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M MODULO N GRACEFUL LABELING OF PATH UNION AND JOIN SUM OF COMPLETE BIPARTITE GRAPHS WITH ITS ALGORITHMS

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Abstract. A graceful labeling of a graph $G(p, q)$ is an injective assignment of labels from the set $\{0, 1, ..., q\}$ to the vertices of G such that when each edge of G has been assigned a label defined by the absolute difference of its end-vertices, the resulting edge labels are distinct. In this paper we used the new labeling technique known as M modulo N graceful labeling and prove that path union of complete bipartite graphs and join sum of complete bipartite graphs are M modulo N graceful labeling. We also give a $C++$ program for finding M modulo N graceful labeling on above said graphs.

Keywords: Complete bipartite graph, Join sum of complete bipartite, Path union of complete bipartite graphs, One modulo N graceful labeling, M modulo N graceful labeling.

AMS Subject Classification: 05C78, 05C85.

1. Introduction

Let $G(V, E)$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$ containing p vertices and q edges respectively. A graceful labeling of a graph G of size q is an injective assignment of labels from the set $\{0, 1, ..., q\}$ to the vertices of G such that when each edge of G has been assigned a label defined by the absolute difference of its end-vertices, the resulting edge labels are distinct. Rosa and Golomb. S.W proved the complete bipartite graph K_{n_1,n_2} has an α-valuation for all $n_1, n_2 \geq 1$ [8, 3] and also its clear that if there exists an α-labeling of graph G, then G is a bipartite graph [8]. Jayanthi, Ramya and Selvi proved that the graph $T@P_n$, $T@2P_n$ and $< T \tilde{o} K_{1,n} >$ are even vertex odd mean graph [5]. Deligen, Lingqi Zhao, Jirimutu discussed that the $k-$gracefulness of $r-$crown $I_r(K_{m,n})$ $(m \leq n, r \geq 2)$ for complete bipartite graph $K_{m,n}$ and proved the conjecture when $m = 5$, for arbitrary $n \geq m$ and $r \geq 2$ [1]. Grady D. Bullington, Linda L. Eroh and Steven J. Winters to studied explicit formulae for the γ-min and γ-max labeling values of a complete bipartite graph [4]. WuZhuang Li, GuangHai Li, and QianTai Yan studied on various Labeling in Complete Bipartite Graphs [12]. Raju. V and Paruvatha Vathana M.

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examined some results on graceful labeling in the join sum of complete bipartite graphs are graceful [7]. Kaneria V. J, Makadia H. M, Jariya M. M. and Meera Meghapara discussed graceful labeling of some graphs obtained by complete bipartite graphs which are the path union of complete bipartite graph and join sum of complete bipartite graphs [6]. Odd gracefulness was introduced by Gnanajothi R. B. [2]. Sekar. C. introduced concept of one modulo three graceful labeling [11]. Ramachandran. V. and Sekar. C. introduced the labeling method of one modulo N graceful where N is a positive integer and showed many graphs are one modulo N graceful graph [9]. Ramachandran. V. proved that cycle C_n is one modulo N graceful labeling if $n \equiv (0\ mod)4$ [10]. Velmurugan. C. and Ramachandran. V. introduced M modulo N graceful labeling and proved that path and star are M modulo N graceful graph [13]. If a graph G is M modulo N graceful labeling, then when M = N =1 the labeling is graceful labeling, when M =1 and N = 2 the labeling is odd graceful labeling, when M =1 and N = 3 the labeling is one modulo 3 graceful labeling and when M = 1 and N=N the labeling is one modulo N graceful labeling.

2. Basic definitions :

2.1. **Definition.** A bipartite graph is a graph in which the vertices can be partitioned into two disjoint sets V_1 and V_2 such that each edge is an edge between a vertex in V_1 to a vertex in V_2.

2.2. **Definition.** A complete bipartite graph is a simple graph in which the vertices can be partitioned into two disjoint sets V_1 and V_2 such that each vertex in V_1 is adjacent to each vertex in V_2 . Take $|V_1| = m$ and $|V_2| = n$, the complete bipartite graph is denoted by $K_{m,n}$.

