TWMS J. App. and Eng. Math. V.12, N.4, 2022, pp. 1228-1238

ACCURATE APPROXIMATE SOLUTION OF CLASSES OF BOUNDARY VALUE PROBLEMS USING MODIFIED DIFFERENTIAL TRANSFORM METHOD

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Abstract. In this paper, a numerical scheme so-called modified differential transformation method (MDTM) based on differential transformation method (DTM), Laplace transform and Padé approximation will be used to obtain accurate approximate solution for a class of boundary value problems (BVP's). The MDTM is employed as an alternative technique to overcome some difficulties in the behavior of the solution and to be valid for a large region. The numerical results obtained demonstrate the applicability and validity of this technique. Numerical comparison is made with existing exact solution.

Keywords: Boundary value problems, Differential transform method, Laplace transform, Padé approximants.

AMS Subject Classification: 41A21, 34B60.

1. Introduction

Boundary value problems are considered as one of an important kind of ordinary differential equations that arise in several branches of engineering, optimization, technology control etc. For instance, in engineering it occurs in beam deflections, optimal control,

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§ Manuscript received: November 07, 2019; accepted: January 06, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.4

fluid dynamics, hydrodynamics and hydro magnetics stability. Obtaining numerical or approximate solutions for BVPs have been proposed in last few years by many researchers, for example see [1, 2, 3, 4]. Another kind of BVPs which attracted the attention of researchers in last years called multipoint BVP, which is an ordinary differential equations with boundary conditions, specified at different points equals to the order of the differential equation. This type of differential equation has received much attention because its wide applications in different area of applied science and engineering. For example, theory of elasticity and fluid flow through porous plate, the vibrations in a wire of uniform cross sections and composed of materials with different densities.

Many research papers have been carried out to find accurate approximate solutions of second order three point BVPs and third order multipoint BVPs via different methods [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 31]. In the last years, DTM has been applied successfully to obtain approximate solution for various types of differential equations such as initial value problems, difference equations, and BVPs. The idea of DTM is based on the concept of Taylor series [16, 17, 18, 19], and it usually gets the solution in a series form. This method constructs an analytical solution in the form of a polynomial. It uses the form of polynomials as the approximation to exact solutions which are sufficiently differentiable. However, DTM has some hitches. By using DTM, we obtain a truncated series solution. This series solution is in a good approximation which converge to the exact solution but in a small region [20].

In order to improve the accuracy of DTM, we use an alternative technique (MDTM) which modifies the series solution for classes of BVPs by applying the Laplace transformation to the truncated series obtained by DTM followed by converting the transformed series into a meromorphic function by Padé approximants, and finally by applying the inverse Laplace transform to the obtained analytic solution, which gives the exact solution or give more accurate solution than the DTM solution large region.

This paper is organized as follows. In section 2, a brief overview of some fundamental concepts and definitions related to differential transformation method, Padé approximantion and Laplace transformation are presented. Section 3 presents several examples to prove and demonstrate the validity and efficiency of our technique. Finally, conclusion and discussion are given in section 4.

2. Preliminaries

In this section, we present some definitions of DTM and Padé approximants.

2.1. Differential Transform.

Definition 2.1. [21]
If a function $f(x)$ is analytical with respect to x in the domain of interest, then

$$F(k) = \frac{f^{(k)}(x_0)}{k!}. \tag{1}$$

The inverse differential transform of $F(k)$ is defined as:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{2}$$

From (1) and (2), we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{3}$$

If $U(k)$, $G(k)$ and $H(k)$ are the differential transforms of $u(x)$, $g(x)$ and $h(x)$ respectively at $x_0 = 0$, then the main operations of the DTM

