuitionistic fuzzy subset. (see Example-1 in* [17] *)*

Definition 2.9. [21] *Let X be the universal set and E be a set of parameters. The pythagorean fuzzy soft set is defined as the pair (F, E) where, $F : E \to PFS(X)$ and $PFS(X)$ is the set of all Pythagorean fuzzy subsets of X. If $\mu_F^2(x) + v_F^2(x) \leq 1$ and $\mu_F(x) + v_F(x) \leq 1$, then pythagorean fuzzy soft sets degenerate into intuitionistic fuzzy soft sets.*

Definition 2.10. [21] *Let $A, B \subseteq E$ and (F, A), (G, B) be two pythagorean fuzzy soft sets over X. (F, A) is said to be pythagorean fuzzy soft subset of (G, B) denoted by $(F, A) \widetilde{\subseteq} (G, B)$ if,*

(1) $A \subseteq B$

(2) $\forall e \in A$, $F(e)$ *is a pythagorean fuzzy subset of* $G(e)$ *that is,* $\forall x \in U$ *and* $\forall e \in A$, $\mu_{F(e)}(x) \leq \mu_{G(e)}(x)$ *and* $v_{F(e)}(x) \geq v_{G(e)}(x)$. *If* $(F,A)\widetilde{\subseteq}(G,B)$ *and* $(G,B)\widetilde{\subseteq}(F,A)$ *then* (F,A), (G,B) *are said to be equal.*

Definition 2.11. [21] *Let* (F,E) *two pythagorean fuzzy soft sets over* X. *The complement of* (F,E) *is denoted by* $(F,E)^c$ *and is defined by*

$$(F,E)^c = \{(e,(x, v_{F(e)}(x), \mu_{F(e)}(x)) : x \in X) : e \in E\}$$

Definition 2.12. [10] *a) A pythagorean fuzzy soft set* (F,E) *over the universe* X *is said to be null pythagorean fuzzy soft set if* $\mu_{F(e)}(x) = 0$ *and* $v_{F(e)}(x) = 1$; $\forall e \in E$, $\forall x \in X$. *It is denoted by* $\widetilde{0}_{(X,E)}$.

b) A pythagorean fuzzy soft set (F,E) *over the universe* X *is said to be absolute pythagorean fuzzy soft set if* $\mu_{F(e)}(x) = 1$ *and* $v_{F(e)}(x) = 0$; $\forall e \in E$, $\forall x \in X$. *It is denoted by* $\widetilde{1}_{(X,E)}$.

Definition 2.13. [10] *Let* (F_1,E) *and* (F_2,E) *be two pythagorean fuzzy soft sets over the universe set* X. *Then,*

a) Restricted union of (F_1,E) *and* (F_2,E) *is denoted by* $(F_1,E)\widetilde{\cup}_R(F_2,E) = (F_3,E)$ *and defined by;*

$$(F_3,E) = \{(e,(x, \mu_{F_3(e)}(x), v_{F_3(e)}(x)) : x \in X) : e \in E\}$$

where $\mu_{F_3(e)}(x) = \max\{\mu_{F_1(e)}(x), \mu_{F_2(e)}(x)\}$ *and* $v_{F_3(e)}(x) = \min\{v_{F_1(e)}(x), v_{F_2(e)}(x)\}$.

b) Restricted intersection of (F_1,E) *and* (F_2,E) *is denoted by* $(F_1,E)\widetilde{\cap}_R(F_2,E) = (F_3,E)$ *and defined by;*

$$(F_3,E) = \{(e,(x, \mu_{F_3(e)}(x), v_{F_3(e)}(x)) : x \in X) : e \in E\}$$

where $\mu_{F_3(e)}(x) = \min\{\mu_{F_1(e)}(x), \mu_{F_2(e)}(x)\}$ *and* $v_{F_3(e)}(x) = \max\{v_{F_1(e)}(x), v_{F_2(e)}(x)\}$.

Definition 2.14. [10] *Let* (F,A) *and* (G,B) *be two pythagorean fuzzy soft sets over the universe set* X, E *be a parameter set and* $A, B \subseteq E$.*Then,*

a) Extended union of (F,A) *and* (G,B) *is denoted by* $(F,E)\widetilde{\cup}_E(G,B) = (H,C)$ *where* $C = A \cup B$ *and* (H,C) *defined by;*

$$(H,C) = \{(e,(x, \mu_{H(e)}(x), v_{H(e)}(x)) : x \in X) : e \in E\}$$

where

$$\mu_{H(e)}(x) = \begin{cases} \mu_{F(e)}(x), & \textit{if } e \in A - B \\ \mu_{G(e)}(x), & \textit{if } e \in B - A \\ \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, & \textit{if } e \in A \cap B \end{cases}$$

$$v_{H(e)}(x) = \begin{cases} v_{F(e)}(x), & \textit{if } e \in A - B \\ v_{G(e)}(x), & \textit{if } e \in B - A \\ \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, & \textit{if } e \in A \cap B \end{cases}$$

b) Extended intersection of (F,A) *and* (G,B) *is denoted by* $(F,E)\widetilde{\cap}_E(G,B) = (H,C)$ *where* $C = A \cup B$ *and* (H,C) *defined by;*

$$(H,C) = \{(e,(x, \mu_{H(e)}(x), v_{H(e)}(x)) : x \in X) : e \in E\}$$

where

$$\begin{aligned} \mu_{H(e)}(x) &= \begin{cases} \mu_{F(e)}(x), \ \textit{if}\ e \in A - B \\ \mu_{G(e)}(x), \ \textit{if}\ e \in B - A \\ \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \ \textit{if}\ e \in A \cap B \end{cases} \\ v_{H(e)}(x) &= \begin{cases} v_{F(e)}(x), \ \textit{if}\ e \in A - B \\ v_{G(e)}(x), \ \textit{if}\ e \in B - A \\ \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \ \textit{if}\ e \in A \cap B \end{cases} \end{aligned}$$

Definition 2.15. [21] *a) Let Let (F, A) and (G, B) be two pythagorean fuzzy soft sets over the universe set X, E be a parameter set and $A, B \subseteq E$. Then "and" operation on them is denoted by $(F, A) \wedge (G, B) = (H, A \times B)$ and defined by;*

$$(H, A \times B) = \{((e_1, e_2), (x, \mu_{H(e_1,e_2)}(x), v_{H(e_1,e_2)}(x)) : x \in X) : e_1, e_2 \in A \times B\}$$

where

$$\begin{aligned} \mu_{H(e_1,e_2)}(x) &= \min\{\mu_{F(e_1)}(x), \mu_{G(e_2)}(x)\} \\ v_{H(e_1,e_2)}(x) &= \max\{v_{F(e_1)}(x), v_{G(e_2)}(x)\}. \end{aligned}$$

b) Let (F, A) and (G, B) be two pythagorean fuzzy soft sets over the universe set X, E be a parameter set and $A, B \subseteq E$. Then "or" operation on them is denoted by $(F, A) \vee (G, B) = (H, A \times B)$ and defined by;

$$(H, A \times B) = \{((e_1, e_2), (x, \mu_{H(e_1,e_2)}(x), v_{H(e_1,e_2)}(x)) : x \in X) : e_1, e_2 \in A \times B\}$$

where

$$\begin{aligned} \mu_{H(e_1,e_2)}(x) &= \max\{\mu_{F(e_1)}(x), \mu_{G(e_2)}(x)\} \\ v_{H(e_1,e_2)}(x) &= \min\{v_{F(e_1)}(x), v_{G(e_2)}(x)\}. \end{aligned}$$