2.3. **Definition.** An α-valuation of a graph G is a graceful valuation of G which also satisfies the following condition: there exists a number γ $(0 \leq \gamma <$ E(G)) such that, for any edge $e \in E(G)$ with the end vertices $u, v \in V(G)$, min { vertex label (v), vertex label (u) } $\leq \gamma <$ max { vertex label (v), vertex label (u) }.

2.4. **Definition.** Consider K copies of a graph G_0 then graph $G = \left\langle G_0^{(1)}, G_0^{(2)}, ..., G_0^{(k)} \right\rangle$ obtained by joining two copies of the graph $G_0^{(i)}$ and $G_0^{(i+1)}$ by a vertex $1 \leq i \leq k-1$ is called join sum of graphs.

2.5. **Definition.** Let G be a graph and $G_1, G_2, ..., G_n, n \geq 2$ be $n-$ copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} (i=1, 2, ..., $n-1$) is called path union of G.

2.6. **Definition.** A graceful labeling of a graph G of size q is an injective assignment of labels from the set $\{0, 1, ..., q\}$ to the vertices of G such that when each edge of G has been assigned a label defined by the absolute difference of its end-vertices, the resulting edge labels are distinct.

2.7. **Definition.** A graph G is said to be one modulo N graceful labeling(where N is a positive integer) if there is a function f from the vertex set of G to {0, 1, N, (N + 1), 2N, (2N + 1), . . ., N(q - 1), N(q - 1) + 1} in such a way that (i) f is 1 - 1 (ii) f induces a bijection f^* from the edge set of G to $\{1, N+1, 2N+1, ..., N(q-1)+1\}$ where $f^*(uv)$ = $|f(u) - f(v)|$ for all $u, v \in V(G)$.

2.8. **Definition.** A graph $G(V(G), E(G))$ with p vertices and q edges is said to be M modulo N graceful labeling (where N is positive integer and M = 1 to N) if there is a function f from the vertex set of G to {0, M, N, N + M, $2N$, ..., N(q - 1), N(q - 1) + M } in such a way that (i) f is 1-1, (ii) f induces a bijection f^* from edge set of G to { M, N + M, $2N$ + M, ..., N(q-1) + M } where $f^*(uv) = |f(u) - f(v)|$ for all $u, v \in V(G)$. A graph G satisfied M modulo N graceful labeling is known as M modulo N graceful graph.

3. Main Result

In this section we focused on to prove that M modulo N Graceful Labeling on path union of complete bipartite graphs and join sum of complete bipartite graphs.

Theorem 3.1. *The Path union of complete bipartite graphs* K_{m_1,n_1}; K_{m_2,n_2}; K_{m_3,n_3}; ...; K_{m_t,n_t} *is* M *modulo* N *graceful labeling, where* $m_1, n_1, ..., m_t, n_t$ *are natural numbers.*

Proof. Let G be a Path union of complete bipartite graphs K_{m_1,n_1}; K_{m_2,n_2}; K_{m_3,n_3}; ...; K_{m_t,n_t}, where m_1, n_1, ..., m_t, n_t are Natural numbers. To produce the path union of this graphs we join u_{m_i} with $v_{(n_i+1)}$ by an edge for all $i = 1$ to $t-1$.

Let G has partion of two disjoint vertex sets named as { u_i, i = 1 to $\sum_{j=1}^{t} m_j$ } and { v_i, i = 1 to $\sum_{j=1}^{t} n_j$ }.

Define M modulo N graceful labeling on vertices of Path union of complete bipartite graphs:

Labeling on vertices named as "u" in the Path union of complete bipartite graphs:

$f(u_i) = (i-1)N$, i = 1 to $\sum_{j=1}^{t} m_j$.

Labeling on vertices named as "v" in the Path union of complete bipartite graphs:

$$f\left(v_{i+\sum_{l=1}^{k-1} n_l}\right) = \left[\sum_{j=1}^{t} m_j n_j - \sum_{r=1}^{k-1} m_r(n_r - 1) - m_k(i-1) + t - k - 1\right] N + \text{M},\ i = 1 \text{ to } n_k,$$

$k = 1$ to t.