TABLE 1. Main operations of the DTM

Original function	Transformed function
$u(x) = g(x) + h(x)$	$U(k) = G(k) + H(k)$
$u(x) = cg(x)$	$U(k) = cG(k)$
$u(x) = \frac{d^n g(x)}{dx^n}$	$U(k) = \frac{(k+n)!}{k!} G(k+n)$
$u(x) = g(x)h(x)$	$U(k) = \sum_{i=0}^{k} G(i)H(k-i)$
$u(x) = x^n$	$U(k) = \delta(k-n)$
$u(x) = \exp(cx)$	$U(k) = \frac{c^k}{k!}$
$u(x) = cos(\omega x)$	$U(k) = \frac{\omega^k}{k!} cos(\frac{k\pi}{2})$
$u(x) = sin(\omega x)$	$U(k) = \frac{\omega^k}{k!} sin(\frac{k\pi}{2})$

Theorem 2.1. [22]
If $f(x) = g(x+a)$, *then*

$$F(k) = \sum_{h=k}^{N} \binom{N}{k} a^{h-k} G(h) \text{ for } N \to \infty.$$

Theorem 2.2. [23]
If $f(y) = y^m$, *then*

$$F(k) = \begin{cases} (Y(0))^m, & k = 0 \\ \frac{1}{Y(0)} \sum_{r=1}^{k} \left(\frac{(m+1)r-k}{k} \right) Y(r) F(k-r), & k \geq 1 \end{cases}$$

Theorem 2.3. [23]
If $f(y) = e^{ay}$, *then*

$$F(k) = \begin{cases} e^{aY(0)}, & k = 0 \\ a \sum_{r=0}^{k-1} \frac{r+1}{k} Y(r+1) F(k-1-r), & k \geq 1 \end{cases}$$

Theorem 2.4. [23]
If $f(y) = \sin(\alpha y)$ *and* $g(y) = \cos(\alpha y)$, *then*

$$F(k) = \begin{cases} \sin(\alpha Y(0)), & k = 0 \\ \alpha \sum_{r=0}^{k-1} \frac{k-r}{k} G(r) Y(k-r), & k \geq 1 \end{cases}$$

and

$$G(k) = \begin{cases} \cos(\alpha Y(0)), & k = 0 \\ -\alpha \sum_{r=0}^{k-1} \frac{k-r}{k} F(r) Y(k-r), & k \geq 1 \end{cases}$$

Using the differential transform, a differential equation in the domain of interest can be transformed into an algebraic equation in the K-domain and $f(t)$ can be obtained by the finite-term Taylor series expansion plus a remainder,as

$$f(t) = \sum_{k=0}^{N} F(k) \frac{(t-t_0)^k}{k!} + R_{N+1}(t) \tag{4}$$

The series solution (4) converges rapidly only in a small region; in the wide region, they may have very slow convergence rates, and then their truncations yield inaccurate results. In the MDTM, we apply a Laplace transform to the series obtained by DTM, then convert the transformed series into a meromorphic function by forming its Padé approximants, and then invert the approximant to obtain an analytic solution, which may be periodic or a better approximation solution than the DTM truncated series solution. For further reference on DTM see [24, 25, 26, 27].

2.2. **Padé approximation.** Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $y(x)$.
The $[L/M]$ Padé approximants to a function $y(x)$ are given by

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$y\left(x\right) = \sum_{i=1}^{\infty} a_i x^i,$$

$$y\left(x\right) - \frac{P_L\left(x\right)}{Q_M\left(x\right)} = \mathrm{O}\left(x^{L+M+1}\right) \tag{5}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation. Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, then we impose the normalization condition

$$Q_M\left(0\right) = 1. \tag{6}$$

Finally, we require that $P_L(x)$ and $Q_M(x)$ have no common factors. If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\begin{cases} P_L\left(x\right) = p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L \\ Q_M\left(x\right) = q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M \end{cases} \tag{7}$$

then, by (6) and (7), we may multiply (5) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (5) in more detail as

$$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0 \\ \cdot \\ \cdot \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{cases} \tag{8}$$

$$\begin{cases} a_0 = p_0 \\ a_0 + a_0 q_1 = p_1 \\ a_2 + a_1 q_1 + a_0 q_2 = p_2 \\ \cdot \\ \cdot \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L \end{cases} \tag{9}$$