Definition 2.16. [17] *Let X and Y be two non-empty sets, let $f : X \to Y$ be a function and let A and B be Pythagorean fuzzy subsets of X and Y, respectively. Then, the membership and non-membership functions of image of A with respect to f that is denoted by $f[A]$ are defined by*

$$\mu_{f[A]}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu_A(z) \ , & \textit{if} \quad f^{-1}(y) \neq \emptyset \\ 0 \ , & \textit{otherwise} \end{cases}$$

and

$$v_{f[A]}(y) = \begin{cases} \inf\limits_{z \in f^{-1}(y)} v_A(z) \ , & \textit{if} \quad f^{-1}(y) \neq \emptyset \\ 0 \ , & \textit{otherwise} \end{cases}$$

respectively. The membership and non-membership functions of pre-image of B with respect to f that is denoted by $f^{-1}[B]$ are defined by

$\mu_{f^{-1}[B]}(x) = \mu_B(f(x))$ and $v_{f^{-1}[B]}(x) = v_B(f(x))$ respectively.

In the study [17], they showed that $\mu^2_{f[A]} + v^2_{f[A]} \leq 1$ pythagorean fuzzy membership condition is provide for pythagorean fuzzy image and pre-image.

3. Pythagorean Fuzzy Soft Topological Spaces

In this section, we introduce some important properties of pythagorean fuzzy soft topological spaces and we define the pythagorean fuzzy soft closure and pythagorean fuzzy soft interior set.

Let X be an initial universe and $PFS(X)$ denote the family of pythagorean fuzzy sets over X and $PFSS(X, E)$ be the family of all pythagorean fuzzy soft sets over X with parameters in E.

Definition 3.1. *Let $X \neq \emptyset$ be a universe set and $\widetilde{\tau} \subset PFSS(X,E)$ be a collection of pythagorean fuzzy soft sets over X, then τ is said to be on pythagorean fuzzy soft topology on X if*

(i) $\widetilde{0}_{(X,E)}$, $\widetilde{1}_{(X,E)}$ belong to $\widetilde{\tau}$,
(ii) The union of any number of pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,
(iii) The intersection of any two pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triple $(X,\widetilde{\tau},E)_p$ is called an pythagorean fuzzy soft tpological space over X. Every member of τ is called a pythagorean fuzzy soft open set in X.

Definition 3.2. *a) Let X be an initial universe set, E be the set of parameters and $\widetilde{\tau} = \left\{\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)}\right\}$. Then $\widetilde{\tau}$ is called a pythagorean fuzzy soft indiscrete topology on X and $(X,\widetilde{\tau},E)_p$ is said to be a pythagorean fuzzy soft indiscrete space over X.*

b) Let X be an initial universe set, E be the set of parameters and $\widetilde{\tau}$ be the collection of all pythagorean fuzzy soft sets which can be defined over X. Then $\widetilde{\tau}$ is called a pythagorean fuzzy soft discrete topology on X and $(X,\widetilde{\tau},E)_p$ is said to be a pythagorean fuzzy soft discrete space over X.

Example 3.1. *Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and*

$$\widetilde{\tau} = \left\{\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)}, (G_1,E), (G_2,E), (G_3,E), (G_4,E)\right\}$$

where $(G_1,E), (G_2,E), (G_3,E), (G_4,E)$ pythagorean fuzzy soft sets over X, defined as;

$$\begin{aligned}
(G_1,E) &= \left\{ \begin{array}{c} (e_1, \{(x_1, 0.32, 0.5), (x_2, 0.5, 0.61)\}) \\ (e_2, \{(x_1, 0.23, 0.4), (x_2, 0.36, 0.6)\}) \end{array} \right\} \\
(G_2,E) &= \left\{ \begin{array}{c} (e_1, \{(x_1, 0.76, 0.6), (x_2, 0.4, 0.2)\}) \\ (e_2, \{(x_1, 0.5, 0.1), (x_2, 0.2, 0.3)\}) \end{array} \right\} \\
(G_3,E) &= \left\{ \begin{array}{c} (e_1, \{(x_1, 0.76, 0.5), (x_2, 0.5, 0.2)\}) \\ (e_2, \{(x_1, 0.5, 0.1), (x_2, 0.36, 0.3)\}) \end{array} \right\} \\
(G_4,E) &= \left\{ \begin{array}{c} (e_1, \{(x_1, 0.32, 0.6), (x_2, 0.4, 0.61)\}) \\ (e_2, \{(x_1, 0.23, 0.4), (x_2, 0.2, 0.6)\}) \end{array} \right\} \\
(\widetilde{0}_{(X,E)})^c &= \widetilde{1}_{(X,E)},\ (\widetilde{1}_{(X,E)})^c = \widetilde{0}_{(X,E)}
\end{aligned}$$

Then $\widetilde{\tau}$ defines a pythagorean fuzzy soft topology on X and hence $(X,\widetilde{\tau},E)_P$ is a pythagorean fuzzy soft topolojical space over X.

Remark 3.1. *As any fuzzy soft subset or intuitionistic fuzzy soft subset of a set can be considered as a Pythagorean fuzzy soft subset, we observe that any fuzzy soft topological space or intuitionistic fuzzy soft topological space is a Pythagorean fuzzy topological space as well. On the other hand, it is obvious that a Pythagorean fuzzy soft topological space needs not to be a fuzzy soft topological space or intuitionistic fuzzy soft topological space. Even an open Pythagorean fuzzy soft subset may be neither a fuzzy soft subset nor an intuitionistic fuzzy soft subset (see Example-3.1).*

Proposition 3.1. *Let $(X,\widetilde{\tau}_1,E)_p$ and $(X,\widetilde{\tau}_2,E)_p$ be two pythagorean fuzzy soft topological space over X. If $\widetilde{\tau}_1 \widetilde{\cap}_E \widetilde{\tau}_2 = \{(H,E) : (H,E) \in \widetilde{\tau}_1$ and $(H,E) \in \widetilde{\tau}_2\}$. Then $\widetilde{\tau}_1 \widetilde{\cap}_E \widetilde{\tau}_2$ is a pythagorean fuzzy soft topology on X.*

