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A MEAN ERGODIC THEOREM VIA WEAK STATISTICAL CONVERGENCE

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Abstract. In this article, firstly we introduce weak statistical compactness and then, we prove a mean ergodic theorem by using this new concept. Since weak convergence implies weak statistical convergence, our result is a more generalization of Cohen,[3].

Keywords: Statistical convergence, weak convergence, mean ergodic theorem.

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1. Introduction

Let K be a subset of $\mathbb{N}$, the set of all positive integers. The natural density δ of K is defined by

$$\delta(K) = \lim_{n} \frac{1}{n} |\{k \le n : k \in K\}|$$

where $|\{k \le n : k \in K\}|$ indicate the number of elements of K not exceeding n. For the details related to the density, we refer [15]. Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A sequence (x_n) in X is said to be statistical convergent to an $x_0 \in X$, if for every ε, $\delta(\{n : \|x_n - x_0\| > \varepsilon\}) = 0$. Then, we can write st-$\lim x_n = x_0$, [11]. Statistical convergence has many applications. For examples; see [4], [14], [19] and the others. Note that a convergent sequence is also statistical convergent and a statistical convergent sequence does not have to be bounded.

Let X' be the continuous dual of X. A sequence (x_n) in X is said to be weak convergent to an x_0 in X if $\lim f(x_n) = f(x_0)$ for every $f \in X'$ and for some x_0 in X. At that rate, we can write $w - \lim x_n = x_0$. It is clear that the weak limit of a sequence is alone. Also; it is shown that a strongly convergent sequence in X is also weak convergent but the converse is not true in the main, [13, p.260-261]. Weak convergence has many applications in normed spaces (see [10], [13]). Similar studies have been done by some authors (see [8, 9, 16, 17, 7]).

Weak statistical convergence in normed spaces has been defined in [2] as follows:

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Definition 1.1. *A sequence (x_n) in X is said to be weak statistical convergent to an x_0 in X if st-$\lim_n f(x_n) = f(x_0)$ for every $f \in X'$ and for some x_0 in X; that is $\delta(\{n : |f(x_n) - f(x_0)| > \varepsilon\}) = 0$. Then, we write $st_w - \lim x_n = x_0$.*

Here note that by the definition, it is obvious that a weak convergent sequence is also weak statistical convergent to the same value. It is shown in [1] that the converse of this claim is not true.

Theorem A.[13, p.115, Th. 7.10(a)]. Let X be a normed space over field $\mathbb{K}$ and $0 \neq x_0 \in X$. Then, there is some $f \in X'$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.

Theorem B.[18] Let $T : X \to X$ be linear. If the iterates T^n of T ($T^n = TT^{n-1}$) are bounded and

$$L_n x = \frac{1}{n}\sum_{i=1}^{n} T^i x,\ n = 1, 2, \cdots$$

is a weakly compact set, then there is an $x_0 \in X$ such that $Tx_0 = x_0$ and $\lim L_n x = x_0$.

Cohen [3] proved a more generalization of the Theorem B, by taking a regular matrix $A = (a_{nk})$. Recall that if $A = (a_{nk})$ is an infinite matrix of real entries a_{nk} and $x = (x_k)$ is a real number sequence, then $Ax = (Ax)_n = \sum_k a_{nk}x_k$ shows the transformed sequence of x, where, $\sum_k$ will denote summation from $k = 1$ to ∞.

A matrix $A = (a_{nk})$ is called regular if $Ax \in c$ and $\lim Ax = \lim x$ for all $x \in c$, where c is the space of all convergent sequences. It is well-known that A matrix $A = (a_{nk})$ is regular (see [5]) if and only if

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty,$$

$$\lim_n a_{nk} = 0 \text{ for each } k \text{ and}$$

$$\lim_n \sum_k a_{nk} = 1.$$

Now let us write the theorem of Cohen.

Theorem C.[3] If T is linear on a Banach space X to X such that $\|T^n\| \leq N$, $A = (a_{nk})$ is a regular matrix such that

$$\lim_k \sum_{j=k}^{\infty} |a_{n\ j+1} - a_{nj}| = 0 \text{ uniformly in } n,$$

and

$$L_n x = \sum_{j=1}^{\infty} a_{nj} T^j x$$

is a weakly compact set, then there is an $x_0 \in X$ such that $Tx_0 = x_0$ and $\lim L_n x = x_0$. For some applications of ergodic theory we refer [6].