To solve these equations, we start with equation (8), which is a set of linear equations for all the unknown $q's$. Once the $q's$ are known, then equation (9) gives an explicit formula for the unknown $p's$, which complete the solution.
If equation (9) and equation (8) are non-singular, then we can solve them directly and

obtain equation (10) [28], where equation (10) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right]=\frac{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \dots & a_{L+1}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ a_L & a_{L+1} & \dots & a_{L+M}\\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \dots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \dots & a_{L+1}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ a_L & a_{L+1} & \dots & a_{L+M}\\ x^M & x^{M-1} & \dots & 1 \end{bmatrix}} \tag{10}$$

To obtain diagonal Padé approximants of different order such as [2 /2], [4/4] or [6/6] we can use the symbolic calculus software, MATLAB.
Note that typically the Padé approximant, obtained from a partial Taylor sum, is more accurate than the latter. However; the Padé, being a rational expression, has poles, which are not present in the original function. It is a simple algebraic task to expand the form of an $[N,\ M]$ Padé in a Taylor series and compute the Padé coefficients by matching with the above [29].

3. Numerical Results

In this section, we have been solved three problems related by BVPs.

3.1. **Problem 1.**

Consider the following second order linear differential equation with boundary conditions at two points [1]

$$y'' + y = 0, \tag{11}$$

subject to the boundary conditions

$$y\left(0\right)=\frac{1}{2},\ y\left(\frac{\pi}{3}\right)=\frac{1}{2}, \tag{12}$$

and the exact solution

$$y\left(t\right)=\frac{1}{2}cos\left(\mathrm{t}\right)+\frac{\sqrt{3}}{6}sin\left(t\right).$$

Transforming Eq.(11) with the boundary conditions Eq.(12), we obtain

$$Y\left(k+2\right)=-\frac{Y(k)}{(k+1)(k+2)}. \tag{13}$$

$$Y\left(0\right)=\frac{1}{2},\ Y\left(1\right)=A. \tag{14}$$

Substituting Eq.(14) in Eq.(13), yields the following:

$$Y\left(2\right)=-\frac{1}{4},\ Y\left(3\right)=-\frac{A}{6},\ Y\left(4\right)=\frac{1}{48},\ Y\left(5\right)=\frac{A}{120},\ Y\left(6\right)=-\frac{1}{1440},\ Y\left(7\right)=-\frac{A}{5040}.$$

Using the inverse transformation rule (2), we obtain an approximate solution of equation (11) in the form

$$y\left(t\right)=\sum_{k=0}^{\infty}Y(k)t^{k}=\frac{1}{2}+At-\frac{1}{4}t^{2}-\frac{A}{6}t^{3}+\frac{1}{48}t^{4}+\frac{A}{120}t^{5}-\frac{1}{1440}t^{6}-\frac{A}{5040}t^{7}+\ldots \quad (15)$$

In order to improve the accuracy of the differential transform solution (15), we implement the modified DTM as follows:
Applying the Laplace transform [30] to the series solution (15) yields

$$\mathcal{L}\left(y\left(t\right)\right)=\frac{1}{2}\frac{1}{s}+\frac{A}{s^{2}}-\frac{1}{2}\frac{1}{s^{3}}-\frac{A}{s^{4}}+\frac{1}{2}\frac{1}{s^{5}}+\frac{A}{s^{6}}-\frac{1}{2}\frac{1}{s^{7}}-\frac{A}{s^{8}}+\ldots.$$

Letting $s = \frac{1}{t}$ gives

$$\mathcal{L}\left(y\left(t\right)\right)=\frac{1}{2}t+At^{2}-\frac{1}{2}t^{3}-At^{4}+\frac{1}{2}t^{5}+At^{6}-\frac{1}{2}t^{7}-At^{8}+\ldots.$$