Proof. Obviously $\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)} \in \widetilde{\tau}_1 \widetilde{\cap}_E \widetilde{\tau}_2$. Let $(H,E), (G,E) \in \widetilde{\tau}_1 \widetilde{\cap}_E \widetilde{\tau}_2$. Then $(H,E), (G,E) \in \widetilde{\tau}_1$ and $(H,E), (G,E) \in \widetilde{\tau}_2$. Since $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ are two pythagorean fuzzy soft topolojies on X, then $(H,E)\widetilde{\cap}_E(G,E)\in\widetilde{\tau}_1$ and $(H,E)\widetilde{\cap}_E(G,E)\in\widetilde{\tau}_2$. Hence $(H,E)\widetilde{\cap}_E(G,E)\in\widetilde{\tau}_1\widetilde{\cap}_E\widetilde{\tau}_2$. Let $\{(F_s,E):s\in S\}\widetilde{\subseteq}\widetilde{\tau}_1\widetilde{\cap}_E\widetilde{\tau}_2$. Then $(F_s,E)\in\widetilde{\tau}_1$ and $(F_s,E)\in\widetilde{\tau}_2$ for any $s\in S$. Since $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ are two pythagorean fuzzy soft topologies on X, $\widetilde{\cup}\{(F_s,E):s\in S\}\in\widetilde{\tau}_1$ and $\widetilde{\cup}\{(F_s,E):s\in S\}\in\widetilde{\tau}_2$. Therefore, $\widetilde{\cup}\{(F_s,E):s\in S\}\in\widetilde{\tau}_1\widetilde{\cap}_E\widetilde{\tau}_2$. □

Proposition 3.2. *Let $(X,\widetilde{\tau},E)_p$ be two pythagorean fuzzy soft topological space over X. For any $e\in E$,*

$$\tau_e=\{f(e):(F,E)\in\widetilde{\tau}\}$$

in a pythagorean fuzzy topology on X.

Proof. *(i)* $\widetilde{0}_{(X,E)},\widetilde{1}_{(X,E)}\in\tau_e$

(ii) Let $A,B\in\tau_e$. Then there exist $(F,E),(G,E)\in\widetilde{\tau}$ such that $A=f(e)$ and $B=g(e)$. By $\widetilde{\tau}$ be a pythagorean fuzzy soft topology on X, $(F,E)\widetilde{\cap}_E(G,E)\in\widetilde{\tau}$. Put $(H,E)=(F,E)\widetilde{\cap}_E(G,E)$. Then $(H,E)\in\widetilde{\tau}$. Note that $A\widetilde{\cap}_EB=f(e)\widetilde{\cap}_Eg(e)=h(e)$ and $\tau_e=\{F(e):(F,E)\in\widetilde{\tau}\}$. Then $A\widetilde{\cap}_EB\in\tau_e$.

(iii) Let $\{A_s:s\in S\}\subseteq\tau_e$. Then for every $s\in\widetilde{\tau}$, there exist $(F_s,E)\in\widetilde{\tau}$ such that $A_s=(F_s,E)$. By $\widetilde{\tau}$ be on pythagorean fuzzy soft topology on X, $\bigcup\{(F_s,E):s\in S\}\in\widetilde{\tau}$. Put $(F,E)=\bigcup\{(F_s,E):s\in S\}$. Then $(F,E)\in\widetilde{\tau}$. Note that $\bigcup\limits_{s\in S}A_s=\bigcup\{(F_s,E):s\in S\}=(F,E)$ and $\tau_e=\{F(e):(F,E)\in\widetilde{\tau}\}=f(e)$. Therefore $\tau_e=\{F(e):(F,E)\in\widetilde{\tau}\}$ is a pythagorean fuzzy soft topology on X. □

Definition 3.3. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X. A pythagorean fuzzy soft set (F,E) over X is said to be a pythagorean fuzzy soft closed set in X, if its complement $(F,E)^c$ belongs to $\widetilde{\tau}$.*

Example 3.2. *Consider the Example-3.1. It is clear that $\left(\widetilde{0}_{(X,E)}\right)^c,\left(\widetilde{1}_{(X,E)}\right)^c,(G_1,E)^c,(G_2,E)^c,(G_3,E)^c$ and $(G_4,E)^c$ are pythagorean fuzzy soft closed set in X.*

$$\begin{aligned}
(G_1,E)^c &= \left\{\begin{array}{c}(e_1,\{(x_1,0.5,0.32),(x_2,0.61,0.5)\})\\(e_2,\{(x_1,0.4,0.23),(x_2,0.6,0.36)\})\end{array}\right\}\\
(G_2,E)^c &= \left\{\begin{array}{c}(e_1,\{(x_1,0.6,0.76),(x_2,0.2,0.4)\})\\(e_2,\{(x_1,0.1,0.5),(x_2,0.3,0.2)\})\end{array}\right\}\\
(G_3,E)^c &= \left\{\begin{array}{c}(e_1,\{(x_1,0.5,0.76),(x_2,0.2,0.5)\})\\(e_2,\{(x_1,0.1,0.5),(x_2,0.3,0.36)\})\end{array}\right\}\\
(G_4,E)^c &= \left\{\begin{array}{c}(e_1,\{(x_1,0.6,0.32),(x_2,0.61,0.4)\})\\(e_2,\{(x_1,0.4,0.23),(x_2,0.6,0.2)\})\end{array}\right\}
\end{aligned}$$

Proposition 3.3. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X. Then, the following properties hold.*

(i) $\widetilde{0}_{(X,E)},\widetilde{1}_{(X,E)}$ are pythagorean fuzzy soft closed set over X.

(ii) The intersection of any number of pythagorean fuzzy soft closed set is a pythagorean fuzzy soft closed set over X.

(iii) The union of any two pythagorean fuzzy soft closed set is a pythagorean fuzzy soft closed set over X.

Proof. Straightforward. □

Definition 3.4. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and (F, E) be a pythagorean fuzzy soft sets over X. The pythagorean fuzzy soft closure of (F, E) denoted by $pcl(F, E)$ is the intersection of all pythagorean fuzzy soft closed super sets of (F, E).*

Clearly $pcl(F, E)$ is the smallest pythagorean fuzzy soft closed set over X which contain (F, E).

Theorem 3.1. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E) \in PFSS(X, E)$. Then the following propeties hold.*

(i) $pcl(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(X,E)}$ and $pcl(\widetilde{1}_{(X,E)}) = \widetilde{1}_{(X,E)}$,
(ii) $(F, E) \widetilde{\subseteq} pcl\,(F, E)$,
(iii) (F, E) is a pythagorean fuzzy soft closed set $\Leftrightarrow pcl(F, E) = (F, E)$,
(iv) $pcl\,(pcl\,(F, E)) = pcl(F, E)$,
(v) $(F, E) \widetilde{\subseteq} (G, E) \Rightarrow pcl\,(F, E) \widetilde{\subseteq} pcl\,(G, E)$,
(vi) $pcl\,\big((F, E) \widetilde{\cup}_E (G, E)\big) = pcl\,(F, E) \widetilde{\cup}_E pcl\,(G, E)$.

Proof. *(i)* and *(ii)* are clear.

(iii) If (F, E) is a pythagorean fuzzy soft closed set over X then $pcl(F, E)$ is itselt a pythagorean fuzzy soft closed set over X which contain (F, E). So $pcl(F, E)$ is the smallest pythagorean fuzzy soft closed set containing (F, E) and $(F, E) = pcl(F, E)$

(iv) Since $pcl(F, E)$ is a pythagorean fuzzy soft closed set therefore by part *(iii)*. We have $pcl(F, E) = pcl\,(pcl\,(F, E))$.