In this paper, we firstly introduce a new concept: weak statistical compactness. Finally, by using this new concept, we prove the statistical version of Theorem C.

2. Main Result

Definition 2.1. *A set $E \subset X$ is said to be weakly statistically compact if every sequence in E has a weak statistically convergent subsequence whose st-limit is in E.*

In what follows for brevity we will write st_w-compact for weakly statistically compactness.

Lemma 2.1. *Let T and L_n $(n = 1, 2, 3, \cdots)$ be linear on X to X. If*

$$TL_n = L_nT, \tag{1}$$

$$st - \lim_n L_n(x - Tx) = 0, \tag{2}$$

$$st_w - \lim_n L_nx = x_0, \tag{3}$$

then $Tx_0 = x_0$.

Proof. Let $f \in X'$. Then by the definition of weak statistical convergence, we have

$$st - \lim_n f(L_nx - x_0) = 0. \tag{4}$$

Since T is linear, this implies that

$$st - \lim_n f(TL_nx - Tx_0) = 0. \tag{5}$$

On the other hand (2) and the continuity of f implies that

$$st - \lim_n f(L_nx - L_nTx) = 0. \tag{6}$$

Now, let us write

$$f(x_0 - Tx_0) = f(x_0 - L_nx + L_nx - L_nTx + L_nTx - Tx_0).$$

Since f is linear the following equality is satisfied

$$f(x_0 - Tx_0) = f(x_0 - L_nx) + f(L_nx - L_nTx) + f(L_nTx - Tx_0).$$

Then, since (1) implies that $L_nTx = TL_nx$, we can write the following equality:

$$f(x_0 - Tx_0) = f(x_0 - L_nx) + f(L_nx - L_nTx) + f(TL_nx - Tx_0).$$

In this equality, (3) implies that

$$st - \lim f(x_0 - L_nx) = 0,$$

(6) implies that

$$st - \lim f(L_nx - L_nTx) = 0$$

and (5) implies that

$$st - \lim f(TL_nx - Tx_0) = 0.$$

So, we can get from (7) that

$$st - \lim f(x_0 - Tx_0) = 0.$$

Since f is arbitrary in X', this means that $Tx_0 = x_0$. □

Now, we are going to give our main result. Firstly, recall that if $A = (a_{nk})$ is an infinite matrix of real entries and $x = (x_k)$ is a bounded sequence of real numbers such that $st - \lim Ax = \lim x$, then the matrix A is said to be st-regular and written $A \in (c, st \cap \ell_\infty)_{reg}$, where $st \cap \ell_\infty$ is the space of all real valued bounded and statistical convergent sequences. It is proved in [12] that A is st-regular if and only if

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty,$$

$$st - \lim_n a_{nk} = 0 \text{ for each } k = 1, 2, \cdots,$$

$$st - \lim \sum_k a_{nk} = 1.$$

Theorem 2.1. *If*

$$\|T^n\| \leq C, \ n = 1, 2, \cdots , \tag{7}$$

$$A \text{ is } st\text{- regular and } st - \lim_n \sum_k |a_{n\ k+1} - a_{nk}| = 0, \tag{8}$$

$$L_n x = \sum_{k=1}^{\infty} a_{nk} T^k x \text{ is a } st_w\text{-compact set}, \tag{9}$$

then there exists an $x_0 \in X$ such that $Tx_0 = x_0$ and st-$\lim L_n x = x_0$.

Proof. Firstly note that as it is shown in [3], by the condition (8), L_n are well defined on X. Also, since for any $z \in X$

$$\begin{aligned} \|L_n z\| &\leq C\|z\| \sum_k |a_{nk}| \\ &\leq C\|z\|.\|A\| \end{aligned}$$

the same condition implies that $\|L_n\| < \infty$. On the other hand, since T is linear, $TL_n z = L_n T z$. Now, for any $z \in X$, we get