The Padé approximants $\left[\frac{4}{4}\right]$ gives

$$\left[\frac{4}{4}\right]=\frac{2At^{2}+t}{2t^{2}+2}$$

Recalling $t = \frac{1}{s}$, we obtain $\left[\frac{4}{4}\right]$ in terms of s

$$\left[\frac{4}{4}\right]=\frac{s+2A}{2s^{2}+2}$$

By using the inverse Laplace transform to the [4/4] Padé approximant, we obtain the modified approximate solution

$$y=\frac{1}{2}cos\left(t\right)+Asin(t). \quad (16)$$

Now, to find the value of constant A, we will back to the boundary condition

$$y\left(\frac{\pi}{3}\right)=\frac{1}{2},$$

and substitute in Eq.(16), to get A=$\frac{\sqrt{3}}{6}$.
Thus, the modified solution will be

$$y\left(t\right)=\frac{1}{2}cos\left(\mathrm{t}\right)+\frac{\sqrt{3}}{6}sin\left(t\right)\ .$$

3.2. **Problem 2.** Consider the following third order linear differential equation with boundary conditions at three points [14]

$$y^{'''}-k^{2}y^{'}+a=0, \quad (17)$$

$$y^{'}(0)=y^{'}(1)=0,y(0.5)=0 \quad (18)$$

Here, the physical constants are $k = 5$ and $a = 1$.
The function $y\left(t\right)$ shows the shear deformation of sandwich beams. The analytic solution of this problem is given by

$$y\left(t\right)=\frac{a}{k^{3}}\left(sinh\left(\frac{k}{2}\right)-sinh\left(kt\right)\}\right)+\frac{a}{k^{2}}\left(t-\frac{1}{2}\right)+\frac{a}{k^{3}}\left(cosh\left(kt\right)-cosh\left(\frac{k}{2}\right)\}\right)tanh\left(\frac{k}{2}\right).$$

Transforming Eq. (17) with the boundary conditions Eq. (18), we obtain

$$Y\left(k+3\right)=\frac{k!}{(k+3)!}\left[25\left(k+1\right)Y\left(k+1\right)-\delta(k)\right]. \quad (19)$$

$$Y(0) = A,\ Y(1) = 0,\ Y(2) = \frac{B}{2!}. \tag{20}$$

Substituting Eq. (20) in Eq. (19), yields the following:

$$Y(3) = -\frac{1}{6},\ Y(4) = \frac{25}{24}B,\ Y(5) = -\frac{5}{24},\ Y(6) = \frac{125}{144}B,\ Y(7) = -\frac{125}{1008}.$$

Using the inverse transformation rule (2), we obtain an approximate solution of equation (17) in the form

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = A + \frac{B}{2}t^2 - \frac{1}{6}t^3 + \frac{25}{24}Bt^4 - \frac{5}{24}t^5 + \frac{125}{144}Bt^6 - \frac{125}{1008}t^7 + \dots. \tag{21}$$

In order to improve the accuracy of the differential transform solution (21), we implement the modified DTM as follows:
Applying the Laplace transform to the series solution (21), yields

$$\mathcal{L}(y(t)) = \frac{A}{s} + \frac{B}{s^3} - \frac{1}{s^4} + \frac{25B}{s^5} - \frac{25}{s^6} + \frac{625B}{s^7} - \frac{625}{s^8} + \dots.$$

For simplicity, let $s = \frac{1}{t}$; then

$$\mathcal{L}(y(t)) = At + Bt^3 - t^4 + 25Bt^5 - 25t^6 + 625Bt^7 - 625t^8 + \dots.$$

The Padé approximants $\left[\frac{4}{4}\right]$ gives

$$\left[\frac{4}{4}\right] = \frac{-At + (25A - B)t^3 + t^4}{25t^2 - 1}$$

Recalling $t = \frac{1}{s}$, we obtain $\left[\frac{4}{4}\right]$ in terms of s

$$\left[\frac{4}{4}\right] = \frac{-As^3 + (25A - B)s + 1}{-s^4 + 25s^2}$$

By using the inverse Laplace transform to the [4/4] Padé approximant, we obtain the modified approximate solution

$$y(t) = A - \frac{B}{25} + \frac{1}{25}t + \left(\frac{1}{50}B + \frac{1}{250}\right)e^{-5t} + \left(\frac{1}{50}B - \frac{1}{250}\right)e^{5t} \tag{22}$$