(v) If $(F, E)\widetilde{\subseteq}(G, E)$, then $(G, E) = (F, E)\widetilde{\cup}_E(G, E) \Rightarrow pcl(G, E) = pcl\,\big((F, E)\widetilde{\cup}_E(G, E)\big) = pcl(F, E)\widetilde{\cup}_E pcl(G, E) \Rightarrow pcl(F, E)\widetilde{\subseteq} pcl(G, E)$

(vi) Since $(F, E)\widetilde{\subseteq}(F, E)\widetilde{\cup}_E(G, E)$ and $(G, E)\widetilde{\subseteq}(F, E)\widetilde{\cup}_E(G, E)$, from the *(v)*,
$pcl\,(F, E) \widetilde{\subseteq} pcl\,\big((F, E) \widetilde{\cup}_E (G, E)\big)$, $pcl\,(G, E) \widetilde{\subseteq} pcl\,\big((F, E) \widetilde{\cup}_E (G, E)\big)$

Therefore $pcl\,(F, E) \widetilde{\cup}_E pcl\,(G, E) \widetilde{\subseteq} pcl\,\big((F, E) \widetilde{\cup}_E (G, E)\big)$

Conversely, suppose that $(F, E) \widetilde{\subseteq} pcl\,(F, E)$ and $(G, E) \widetilde{\subseteq} pcl\,(G, E)$. From the proposition 3.3, $pcl\,(F, E) \widetilde{\cup}_E pcl\,(G, E)$ is a pythagorean fuzzy soft closed set over X being the union of two pythagorean fuzzy soft closed set.

Then, $pcl\,\big((F, E) \widetilde{\cup}_E (G, E)\big) \widetilde{\subseteq} pcl\,(F, E) \widetilde{\cup}_E pcl\,(G, E)$. Hence $pcl\,\big((F, E) \widetilde{\cup}_E (G, E)\big) = pcl\,(F, E) \widetilde{\cup}_E pcl\,(G, E)$ is obtained. □

Definition 3.5. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(H, E) \in PFSS(X, E)$. The pythagorean fuzzy soft interior of (H, E), denoted by $pint(H, E)$, is the union of all the pythagorean fuzzy soft open sets contained in (H, E).*

Theorem 3.2. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(H, E) \in PFSS(X, E)$. Then the following properties hold.*
(i) $pint(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(X,E)}$ *and* $pint(\widetilde{1}_{(X,E)}) = \widetilde{1}_{(X,E)}$,
(ii) $pint\,(H, E) \widetilde{\subseteq} (H, E)$,
(iii) (H, E) *is a pythagorean fuzzy soft open set* $\Leftrightarrow pint(H, E) = (H, E)$,
(iv) $pint\,(pint\,(H, E)) = pint(H, E)$,
(v) $(H, E) \widetilde{\subseteq} (G, E) \Rightarrow pint\,(H, E) \widetilde{\subseteq} pint\,(G, E)$,
(vi) $pint\,\big((H, E) \widetilde{\cap}_E (G, E)\big) = pint\,(H, E) \widetilde{\cap}_E pint\,(G, E)$.

Proof. *(i)* and *(ii)* are clear.

(iii) If (H, E) is pythagorean fuzzy soft open set over X, then $pint(H, E)$ is a larger pythagorean fuzzy soft open set which contained in (H, E). So $(H, E) = pint(H, E)$ is obtained.

(iv) Since $pint(H,E)$ is a pythagorean fuzzy soft open set, then by part *(iii)*, we have $pint(pint(H,E)) = pint(H,E)$.

(v) By the part *(ii)*, $pint(H,E)\widetilde{\subseteq}(H,E)$ and $pint(H,E)\widetilde{\subseteq}(H,E)\widetilde{\subseteq}(G,E)$ is obtained. Since $pint(h,E)$ is pythagorean fuzzy soft open set and $pint(G,E)$ is largest pythagorean fuzzy soft open set which contained in (G,E), then $pint(H,E)\widetilde{\subseteq}pint(G,E)\widetilde{\subseteq}(G,E)$ and $pint(H,E)\widetilde{\subseteq}pint(G,E)$ is obtained.

(vi) By the part *(v)*,

$$\begin{aligned}(H,E)\widetilde{\cap}_E(G,E)\widetilde{\subseteq}(H,E) &\Rightarrow pint\left((H,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}pint(H,E)\\(H,E)\widetilde{\cap}_E(G,E)\widetilde{\subseteq}(G,E) &\Rightarrow pint\left((H,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}pint(G,E)\end{aligned}$$

Therefore $pint\left((H,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}pint(H,E)\widetilde{\cap}_E pint(G,E)$.

Conversely, $pint\left((H,E)\widetilde{\cap}_E(G,E)\right)$ is largest pythagorean fuzzy soft open set which contained in $(H,E)\widetilde{\cap}_E(G,E)$. Then $pint(H,E)\widetilde{\cap}_E pint(G,E)\widetilde{\subseteq}pint\left((H,E)\widetilde{\cap}_E(G,E)\right)$. Hence $pint\left((H,E)\widetilde{\cap}_E(G,E)\right) = pint(H,E)\widetilde{\cap}_E pint(G,E)$ is obtained. □

Example 3.3. *Let* $X=\{x_1,x_2\}$, $E=\{e_1,e_2\}$

$$\begin{aligned}(H_1,E) &= \left\{\begin{array}{c}(e_1,\{(x_1,0.6,0.3),(x_2,0.8,0.4)\})\\(e_2,\{(x_1,0.5,0.1),(x_2,0.6,0.2)\})\end{array}\right\}\\(H_2,E) &= \left\{\begin{array}{c}(e_1,\{(x_1,0.5,0.4),(x_2,0.3,0.5)\})\\(e_2,\{(x_1,0.4,0.3),(x_2,0.5,0.6)\})\end{array}\right\}\end{aligned}$$

Obviously $\widetilde{\tau}=\left\{\widetilde{0}_{(X,E)},\widetilde{1}_{(X,E)},(H_1,E),(H_2,E)\right\}$ *is a pythagorean fuzzy soft topology on* X.

(i) Suppose that any $(G,E)\in PFSS(X,E)$ *is defined as following:*

$$(G,E)=\left\{\begin{array}{c}(e_1,\{(x_1,0.7,0.2),(x_2,0.8,0.3)\})\\(e_2,\{(x_1,0.6,0.1),(x_2,0.7,0.1)\})\end{array}\right\}$$

Then $\widetilde{0}_{(X,E)},(H_1,E),(H_2,E)\widetilde{\subseteq}(G,E)$. *Therefore* $pint(G,E)=\widetilde{0}_{(X,E)}\widetilde{\cup}_E(H_1,E)\widetilde{\cup}_E(H_2,E)=(H_1,E)$.