From the definition of f it's clear that $\left\{\left\{f(u_i),\ i = 1 \text{ to } \sum_{j=1}^{t} m_j\right\} \cup \left\{ f\left(v_{i+\sum_{l=1}^{k-1} n_l}\right),\ i = 1 \text{ to } n_k,\ k = 1 \text{ to } t\right\}\right\} = \left\{\left\{0, N, 2N, ..., \left[\sum_{j=1}^{t} m_j - 1\right] N\right\} \cup \left\{\left[\sum_{j=1}^{t} m_j n_j + t - 2\right] N + \text{M}, \left[\sum_{j=1}^{t} m_j n_j - m_1 + t - 2\right] N + \text{M}, ..., \left[\sum_{j=1}^{t} m_j n_j - \sum_{r=1}^{t-1} m_r(n_r - 1) - m_t(n_t - 1) - 1\right] N + \text{M}\right\}\right\} \subseteq \{0, \text{M}, N, N + \text{M}, 2N, ..., N(q-1), N(q-1) + \text{M}\}$. Hence the vertices has distinct M modulo N graceful labeling.

Define M modulo N graceful labeling on edges of Path union of complete bipartite graphs:

$$f^*\left(e_{\sum\limits_{h=1}^{k} m_{h-1}n_{h-1}+(i-1)m_k+(k-1)+j}\right) = \left[\sum_{d=1}^{t} m_d n_d - \sum_{r=1}^{k-1} m_r n_r \text{ - } m_k(i-1) + t - k - j\right] N$$

$+ \text{M}$, $j = 1$ to m_k, $i = 1$ to n_k, $k = 1$ to t, $m_0 = 0$, $n_0 = 0$.

$$f^*\left(e_{\sum\limits_{h=1}^{k} m_h n_h + k}\right) = \left[\sum_{d=1}^{t} m_d n_d - \sum_{r=1}^{k} m_r n_r + t - k - 1\right] N + \text{M},$$

$k = 1$ to $t - 1$.

From the definition of f^* it's clear that $\left\{ \left\{ f^*\left(e_{\sum\limits_{h=1}^{k} m_{h-1}n_{h-1}+(i-1)m_k+(k-1)+j}\right), j = 1 \text{ to } m_k, i = 1 \text{ to } n_k, k = 1 \text{ to } t, m_0 = 0, n_0 = 0 \right\} \cup \left\{ f^*\left(e_{\sum\limits_{h=1}^{k} m_h n_h + k}\right), k = 1 \text{ to } t - 1 \right\} \right\} = \left\{ \left\{ \left[\sum\limits_{d=1}^{t} m_d n_d + \text{t-2}\right] N + \text{M}, \left[\sum\limits_{d=1}^{t} m_d n_d + \text{t-3}\right] N + \text{M}, ..., \left[\sum\limits_{d=1}^{t} m_d n_d - \sum\limits_{r=1}^{t} m_r n_r\right] N + \text{M} \right\} \cup \left\{ \left[\sum\limits_{d=1}^{t} m_d n_d - m_1 n_1 + t - 2\right] N + \text{M}, \left[\sum\limits_{d=1}^{t} m_d n_d - \sum\limits_{r=1}^{2} m_r n_r + t - 3\right] N + \text{M}, ..., \left[\sum\limits_{d=1}^{t} m_d n_d - \sum\limits_{r=1}^{t-1} m_r n_r\right] N + \text{M} \right\} \right\} = \{ \text{M}, N + \text{M}, 2N + \text{M}, ..., N(\text{q-1}) + \text{M} \}$.
Hence each edges has distinct labelings and so induced function f^* is 1-1 .