Now, to find the values of constants of A and B, we will back to the boundary conditions

$$y'(1) = 0,\ y(0.5) = 0,$$

and substitute in Eq.(22), to get

$$A = -0.0121070856147,\ B = 0.19732285963.$$

Thus, the modified approximate solution will be

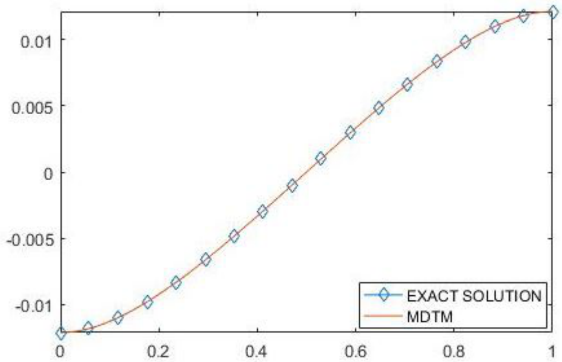
$$y(t) = -\frac{1}{50} + \frac{1}{25}t + \frac{19435}{2445744}e^{-5t} - 0.0000535428074e^{5t}.$$

Table 2 exhibits the approximate solution obtained by using the MDTM, DTM [14] and ADM [14]. It is clear that the obtained results in a our method are more accurate than the other method in literature and in a good agreement with the exact solution, this obviously noted from the absolute error which leads to conclude that this technique is effective and more reliable. In Fig. 1, we plot the approximate solution and the exact solution.

TABLE 2. 1: Comparison of the exact solution and the MDTM solution for Problem 1

x	Exact Value	MDTM Solution	Absolute Error	DTM Absolute Error [14]	ADM Absolute Error [14]
0.0	-0.012107085615	-0.012107085615	3.53×10^{-14}	5.21×10^{-12}	3.52×10^{-11}
0.2	-0.0092222062091	-0.0092222062091	2.64×10^{-14}	5.1×10^{-12}	3.59×10^{-11}
0.4	-0.003320194773	-0.003320194773	4.63×10^{-14}	9.04×10^{-12}	3.9×10^{-11}
0.6	0.003320194773	0.003320194773	1.16×10^{-13}	1.89×10^{-11}	4.89×10^{-11}
0.8	0.009222206209	0.009222206209	3.13×10^{-13}	4.69×10^{-11}	7.69×10^{-11}
1.0	0.012107085615	0.012107085614	8.49×10^{-13}	1.21×10^{-10}	1.51×10^{-10}

FIGURE 1. The graphs of approximated and exact solution $y(t)$ for problem 3.2



3.3. **Problem 3.** Consider the fifth order nonlinear differential equation [4]

$$y^{(5)} = e^{-t}y^2, \ 0 \le x \le 1. \tag{23}$$

with boundary conditions

$$y(0) = y^{'}(0) = y^{''}(0) = 1, y(1) = y^{'}(1) = e. \tag{24}$$

It is easy to see that the exact solution is $y\left(t\right) = e^t$.
Transforming Eq.(23) with the boundary conditions Eq. (24), we obtain

$$Y\left(k+5\right) = \frac{k!}{(k+5)!}\sum_{i=0}^{k}\frac{(-1)^i}{i!}G(k-i). \tag{25}$$

Where $G(k)$ is the differential transform of $g\left(y\right) = y^2$.

$$Y\left(0\right) = 1,\ Y\left(1\right) = 1, \quad Y\left(2\right) = \frac{1}{2}, \quad Y\left(3\right) = \frac{A}{3!}, \quad Y\left(4\right) = \frac{B}{4!}. \tag{26}$$

By theorem (2.2), the differential transform $G\left(k\right)$ in equation (25) is

$$G\left(0\right) = \left(Y\left(0\right)\right)^3 = 1, \tag{27}$$

$$G\left(k\right) = \sum_{r=1}^{k}\left(\frac{3r-k}{k}\right)Y\left(r\right)G\left(k-r\right), \quad k\geq 1. \tag{28}$$