(ii) Suppose that any $(F,E)\in PFSS(X,E)$ *is defined as following:*

$$(F,E)=\left\{\begin{array}{c}(e_1,\{(x_1,0.2,0.7),(x_2,0.3,0.8)\})\\(e_2,\{(x_1,0.1,0.6),(x_2,0.1,0.7)\})\end{array}\right\}$$

Now, we find the complement of $(H_1,E),(H_2,E)$.

$$\begin{aligned}(H_1,E)^c &= \left\{\begin{array}{c}(e_1,\{(x_1,0.3,0.6),(x_2,0.4,0.8)\})\\(e_2,\{(x_1,0.1,0.5),(x_2,0.2,0.6)\})\end{array}\right\}\\(H_2,E)^c &= \left\{\begin{array}{c}(e_1,\{(x_1,0.4,0.5),(x_2,0.5,0.3)\})\\(e_2,\{(x_1,0.3,0.4),(x_2,0.6,0.5)\})\end{array}\right\}\\(\widetilde{0}_{(X,E)})^c &= \widetilde{1}_{(X,E)},\ (\widetilde{1}_{(X,E)})^c=\widetilde{0}_{(X,E)}\end{aligned}$$

Obviously, $(\widetilde{0}_{(X,E)})^c$, $(\widetilde{1}_{(X,E)})^c,(H_1,E)^c,(H_2,E)^c$ *are all pythagorean fuzzy soft closed sets over* $(X,\widetilde{\tau},E)_p$. *Then* $(\widetilde{1}_{(X,E)})^c,(H_1,E)^c,(H_2,E)^c\widetilde{\supseteq}(F,E)$. *Therefore* $pcl(F,E)=(\widetilde{1}_{(X,E)})^c\widetilde{\cap}_E(H_1,E)^c\widetilde{\cap}_E(H_2,E)^c=(H_1,E)^c$

Theorem 3.3. *Let* $(X,\widetilde{\tau},E)_p$ *be a pythagorean fuzzy soft topological space over* X *and* $(H,E)\widetilde{\subset}PFSS(X,E)$. *Then*

(i) $(pcl(H,E))^c=pint((H,E)^c)$

(ii) $pint((H,E))^c=pcl((H,E)^c)$

Proof. *(i)*

$$\begin{aligned} pcl(H,E) &= \widetilde{\cap}_E \left\{(F,E)\in\tau^c : (F,E)\widetilde{\supseteq}(H,E)\right\} \\ &\Rightarrow (pcl(H,E))^c = \left(\left\{\widetilde{\cap}_E(F,E)\in\tau^c : (F,E)\widetilde{\supseteq}(H,E)\right\}\right)^c \\ &= \widetilde{\cup}_E \left\{(F,E)\in\tau : (F,E)^c\widetilde{\subseteq}(H,E)^c\right\} = pint((H,E)^c) \end{aligned}$$

(ii)

$$\begin{aligned} pint(H,E) &= \widetilde{\cup}_E \left\{(F,E)\in\tau : (F,E)\widetilde{\subseteq}(H,E)\right\} \\ &\Rightarrow (pint(F,E))^c = \left(\left\{\widetilde{\cup}_E(F,E)\in\tau : (F,E)\widetilde{\subseteq}(H,E)\right\}\right)^c \\ &= \widetilde{\cap}_E \left\{(F,E)^c\in\tau^c : (F,E)^c\widetilde{\supseteq}(H,E)^c\right\} = pcl((H,E)^c) \end{aligned}$$

□

4. Pythagorean Fuzzy Soft Continuity

Let X and Y be non-empty initial universe and E,E' be parameter set and $A \subseteq E$, $B \subseteq E'$.

Definition 4.1. *Let $(X, \widetilde{\tau}_1, E)_p$, $(Y, \widetilde{\tau}_2, E')_p$ be two pythagorean fuzzy soft topological spaces and $\psi : X \to Y$, $\sigma : E \to E'$ be mappings. Then a mapping $f = (\psi, \sigma) : PFSS(X,E) \to PFSS(Y,E')$ is defined as: for $(H,A) \in PFSS(X,E)$ the image of (H,A) under f, denoted by $f((H,A))$, is a pythagorean fuzzy soft set in $PFSS(Y,E')$ given by*

$$\mu_{\psi(H)}(e')(y) = \begin{cases} \sup\limits_{e\in\sigma^{-1}(e')\cap A, x\in\psi^{-1}(y)} \mu_{H(e)}(x), & \text{if } \psi^{-1}(y)\neq\emptyset \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$v_{\psi(H)}(e')(y) = \begin{cases} \inf\limits_{e\in\sigma^{-1}(e')\cap A, x\in\psi^{-1}(y)} v_{H(e)}(x), & \text{if } \psi^{-1}(y)\neq\emptyset \\ 1 \quad , & \text{otherwise} \end{cases}$$

For $(F,B) \in PFSS(Y,E')$, the inverse image of (F,B) under f, denoted by $f^{-1}((F,B))$, is a pythagorean fuzzy soft set in $PFSS(X,E)$ given by:

$$\begin{aligned} \mu_{\psi^{-1}(F)}(e)(x) &= \mu_{F(\sigma(e))}(\psi(x)), \\ v_{\psi^{-1}(F)}(e)(x) &= v_{F(\sigma(e))}(\psi(x)) \end{aligned}$$

for all $e \in E$ and $x \in X$.

Remark 4.1. *$f((H,A))$ and $f^{-1}((F,B))$ are pythagorean fuzzy soft subsets. We can obtain*

$$\begin{aligned} \mu^2_{\psi(H)}(e')(y) + v^2_{\psi(H)}(e')(y) &= \left(\sup_{e\in\sigma^{-1}(e')\cap A, x\in\psi^{-1}(y)} \mu_{H(e)}(x)\right)^2 + \left(\inf_{e\in\sigma^{-1}(e')\cap A, x\in\psi^{-1}(y)} v_{H(e)}(x)\right)^2 \\ &= \sup_{e\in\sigma^{-1}(e')\cap A, x\in\psi^{-1}(y)} \mu^2_{H(e)}(x) + \inf_{e\in\sigma^{-1}(e')\cap A, x\in\psi^{-1}(y)} v^2_{H(e)}(x) \\ &\leq \sup_{e\in\sigma^{-1}(e')\cap A, x\in\psi^{-1}(y)} (1 - v^2_{H(e)}(x)) + \inf_{e\in\sigma^{-1}(e')\cap A, x\in\psi^{-1}(y)} v^2_{H(e)}(x) = 1, \end{aligned}$$

whenever $\psi^{-1}(y) \neq \emptyset$ is non-empty. On the other hand if $\psi^{-1}(y) = \emptyset$, then we have $\mu^2_{\psi(H)}(e')(y) + v^2_{\psi(H)}(e')(y) = 1$. Thus, we showed pythagorean fuzzy membership grade condition is provide for pythagorean fuzzy soft image and pre-image.

The proof for $f^{-1}((F,B))$ is similar. The following proposition gives some basic properties of image f.

Proposition 4.1. *Let $f : PFSS(X,E) \to PFSS(Y,E')$ be a pythagorean fuzzy soft mapping. Then for pythagorean fuzzy soft sets (H,A) and (G,B) in $PFSS(X,E)$, we have,*

(1) $f(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(Y,E)}$
(2) $f(\widetilde{1}_{(X,E)}) \widetilde{\subseteq} \widetilde{1}_{(Y,E)}$
(3) $f\left((H,A)\widetilde{\cup}_E(G,B)\right) = f((H,A))\widetilde{\cup}_E f((G,B))$
(4) $f\left((H,A)\widetilde{\cap}_E(G,B)\right) \widetilde{\subseteq} f((H,A))\widetilde{\cap}_E f((G,B))$
(5) $f\left((H,A)\widetilde{\cup}_R(G,B)\right) = f((H,A))\widetilde{\cup}_R f((G,B))$
(6) $f\left((H,A)\widetilde{\cap}_R(G,B)\right) \widetilde{\subseteq} f((H,A))\widetilde{\cap}_R f((G,B))$
(7) *If $(H,A)\widetilde{\subseteq}(G,B)$, then $f((H,A))\widetilde{\subseteq} f((G,B))$*

Proof. We only proof (3), (4) and (7). The others can be similarly proved.