□

Theorem 3.2. *C++ program for Designing* M *modulo N graceful labeling of Path union of complete bipartite graphs* $K_{m_1,n_1}; K_{m_2,n_2}; K_{m_3,n_3}; ...; K_{m_t,n_t}$*, where* $m_1, n_1, ..., m_t, n_t$ *are Natural numbers.*

Proof.
```
♯include<iostream.h>
♯include<conio.h>
void main()
{
clrscr();
int a,b,i,j,t,l,k,r,h,d,s,N,M,x,y,m[10],n[10],z,P,Q,R,Y,I,E,F,G,Z,X,H;
cout≪"Enter N value: N = ";
cin≫ N;
cout≪ "Enter t <10 : t = ";
cin≫ t;
cout≪"Enter m[t] and n[t] value upto t= 1 to "≪t≪endl;
for(i = 1; i¡= t; i++)
{
```

```
cout≪” m[”≪i≪”] =”;
cin≫ m[i];
cout≪” n[ ”≪ i≪ ”] =”;
cin≫ n[i];
}
cout≪ ” M modulo N graceful labeling of G = Km[1], n[1]; Km[2], n[2]; .....; Km[t], n[t];
path union complete bipartite graphs ”≪endl;
cout≪”Want to find particular M value say Yes type 1 :Y=”;
cin≫ Y;
if(Y==1)
{
cout≪”Eenter M value: M=”;
cin≫M;
goto I;
}
for(M=1;M<=N;M++)
{
I:
cout≪endl≪ M ≪” modulo ”≪ N ≪” graceful labeling for Vertices”≪endl;
b=0;m[0]=0;
for(i=1;i<=t;i++)
{
b=b+m[i-1];
for(j=1;j<=m[i];j++)
{
a=b+j;
cout≪” f(u”≪a≪”)=”≪(a-1)*N;
} }
y=0;
for(j=1;j<=t;j++)
{
y=y+(m[j]*n[j]);
}
for(k=1;k<=t;k++)
{
z=0;x=0;
for(l=1;l<=k-1;l++)
{
z=z+n[l];
x=x+(m[l]*(n[l]-1));
}
for(i=1;i<=n[k];i++)
{
cout≪” f(v”≪i+z≪”)=”≪(y-x-m[k]*(i-1)+t-k-1)*N + M;
} }
cout≪endl≪M≪” modulo ”≪ N ≪” graceful labeling for edges”≪endl;
for(k=1;k<=t;k++)
{
E=0;F=0;G=0;n[0]=0;m[0]=0;
```

```
for(h=1;h<=k;h++)
{
E=E+(m[h-1]*n[h-1]);
}
for(s=1;s<=k-1;s++)
{
F=F+(m[s]*n[s]);
}
for(i=1;i<=n[k];i++)
{
H=(i-1)*m[k];
for(j=1;j<=m[k];j++)
{
cout<<" f*(e"<<E+H+(k-1)+j<<")="<<(y-F-m[k]*(i-1)+t-k-j)*N + M;
} } }
for(k=1;k<=t-1;k++)
{
P=0;Q=0;R=0;m[0]=0;n[0]=0;
for(s=1;s<=k;s++)
{
P=P+(m[s]*n[s]);
Q=Q+(m[s]*n[s]);
}
cout<<" f*(e"<<P+k<<")="<<(y-Q+t-k-1)*N + M;
}
if(Y==1)
{
cout<<endl<<"G=Km[1],n[1];Km[2],n[2];.....;Km[t],n[t];path union of complete bipartite graphs
is"<<M<<" modulo "<<N<<" graceful labeling";
goto J;
} }
cout<<endl<<"G=Km[1],n[1];Km[2],n[2];.....;Km[t],n[t];path union of complete bipartite graphs
is M modulo N graceful labeling";
J:
getch();

}
```

□

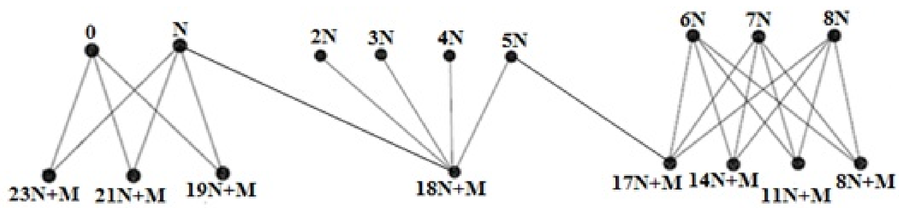

FIGURE 1. M modulo N graceful Labeling of Path union of complete bipartite graphs $K_{2,4}; K_{4,1}; K_{3,4}$