Therefore,

$$\mathrm{G}\left(1\right) = 2,\ \mathrm{G}\left(2\right) = 2,$$

then

$$Y(5)=\frac{1}{5!},\ Y(6)=\frac{1}{6!},\ Y(7)=\frac{1}{7!}.$$

Using the inverse transformation rule (2), we obtain an approximate solution of equation (23) in the form

$$y(t)=\sum_{k=0}^{\infty}Y(k)t^k=1+t+\frac{1}{2}t^2+\frac{A}{3!}t^3+\frac{B}{4!}t^4+\frac{1}{5!}t^5+\frac{1}{6!}t^6+\frac{1}{7!}t^7+.. \tag{29}$$

In order to improve the accuracy of the differential transform solution (29), we implement the modified DTM as follows:
Applying the Laplace transform to the series solution (29), yields

$$\mathcal{L}(y(t))=\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{A}{s^4}+\frac{B}{s^5}+\frac{1}{s^6}+\frac{1}{s^7}+\frac{1}{s^8}+\ldots.$$

For simplicity, let $s=\frac{1}{t}$; then

$$\mathcal{L}(y(t))=t+t^2+t^3+At^4+Bt^5+t^6+t^7+t^8+\ldots, \tag{30}$$

we can write (30) as

$$\mathcal{L}(y(t))=\left(t+t^3+Bt^5+t^7\right)+\left(t^2+At^4+t^6+t^8\right)+\ldots, \tag{31}$$

The Padé Approximants $\left[\frac{3}{3}\right]$ gives

$$\left[\frac{3}{3}\right]=-\frac{t^2}{at^2-1}+\frac{(B-1)t^3-t}{Bt^2-1}$$

Recalling $t=\frac{1}{s}$, we obtain $\left[\frac{3}{3}\right]$ in terms of s

$$\left[\frac{3}{3}\right]=\frac{1}{s^2-a}+\frac{s^2-(B-1)}{s^3-Bs}$$

By using the inverse Laplace transform to the $\left[\frac{3}{3}\right]$ Padé approximant, we obtain the modified approximate solution

$$y=\frac{1}{\sqrt{A}}sinh(\left(\sqrt{A}t\right)+\frac{1}{B}cosh\left(\sqrt{B}t\right)+\frac{B-1}{B}. \tag{32}$$

Now, to find the values of constants A and B, we will back to the initial condition $y(1)=e$, and substitute in Eq.(32), to get

$$1+1+\frac{1}{2}+\frac{A}{6}+\frac{B}{24}+\sum_{n=5}^{N}\frac{1}{n!}+R_{N+1}(t)=e,$$

when $N\to\infty$, then

$$\frac{5}{2}+\frac{A}{6}+\frac{B}{24}+\left(\sum_{n=0}^{\infty}\frac{1}{n!}-\sum_{n=0}^{4}\frac{1}{n!}\right)=e,$$

that gives

$$\frac{5}{2}+\frac{A}{6}+\frac{B}{24}+e-(1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24})=e,$$

then

$$4A+B=5 \tag{33}$$

Now,

$$y^{'}=1+t+\frac{A}{2}t^2+\frac{B}{3!}t^3+\frac{1}{4!}t^4+\frac{1}{5!}t^5+\frac{1}{6!}t^6+\ldots,$$

and $y^{'}(1)=e,$ then

$$1 + 1 + \frac{A}{2} + \frac{B}{6} + \sum_{n=4}^{\infty} \frac{1}{n!} = e,$$

thus

$$3A + B = 4 \tag{34}$$

From (33) and (34), $A = B = 1$. Thus, the modified solution will be

$$y(t) = sinh(t) + cosh(t) = e^t.$$

4. Conclusions

In this paper, the MDTM is employed successfully for obtaining a new accurate approximate solution of a class of BVP's. The applicability and efficiency of this technique is tested throughout several examples. The results obtained are converge to the exact solution and in most cases it gives as the exact solution. This rereads that the MDTM is very strong technique and gives us more accurate results as compared by other methods.

Acknowledgement. The authors would like to express their sincere gratitude to the anonymous referees for their useful and valuable comments.

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