(3) We consider $(H,A)\widetilde{\cup}_E(G,B) = (O, A\cup B)$ and
$f\left((H,A)\widetilde{\cup}_E(G,B)\right) = (\psi(H),\sigma(A))\widetilde{\cup}_E(\psi(G),\sigma(B)) = (I,\sigma(A)\cup\sigma(B))$. Then
$f\left((H,A)\widetilde{\cup}_E(G,B)\right) = (\psi(O),\sigma(A\cup B)) = (\psi(O),\sigma(A)\cup\sigma(B))$. For any $y\in Y$ and $e'\in\sigma(A)\cup\sigma(B)$, if $\psi^{-1}(y)\neq\emptyset$, then $\mu_I(e')(y) = \mu_{\psi(O)}(e')(y) = 0$ and $v_I(e')(y) = v_{\psi(O)}(e')(y) = 1$. Otherwise,

(i) If $e'\in\sigma(A)-\sigma(B)$, then $I(e') = \psi(H)(e')$. On the other hand, $e'\in\sigma(A)-\sigma(B)$ implies that there does not exist $e\in B$ such that $\sigma(e) = e'$. That is, for any $e\in\sigma^{-1}(e')\cap(A\cup B)$, we have $e\in\sigma^{-1}(e')\cap(A-B)$. Hence by Definition 4.1, we have

$$\begin{aligned}\mu_{\psi(O)}(e')(y) &= \sup_{e\in\sigma^{-1}(e')\cap(A\cup B),\ x\in\psi^{-1}(y)}\mu_{O(e)}(x) = \sup_{e\in\sigma^{-1}(e')\cap(A-B),\ x\in\psi^{-1}(y)}\mu_{O(e)}(x)\\ &= \sup_{e\in\sigma^{-1}(e')\cap(A\cup B),\ x\in\psi^{-1}(y)}\mu_{H(e)}(x) = \mu_{I(e')}(y)\end{aligned}$$

Similarly, we obtain $v_{\psi(O)}(e')(y) = v_{I(e')}(y)$.

(ii) If $e'\in\sigma(B)-\sigma(A)$, analogous to *(i)*, we have $\mu_{\psi(O)}(e')(y) = \mu_{I(e')}(y)$, $v_{\psi(O)}(e')(y) = v_{I(e')}(y)$.

(iii) If $e'\in\sigma(A)\cap\sigma(B)$, then

$$\begin{aligned}\mu_{\psi(O)}(e')(y) &= \sup_{e\in\sigma^{-1}(e')\cap(A\cup B),\ x\in\psi^{-1}(y)}\mu_{O(e)}(x) = \sup_{e\in(\sigma^{-1}(e')\cap A)\cup(\sigma^{-1}(e')\cap B),\ x\in\psi^{-1}(y)}\mu_{O(e)}(x)\\ &= \left(\sup_{e\in(\sigma^{-1}(e')\cap A)-(\sigma^{-1}(e')\cap B),\ x\in\psi^{-1}(y)}\mu_{H(e)}(x)\right)\\ &\quad\bigvee\left(\sup_{e\in(\sigma^{-1}(e')\cap A)\cap(\sigma^{-1}(e')\cap B),\ x\in\psi^{-1}(y)}\max\{\mu_{H(e)}(x),\mu_{G(e)}(x)\}\right)\\ &\quad\bigvee\left(\sup_{e\in(\sigma^{-1}(e')\cap B)-(\sigma^{-1}(e')\cap A),\ x\in\psi^{-1}(y)}\mu_{G(e)}(x)\right)\\ &= \max\left\{\sup_{e\in(\sigma^{-1}(e')\cap A),\ x\in\psi^{-1}(y)}\mu_{H(e)}(x),\ \sup_{e\in(\sigma^{-1}(e')\cap B),\ x\in\psi^{-1}(y)}\mu_{G(e)}(x)\right\}\\ &= \max\left\{\mu_{\psi(H)}(e')(y),\mu_{\psi(G)}(e')(y)\right\} = \mu_{I(e')}(y)\end{aligned}$$

Similarly, we obtain $v_{\psi(O)}(e')(y) = v_{I(e')}(y)$. Therefore
$\left((H,A)\widetilde{\cup}_E(G,B)\right) = f((H,A))\widetilde{\cup}_E f((G,B))$.

(4) Suppose that $(H,A)\widetilde{\cap}_E(G,B) = (H,A\cup B)$ and ($f\left((H,A)\widetilde{\cap}_E(G,B)\right) =$
$(\psi(H),\sigma(A))\widetilde{\cap}_E(\psi(G),\sigma(B)) = (I,\sigma(A)\cup\sigma(B))$. Then

$f\left((H,A)\widetilde{\cap}_E(G,B)\right) = (\psi(O),\sigma(A\cup B)) = (\psi(O),\sigma(A)\cup\sigma(B))$. For any $y\in Y$ and $e'\in\sigma(A)\cup\sigma(B)$, if $\psi^{-1}(y)\neq\emptyset$, then $\mu_I(e')(y) = \mu_{\psi(O)}(e')(y) = 0$ and $v_I(e')(y) = v_{\psi(O)}(e')(y) = 1$. Otherwise,

(i) If $e'\in\sigma(A)-\sigma(B)$, then $I(e') = \psi(H)(e')$. On the other hand, $e'\in\sigma(A)-\sigma(B)$ implies that there does not exist $e\in B$ such that $\sigma(e)=e'$. That is, for any $e\in\sigma^{-1}(e')\cap(A\cup B)$, we have $e\in\sigma^{-1}(e')\cap(A-B)$. Hence by Definition 4.1, we have

$$\begin{aligned}
\mu_{\psi(O)}(e')(y) &= \sup_{e\in\sigma^{-1}(e')\cap(A\cap B),\ x\in\psi^{-1}(y)} \mu_{O(e)}(x) = \sup_{e\in\sigma^{-1}(e')\cap(A-B),\ x\in\psi^{-1}(y)} \mu_{O(e)}(x)\\
&= \sup_{e\in\sigma^{-1}(e')\cap(A-B),\ x\in\psi^{-1}(y)} \mu_{H(e)}(x)\\
&\leq \min\left\{\sup_{e\in(\sigma^{-1}(e')\cap A),\ x\in\psi^{-1}(y)} \mu_{H(e)}(x), \sup_{e\in(\sigma^{-1}(e')\cap B),\ x\in\psi^{-1}(y)} \mu_{G(e)}(x)\right\}\\
&= \min\left\{\mu_{\psi(H)}(e')(y), \mu_{\psi(G)}(e')(y)\right\} = \mu_{I(e')}(y)
\end{aligned}$$

Similarly, we obtain $v_{\psi(O)}(e')(y)\geq v_{I(e')}(y)$.

(ii) If $e'\in\sigma(B)-\sigma(A)$, analogous to *(i)*, we have $\mu_{\psi(O)}(e')(y) = \mu_{I(e')}(y)$, $v_{\psi(O)}(e')(y) = v_{I(e')}(y)$.