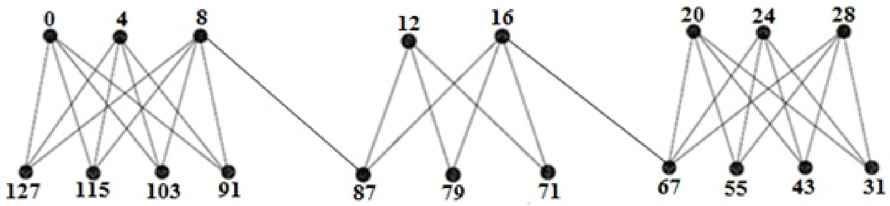

FIGURE 2. 3 modulo 4 graceful Labeling of Path union of complete bipartite graphs $K_{3,4}; K_{2,3}; K_{3,4}$

Note: Path union of complete bipartite graphs $K_{3,4}; K_{2,3}; K_{3,4}$ in Figure 2 has an α-labeling with $\gamma = 28$.

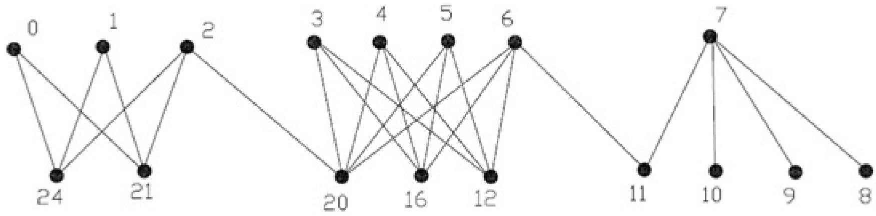

FIGURE 3. 1 modulo 1 graceful Labeling of Path union of complete bipartite graphs $K_{3,2}; K_{4,3}; K_{1,4}$

Note: Path union of complete bipartite graphs $K_{3,2}; K_{4,3}; K_{1,4}$ in Figure 3 has an α-labeling with $\gamma = 7$.

Theorem 3.3. *Let $G = < K_{m_1,n_1}; K_{m_2,n_2}; K_{m_3,n_3}; ...; K_{m_t,n_t} >$ be the join sum of complete bipartite graphs is* M *modulo N graceful labeling, where $m_1, n_1, ..., m_t, n_t$ are Natural numbers.*

Proof. Let G be the join sum of complete bipartite graphs $< K_{m_1,n_1}$; K_{m_2,n_2}; K_{m_3,n_3}; ...; $K_{m_t,n_t} >$, where m_1, n_1,..., m_t, n_t are Natural numbers. Let two disjoint sets u_1, u_2, ..., u_{m_1}, u_{m_1+1}, ..., $u_{m_1+m_2}$, ..., $u_{m_1+m_2+...+m_t}$ and v_1, v_2,..., v_{n_1}, v_{n_1+2}, ..., $v_{n_1+n_2+1}$, $v_{n_1+n_2+3}$, ..., $v_{n_1+n_2+...+n_t+(t-1)}$. Also include v_{n_1+1}, $v_{n_1+n_2+2}$, ..., $v_{n_1+n_2+...+n_{t-1}+(t-1)}$ be the vertices for join sum of complete bipartite graphs and join vertices are ($v_{n_1+n_2+...+n_i+(i-1)}$, $v_{n_1+n_2+...+n_i+(i)}$) and $\left(v_{n_1+n_2+...+n_i+(i)}, v_{n_1+n_2+...+n_i+(i+1)}\right)$, $1 \leq i \leq t-1$ by on edge to produced join sum of complete bipartite graph G.

Define M modulo N graceful Labeling for vertices of join sum of complete bipartite graphs

Labeling on the vertices naming as "u" in join sum of complete bipartite graphs:
$f(u_i)= (i-1)N$, $i = 1$ to $\sum\limits_{j=1}^{t} m_j$.