$$\begin{aligned} \{n : \|L_n(z - Tz)\| > \varepsilon\} &= \Big\{n : \|a_{n_1} Tz + \sum_k (a_{n\ k+1} - a_{nk}) T^{k+1} z\| > \varepsilon\Big\} \\ &\subseteq \Big\{n : C\|z\| \Big(|a_{n_1}| + \sum_{k=1}^{N} |a_{n\ k+1} - a_{nk}| + \sum_{k \geq N} |a_{n\ k+1} - a_{nk}|\Big) > \varepsilon\Big\} \\ &\subseteq \Big\{n : C\|z\| \Big(|a_{n_1}| + \sum_{k=1}^{N} |a_{n\ k+1} - a_{nk}|\Big) > \varepsilon\Big\} \cup \Big\{n : \sum_{k=1}^{N} |a_{n\ k+1} - a_{nk}| > \varepsilon\Big\} \\ &\subseteq \Big\{n : C\|z\| \Big(|a_{n_1}| + \sum_{k=1}^{N} |a_{n\ k+1} - a_{nk}|\Big) > \varepsilon\Big\} \cup \Big\{n : \sum_{k=1}^{\infty} |a_{n\ k+1} - a_{nk}| > \varepsilon\Big\}. \end{aligned}$$

Then, by using the condition (8), we have

$$\begin{aligned} \delta(\{n : \|L_n(z - Tz)\| > \varepsilon\}) &\leq \delta\Big(\Big\{n : C\|z\| \Big(|a_{n_1}| + \sum_{k=1}^{N} |a_{n\ k+1} - a_{nk}|\Big) > \varepsilon\Big\}\Big) \\ +\delta\Big(\Big\{n : \sum_{k=1}^{\infty} |a_{n\ k+1} - a_{nk}| > \varepsilon\Big\}\Big) & \\ &= 0. \end{aligned}$$

Thus, $st - \lim L_n(z - Tz) = 0$.

On the other hand, from (9), there exists an $x_0 \in X$ such that $st_w - \lim_i L_{n_i} x = x_0$. So, T and L_{n_i} satisfy the conditions of Lemma 2.1 and hence $Tx_0 = x_0$.

Since $Tx_0 = x_0$, $T^k x_0 = x_0$, $k = 1, 2, \cdots$ and

$$\begin{aligned} L_n x_0 &= \sum_k a_{nk} T^k x \\ &= \Big(\sum_k a_{nk}\Big) x_0, \end{aligned}$$

from (8), $st - \lim L_n x_0 = x_0$. Now, by writing $x = x_0 + (x - x_0)$, we get that $st - \lim L_n x = x_0 + st - \lim L_n(x - x_0)$.

To complete the proof, we are going to show that $st-\lim L_n(x-x_0)=0$. Suppose that this is false. Then, by Theorem A, we can find an $f_0 \in X'$ such that $f_0(x-x_0)=1$ and $f_0(y-Ty)=0, y \in X$. Then, since $(T^kx-T^{k+1}x)\in X,\ k=1,2,\cdots$, we have $f_0(T^kx-f_0T^{k+1}x)=0$. Also, since $(x-Tx)\in X$ and

$$\begin{aligned} f_0(x-T^kx) &= f_0(x-T^{k-1}x)+f_0(T^{k-1}x-T^kx) \\ &= f_0(x-T^{k-1}x), \end{aligned} \tag{10}$$

we have $f_0(x)=f_0(T^kx)$, $k=1,2,\cdots$.

By the linearity of f_0, from (8) and (10), we get that

$$\begin{aligned} f_0(L_nx) &= \sum_k a_{nk}f_0(T^kx) \\ &= \Big(\sum_k a_{nk}\Big)f_0(x). \end{aligned}$$

So, the condition (8) implies that

$$st-\lim f_0(L_nx)=f_0(x).$$

Now, since L_nx is st_w-compact, there exists a subsequence L_{n_i} and an $x_0\in X$ such that $L_{n_i}x$ st-weakly converges to x_0. Thus, we have

$$\begin{aligned} 0 &= st-\lim_i f_0(x_0-L_{n_i}x) \\ &= f_0(x_0)-(st-\lim_i L_{n_i}x) \\ &= f_0(x_0)-f_0(x) \end{aligned}$$

which is a contradiction to $f_0(x-x_0)=1$. This completes the proof. □

Note that the classical Cesáro matrix $(C,1)$ of the first type gratify the conditions of the Theorem C. Now we will give an example which compensate the conditions of our theorem but do not gratify the conditions of the Theorem C. Let us define a matrix $A=(a_{nk})$ by

$$a_{nk}=\begin{cases} 0 \quad , & n=k^2 \\ 1/n \quad , & n\neq k^2 \text{ and } 1\leq k\leq n. \end{cases}$$

Then, $A=(a_{nk})$ is st-regular and $st-\lim_n\sum_k|a_{n\ k+1}-a_{nk}|=0$. But, since $\lim_n\sum_k a_{nk}$ does not exist, $A=(a_{nk})$ is not regular.

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