(iii) If $e'\in\sigma(A)\cap\sigma(B)$, then

$$\begin{aligned}
\mu_{\psi(O)}(e')(y) &= \sup_{e\in\sigma^{-1}(e')\cap(A\cap B),\ x\in\psi^{-1}(y)} \mu_{O(e)}(x) = \mu_{\psi(O)}(e')(y)\\
&= \sup_{e\in\sigma^{-1}(e')\cap(A\cap B),\ x\in\psi^{-1}(y)} \max\left\{\mu_{H(e)}(x), \mu_{G(e)}(x)\right\}\\
&\leq \max\left\{\sup_{e\in(\sigma^{-1}(e')\cap A),\ x\in\psi^{-1}(y)} \mu_{H(e)}(x), \sup_{e\in(\sigma^{-1}(e')\cap B),\ x\in\psi^{-1}(y)} \mu_{G(e)}(x)\right\}\\
&= \max\left\{\mu_{\psi(H)}(e')(y), \mu_{\psi(G)}(e')(y)\right\} = \mu_{I(e')}(y)
\end{aligned}$$

Similarly, we obtain $v_{\psi(O)}(e')(y) = v_{I(e')}(y)$. Therefore $f\left((H,A)\widetilde{\cap}_E(G,B)\right)\widetilde{\subseteq} f((H,A))\widetilde{\cap}_E f((G,B))$.

(7) Suppose that $(H,A)\widetilde{\subseteq}(G,B)$. Then $A\subseteq B$ and for any $e\in A$ and $x\in X$, we have $\psi(A)\subseteq\psi(B)$.We have,

$$\begin{aligned}
\mu_{\psi(H)}(e')(y) &= \begin{cases} \sup_{e\in(\sigma^{-1}(e')\cap A),\ x\in\psi^{-1}(y)} \mu_{H(e)}(x), & \text{if } \psi^{-1}(y)\neq\emptyset\\ 0 \qquad , & \text{otherwise}\end{cases}\\
&\leq \begin{cases} \sup_{e\in(\sigma^{-1}(e')\cap B),\ x\in\psi^{-1}(y)} \mu_{H(e)}(x), & \text{if } \psi^{-1}(y)\neq\emptyset\\ 0 \qquad , & \text{otherwise}\end{cases}\\
&= \mu_{\psi(G)}(e')(y)
\end{aligned}$$

Similarly, $v_{\psi(H)}(e')(y)\geq v_{\psi(G)}(e')(y)$. Therefore $f((H,A))\widetilde{\subseteq} f((G,B))$. □

Proposition 4.2. *Let $f: PFSS(X,E)\to PFSS(Y,E')$ be a pythagorean fuzzy soft mapping. Then for pythagorean fuzzy soft sets $(H,A),(G,B)\in PFSS(X,E)$, we have,*

(1) $f^{-1}(\widetilde{0}_{(Y,E')}) = \widetilde{0}_{(X,E)}$
(2) $f^{-1}(\widetilde{1}_{(Y,E')})\widetilde{\subseteq}\widetilde{1}_{(X,E)}$
(3) $f^{-1}\left((H,A)\widetilde{\cup}_E(G,B)\right) = f^{-1}((H,A))\widetilde{\cup}_E f^{-1}((G,B))$
(4) $f^{-1}\left((H,A)\widetilde{\cap}_E(G,B)\right)\widetilde{\subseteq} f^{-1}((H,A))\widetilde{\cap}_E f^{-1}((G,B))$

(5) *If* $(H,A)\widetilde{\subseteq}(G,B)$, *then* $f^{-1}((H,A))\widetilde{\subseteq}f^{-1}((G,B))$
(6) $(H,A)\widetilde{\subseteq}f^{-1}\left(f\left((H,A)\right)\right)$, *then* $f(f^{-1}\left((G,B)\right)) = (G,B)\widetilde{\cap}_E f(\widetilde{1}_X)$

Proof. Straightforward □

Definition 4.2. *Let* $(X,\widetilde{\tau}_1,E)$ *and* $(Y,\widetilde{\tau}_2,E')$ *be two pythagorean fuzzy soft topological spaces, a pythagorean fuzzy soft mapping* $f=(\psi,\sigma): PFSS(X,E)\rightarrow PFSS(Y,E')$ *is called a pythagorean fuzzy soft continuous if* $f^{-1}((G,B))\in\widetilde{\tau}_1$ *for all* $(G,B)\in\widetilde{\tau}_2$

Example 4.1. *Let* $X=\{x_1,x_2\}$, $Y=\{y_1,y_2\}$, $E=\{e_1,e_2\}$ *and* $E'=\{e_1',e_2'\}$.

$\widetilde{\tau}_1=\left\{\widetilde{0}_X,\widetilde{1}_X,(H_1,E),(H_2,E)\right\}$ *where* $(H_1,E),(H_2,E)$, *pythagorean fuzzy soft sets over* X, *defined as follows.*

$$(H_1,E) = \left\{\begin{array}{l}(e_1,\{(x_1,0.8,0.5),(x_2,0.7,0.4)\})\\(e_2,\{(x_1,0.4,0.3),(x_2,0.5,0.6)\})\end{array}\right\}$$

$$(H_2,E) = \left\{\begin{array}{l}(e_1,\{(x_1,0.6,0.7),(x_2,0.3,0.5)\})\\(e_1,\{(x_1,0.1,0.5),(x_2,0.2,0.8)\})\end{array}\right\}$$

Then $\widetilde{\tau}_1$ *is a pythagorean fuzzy soft topology over* X *and hence* $(X,\widetilde{\tau}_1,E)$ *is a pythagorean fuzzy soft topological space.*

$\widetilde{\tau}_2=\left\{\widetilde{0}_{(Y,E')},\widetilde{1}_{(Y,E')},(G_1,E'),(G_2,E')\right\}$ *where* $(G_1,E'),(G_2,E')$, *pythagorean fuzzy soft sets over* Y, *defined as follows.*

$$(G_1,E') = \left\{\begin{array}{l}(e_1',\{(y_1,0.7,0.4),(y_2,0.8,0.5)\})\\(e_2',\{(y_1,0.5,0.6),(y_2,0.4,0.3)\})\end{array}\right\}$$

$$(G_2,E) = \left\{\begin{array}{l}(e_1',\{(y_1,0.3,0.5),(y_2,0.6,0.7)\})\\(e_2',\{(y_1,0.2,0.8),(y_2,0.1,0.5)\})\end{array}\right\}$$

Then $\widetilde{\tau}_2$ *is a pythagorean fuzzy soft topology over* Y *and hence* $(Y,\widetilde{\tau}_1,E')$ *is a pythagorean fuzzy soft topological space.*

If $f=(\psi,\sigma): PFSS(X,E)\rightarrow PFSS(Y,E')$ *defined as follows:*

$$\begin{array}{lllll}\psi(x_1) & = & y_2 & & \sigma(e_1)=e_1'\\ \psi(x_2) & = & y_1 & & \sigma(e_2)=e_2'\end{array}$$

Then it is easy to verify that $f^{-1}((G,E'))\in\widetilde{\tau}_1$ *for all* $(G,E')\in\widetilde{\tau}_2$. *Thus* $f=(\psi,\sigma)$ *is a pythagorean fuzzy soft continuous mapping.*