Labeling on the vertices naming as "v" except join vertices in join sum of complete bipartite graphs:

$$f\left(v_{i+\sum_{l=1}^{k-1} n_l+(k-1)}\right)=\left[\sum_{j=1}^{t} m_j n_j - \sum_{r=1}^{k-1} m_r(n_r-1) - m_k(i-1)+2t-2k-1\right]N+\text{M},$$

$i=1$ to n_k, $k=1$ to t.

Labeling on the join vertices in join sum of complete bipartite graphs:

$$f\left(v_{\sum_{l=1}^{k} n_l+(k)}\right)=\left[\sum_{j=1}^{t} m_j n_j - \sum_{r=1}^{k} m_r(n_r-1)+2t-2k-2\right]N+\text{M},\ k=1 \text{ to } t-1.$$

From the definition of f it's clear that $\Bigg\{\Bigg\{ f(u_i), i=1 \text{ to } \sum_{j=1}^{t} m_j \Bigg\}\cup\Bigg\{ f\left(v_{i+\sum_{l=1}^{k-1} n_l+(k-1)}\right),$ $i=1$ to n_k, $k=1$ to $t\Bigg\}\cup\Bigg\{ f\Big(v_{\sum_{l=1}^{k} n_l+(k)}$, $k=1$ to t-1 $\Big)\Bigg\}\Bigg\}=\Bigg\{\Bigg\{$ 0, N, $2N$, ..., $\left[\sum_{j=1}^{t} m_j -1\right]N\Bigg\}\cup\Bigg\{\left[\sum_{j=1}^{t} m_j n_j+2t-3\right]N+\text{M}$, $\left[\sum_{j=1}^{t} m_j n_j - m_1+2t-3\right]N+\text{M}$, ..., $\left[\sum_{j=1}^{t} m_j n_j - \sum_{r=1}^{t-1} m_r(n_r-1) - m_t(n_t-1)-1\right]N+\text{M}\Bigg\}\cup\Bigg\{\left[\sum_{j=1}^{t} m_j n_j - m_1(n_1-1)+2t-4\right]N$ $+\text{M}$, $\left[\sum_{j=1}^{t} m_j n_j - \sum_{r=1}^{2} m_r(n_r-1)+2t-6\right]N+\text{M}$, ..., $\left[\sum_{j=1}^{t} m_j n_j - \sum_{r=1}^{t-1} m_r(n_r-1)\right]N+\text{M}$ $\Bigg\}\Bigg\}\subseteq\{0, \text{M}, N, N+\text{M}, 2N, ..., N(\text{q}-1), N(\text{q}-1)+\text{M}\}$. Hence the vertices have distinct M modulo N graceful labeling.

Define M modulo N graceful Labeling an edges on join sum of complete bipartite graphs

Labeling an edges incident with join vertices of join sum of complete bipartite graphs:

$$f^*\left(e_{\sum_{h=1}^{k} m_h n_h+2(k-1)+j}\right)=\left[\sum_{d=1}^{t} m_d n_d - \sum_{r=1}^{k} m_r n_r+2t-2k-j\right]N+\text{M},\ j=1 \text{ to } 2,$$

$k=1$ to $t-1$.

Labeling an edges not incident with join vertices of join sum of complete bipartite graphs:

$$f^*\left(e_{\sum_{h=1}^{k} m_{h-1}n_{h-1}+(i-1)m_k+2(k-1)+j}\right)=\left[\sum_{d=1}^{t} m_d n_d - \sum_{r=1}^{k-1} m_r n_r - m_k(i-1)+ 2t-2k-j\right]N+\text{M},$$

$j=1$ to m_k, $i=1$ to n_k, $k=1$ to t, $m_0=0$ and $n_0=0$.

From the definition of f^* it's clear that $\left\{ \left\{ f^*\left(e_{\sum\limits_{h=1}^{k} m_h n_h + 2(k-1)+j}\right), j = 1 \text{ to } 2, k = 1 \text{ to } t-1 \right\} \cup \left\{ f^*\left(e_{\sum\limits_{h=1}^{k} m_{h-1}n_{h-1}+(i-1)m_k+2(k-1)+j}\right), j = 1 \text{ to } m_k, i = 1 \text{ to } n_k, k = 1 \text{ to } t, m_0 = 0 \text{ and } n_0 = 0 \right\} \right\} = \left\{ \left\{ \left[\sum\limits_{d=1}^{t} m_d n_d - m_1 n_1 + 2t \text{ - } 3\right] N + \text{M}, \left[\sum\limits_{d=1}^{t} m_d n_d - m_1 n_1 + 2t \text{ - } 4\right] N + \text{M}, ..., \left[\sum\limits_{d=1}^{t} m_d n_d - \sum\limits_{r=1}^{t-1} m_r n_r\right] N + \text{M} \right\} \cup \left\{ \left[\sum\limits_{d=1}^{t} m_d n_d + 2t - 3\right] N + \text{M}, \left\{ \left[\sum\limits_{d=1}^{t} m_d n_d + 2t - 4\right] N + \text{M}, ..., \left[\sum\limits_{d=1}^{t} m_d n_d - \sum\limits_{r=1}^{t} m_r n_r\right] N + \text{M} \right\} \right\} = \{ \text{M}, N + \text{M}, 2N + \text{M}, ..., N(\text{q-1}) + \text{M} \}$. Hence each edges has distinct labeling and induced function f^* is 1-1 .

□

Theorem 3.4. *C++ program for* M *modulo* N *graceful labeling of* $G = < K_{m_1,n_1}; K_{m_2,n_2}; K_{m_3,n_3}; ...; K_{m_t,n_t} >$ *the join sum of complete bipartite graphs, where* $m_1, n_1, ..., m_t, n_t$ *are Natural numbers.*

Proof.
```
♯include<iostream.h>
♯include<conio.h>
void main()
{
clrscr();
int a,b,i,j,t,l,k,r,h,d,s,N,M,x,y,m[10],n[10],z,P,Q,R,Y,I,E,F,G,Z,X,H;
cout≪"Enter N value: N = ";
cin≫N;
cout≪"Enter t<10 : t = ";
cin≫t;
cout≪"Enter m and n value upto t="≪t≪endl;
for(i=1;i<=t;i++)
{
cout≪" m["≪i≪"]=";
cin≫m[i];
cout≪" n["≪i≪"]=";
cin≫n[i]; }
cout≪"M modulo N graceful labeling of G=<Km[1],n[1];Km[2],n[2];.....;Km[t],n[t]>"≪endl;
cout≪"Want to find particular M value say Yes type 1 :Y="; cin≫Y; if(Y==1) {
cout≪"Eenter M value: M=";
cin≫M;
goto I;
}
for(M=1;M<=N;M++)
{
I:
cout≪endl≪M≪" modulo "≪ N ≪" graceful labeling for Vertices"≪endl;
```

```
b=0;
m[0]=0;
for(i=1;i<=t;i++)
{
b=b+m[i-1];
for(j=1;j<=m[i];j++)
{
a=b+j;
cout≪" f(u"≪a≪")="≪(a-1)*N;
} }
y=0;
for(j=1;j<=t;j++)
{
y=y+(m[j]*n[j]);
}
for(k=1;k<=t;k++)
{
z=0;x=0;
for(l=1;l<=k-1;l++)
{
z=z+n[l];
x=x+(m[l]*(n[l]-1));
}
for(i=1;i<=n[k];i++)
{
cout≪" f(v"≪i+z+(k-1)≪")="≪(y-x-m[k]*(i-1)+2*t-2*k-1)*N + M;
} }
for(k=1;k<=t-1;k++)
{
Z=0;X=0;
for(l=1;l<=k;l++)
{
Z=Z+n[l];
X=X+(m[l]*(n[l]-1));
}
cout≪" f(v"≪Z+k≪")="≪(y-X+2*t-2*k-2)*N + M;
}
cout≪endl≪M≪" Modulo "≪ N ≪" graceful labeling for edges"≪endl;
for(k=1;k<=t;k++)
{
E=0;F=0;G=0;n[0]=0;m[0]=0;
for(h=1;h<=k;h++)
{
E=E+(m[h-1]*n[h-1]);
}
for(s=1;s<=k-1;s++)
{
F=F+(m[s]*n[s]);
}
```

```
for(i=1;i<=n[k];i++)
{
H=(i-1)*m[k];
for(j=1;j<=m[k];j++)
{
cout≪” f*(e”≪E+H+2*(k-1)+j≪”)=”≪(y-F-m[k]*(i-1)+2*t-2*k-j)*N + M;
} } }
for(k=1;k<=t-1;k++)
{
P=0;Q=0;R=0;m[0]=0;n[0]=0;
for(s=1;s<=k;s++)
{
P=P+(m[s]*n[s]);
Q=Q+(m[s]*n[s]);
}
for(j=1;j<=2;j++)
{
cout≪” f*(e”≪P+2*(k-1)+j≪”)=”≪(y-Q+2*t-2*k-j)*N + M;
} }
if(Y==1)
{
cout≪endl≪”G=<Km[1],n[1];Km[2],n[2];.....;Km[t],n[t]> the join sum of complete bipar-
tite graphs is ”≪ M≪” modulo ”≪ N ≪ ” graceful labeling”;
goto J;
} }
cout≪endl≪”G=<Km[1],n[1];Km[2],n[2];.....;Km[t],n[t]> the join sum of complete bipar-
tite graphs is M modulo N graceful labeling”;
J:
getch();
}
```