Theorem 4.1. *Let* $(X,\widetilde{\tau}_1,E)_p$ *and* $(Y,\widetilde{\tau}_2,E')_p$ *be two pythagorean fuzzy soft topological spaces,* $f=(\psi,\sigma): PFSS(X,E)\rightarrow PFSS(Y,E')$ *be a pythagorean fuzzy soft continuous mapping. If* $(G,E')\in PFSS(Y,E')$ *is a pythagorean fuzzy soft closed set, then* $f^{-1}((G,E'))\in PFSS(X,E)$ *is a pythagorean fuzzy soft closed set.*

Proof. Let $f: PFSS(X,E)\rightarrow PFSS(Y,E')$ be a pythagorean fuzzy soft continuous mapping on pythagorean fuzzy soft topological space $(X,\widetilde{\tau}_1,E)_p$ and (G,E') be any pythagorean fuzzy soft closed set in Y. Then, since $f^{-1}((G,E')^c)=(\ f^{-1}((G,E')))^c$ and $(G,E')^c$ is pythagorean fuzzy soft open set are obtained. Therefore $f^{-1}((G,E')))^c$ is pythagorean fuzzy soft openset in X. This means that $f^{-1}((G,E'))$ is a pythagorean fuzzy soft closed set in X.

Conversely, suppose that $f^{-1}((G,E'))$ is a pythagorean fuzzy soft closed set in X whenever (G,E') is a pythagorean fuzzy soft closed set in Y. For any pythagorean fuzzy soft open set (H,E') in Y. $f^{-1}((H,E')^c)=f^{-1}((H,E')))^c$. From the hypothesis, $f^{-1}((H,E')^c)$

is a pythagorean fuzzy soft closed set in X. Therefore $f^{-1}((H,E'))$ is a pythagorean fuzzy soft continuous mapping on pythagorean fuzzy soft topological space $(X,\widetilde{\tau}_1,E)_p$. $\square$

Theorem 4.2. *Let $(X,\widetilde{\tau}_1,E)_p$ and $(Y,\widetilde{\tau}_2,E')_p$ be two pythagorean fuzzy soft topological spaces and $f=(\psi,\sigma):PFSS(X,E)\to PFSS(Y,E')$ be a pythagorean fuzzy soft continuous mapping. Then f is a pythagorean fuzzy soft continuous mapping on pythagorean fuzzy soft topological space $(X,\widetilde{\tau}_1,E)_p$ if and only if $f^{-1}(pint(G,E'))\widetilde{\subseteq}$ $pint\left(f^{-1}((H,E'))\right)$ for each $(G,E')\in PFSS(Y,E')$.*

Proof. ($:\Rightarrow$) Suppose that f be a pythagorean fuzzy soft continuous mapping and $(G,E')\in PFSS(Y,E')$. Then $f^{-1}(pint(G,E')^o)\in\widetilde{\tau}_1$ and from $pint(G,E')\widetilde{\subseteq}(G,E')$ we have $f^{-1}(pint(G,E'))\widetilde{\subseteq}\ f^{-1}((G,E'))$. Because of $pint(f^{-1}((G,E')))$ is a largest pythagorean fuzzy soft open set contained by $f^{-1}((G,E'))$, $f^{-1}(pint(G,E'))\widetilde{\subseteq}pint(f^{-1}((G,E')))$.

($\Leftarrow$:) Conversely, suppose that $f^{-1}(pint(G,E'))\widetilde{\subseteq}pint(f^{-1}((G,E')))$, for all $(G,E')\in PFSS(Y,E')$. If $(G,E')\in\widetilde{\tau}_2$, then we have, $f^{-1}((G,E'))=f^{-1}(pint(G,E'))\widetilde{\subseteq}$ $pint(f^{-1}((G,E')))\widetilde{\subseteq}f^{-1}((G,E'))$. So, $f^{-1}((G,E'))\in\widetilde{\tau}_1$. It means that, f is pythagorean fuzzy soft continuous mapping. $\square$

Theorem 4.3. *Let $(X,\widetilde{\tau}_1,E)_p$ and $(Y,\widetilde{\tau}_2,E')_p$ be two pythagorean fuzzy soft topological spaces and $f:PFSS(X,E)\to PFSS(Y,E')$ be a pythagorean fuzzy soft continuous mapping. Then $f=(\psi,\sigma)$ is a pythagorean fuzzy soft continuous mapping on pythagorean fuzzy soft topological space $(X,\widetilde{\tau}_1,E)_p$ if and only if $f(pcl(H,E))\widetilde{\subseteq}pcl(f((H,E)))$ for $(H,E)\in PFSS(X,E)$.*

Proof. ($:\Rightarrow$) Suppose that f be a pythagorean fuzzy soft continuous mapping on pythagorean fuzzy soft topological space $(X,\widetilde{\tau}_1,E)_p$ and $(H,E)\in PFSS(X,E)$. Since $pcl(f((H,E)))$ is a pythagorean fuzzy soft closed set in, $f^{-1}(pcl(f((H,E))))$ is a pythagorean fuzzy soft closed set in X. Then $pcl(f^{-1}(pcl(f((H,E)))))=f^{-1}(pcl(f((H,E))))$ and $f((H,E))\widetilde{\subseteq}pcl(f((H,E)))$. Thus $(H,E)\widetilde{\subseteq}f^{-1}(f((F,E)))\widetilde{\subseteq}f^{-1}\,(pcl(f((H,E))))$ then, we obtain $(H,E)\widetilde{\subseteq}pcl(f^{-1}(pcl(f((H,E)))))\widetilde{\subseteq}f^{-1}(pcl(f((H,E))))$. Hence $f(pcl(H,E))\widetilde{\subseteq}pcl(f((H,E)))$.

($\Leftarrow$:) Suppose that $f(pcl(H,E))\widetilde{\subseteq}pcl(f((H,E)))$ for every $(H,E)\in PFSS(X,E)$ and (G,E') be any pythagorean fuzzy soft closed set in Y. So $pcl(G,E')=(G,E')$. From the hypothesis, $f(pcl(f^{-1}((G,E'))))\widetilde{\subseteq}pcl(f(f^{-1}((G,E'))))\widetilde{\subseteq}pcl(G,E')=(G,E')$ is obtained. Hence $pcl(f^{-1}((G,E')))=f^{-1}((G,E'))$ and $f^{-1}((G,E'))\widetilde{\subseteq}pcl(f^{-1}((G,E')))$. That is, $pcl(f^{-1}((G,E')))=f^{-1}((G,E'))$ and, so $f^{-1}((G,E'))$ is a pythagorean fuzzy soft closed set in X. Thus f is a pythagorean fuzzy soft continuous mapping on pythagorean fuzzy soft topological space $(X,\widetilde{\tau}_1,E)_p$. $\square$

5. Conclusion

The theories of soft set, fuzzy set, intuitionistic fuzzy soft set and pythagorean fuzzy set all are important mathematical tools for dealing with uncertainties. In this study, we have presented pythagorean fuzzy soft topological spaces. Then, we investigated pythagorean fuzzy soft interior and pythagorean fuzzy soft closure. Moreover, we define the image and pre-image of pythagorean fuzzy soft sets. Using this notion, we introduced pythagorean fuzzy soft continuous mapping.

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