□

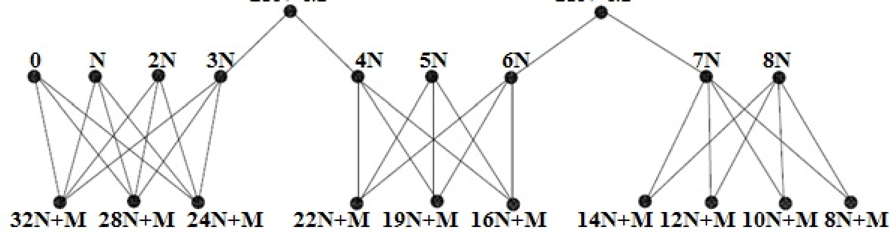

FIGURE 4. $G =< K_{4,3}; K_{3,3}; K_{2,4} >$ the join sum of complete bipartite graphs is M modulo N graceful labeling

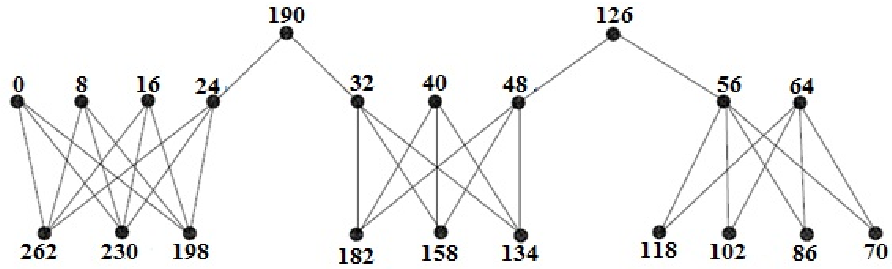

FIGURE 5. $G =< K_{4,3}; K_{3,3}; K_{2,4} >$ the join sum of complete bipartite graphs is 6 modulo 8 graceful labeling

Note: $G =< K_{4,3}; K_{3,3}; K_{2,4} >$ the join sum of complete bipartite graphs in Figure 5 has an α-labeling with $\gamma = 64$.

4. Conclusion

In this study, we have proved that the path union of complete bipartite graphs and join sum of complete bipartite graphs are M modulo N graceful labeling. The given $C++$ program are especially helpful to invent M modulo N graceful labeling for above said graphs for all M and N. Also M modulo N graceful labeling of the path union of complete bipartite graphs and join sum of complete bipartite graphs admits α-labeling. In our future work we are going to focus on if every graph G with α - labeling admits M modulo N graceful labeling or not.

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