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THE EDGE-TO-VERTEX STEINER DOMINATION NUMBER OF A GRAPH

J. JOHN[1*], S. ANCY MARY[2], §

ABSTRACT. A set $W \subseteq E$ is said to be an edge-to-vertex Steiner dominating set of G if W is both an edge-to-vertex dominating set and a edge-to-vertex Steiner set of G. The edge-to-vertex Steiner domination number $\gamma_{sev}(G)$ of G is the minimum cardinality of its edge-to-vertex Steiner dominating set of G and any edge-to-vertex Steiner dominating set of cardinality $\gamma_{sev}(G)$ is a γ_{sev}-set of G. Some general properties satisfied by this concept are studied. The edge-to-vertex Steiner domination number of certain classes of graphs are determined. Connected graph of size $q \geq 3$ with edge-to-vertex Steiner domination number q or $q-1$ are characterized. It is shown for every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_{ev}(G) = a$ and $\gamma_{sev}(G) = b$.

Keywords: Edge-to-vertex Steiner domination number, Edge-to-vertex Steiner number, Edge-to-vertex Steiner distance, Edge-to-vertex domination number.

AMS Subject Classification: 05C12, 05C69.

1. Introduction

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic definitions and terminologies we refer to [1]. Two vertices u and v are said to be adjacent if uv is an edge of G. The open neighbourhood of a vertex v in a graph G is defined as the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, while the closed neighbourhood of v in G is defined as $N_G[v] = N_G(v) \cup \{v\}$. For any vertex v in a graph G, the number of vertices adjacent to v is called the degree of v in G, denoted by $deg_G(v)$. If the degree of a vertex is 0, it is called an isolated vertex, while if the degree is 1, it is called an end-vertex. The minimum degree of vertices in G is defined by $\delta(G) = min\{deg(v) : v \in V(G)\}$. The maximum degree of vertices in G is defined by $\triangle(G) = max\{deg(v) : v \in V(G)\}$. A cut-vertex (cut-edge) of a graph G is a vertex (edge) whose removal increases the number of components. For a cut-vertex v in a connected graph G and a component H of G v, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a

[1] Department of Mathematics, Government College of Engineering, Tirunelveli-627 007, India.
e-mail: john@gcetly.ac.in; ORCID: https://orcid.org/0000-0001-5528-4387.
[*] Corresponding author.
[2] Department of Mathematics, St. John's College of Arts and Science, Nagercoil, 629 204, India.
e-mail: ancymary369@gmail.com; ORCID: https://orcid.org/0000-0002-9276-8329.
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branch of G at v. A vertex v is called a universal vertex if $deg_G(v) = p - 1$. For any set S of vertices of G, the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with vertex set S. A vertex v is an extreme vertex of G if $\langle N(v) \rangle$ is complete. An edge of a connected graph G is called an extreme edge of G if one of its ends is an extreme vertex of G.

A subset $S \subseteq V(G)$ is called a dominating set if every vertex $v \in V(G) \backslash S$ is adjacent to a vertex $u \in S$. The domination number $\gamma(G)$ of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-*set* of G. The domination concept was studied in [4]. A subset $S \subseteq E(G)$ is said to be an edge-to-vertex-dominating set of G if every vertex in G is dominated by an edge in S. The edge-to-vertex domination number $\gamma_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex dominating sets. Any edge-to-vertex domination set of cardinality $\gamma_{ev}(G)$ is called a γ_{ev}-*set* of G. The edge-to-vertex domination number of a graph was studied in [12-15]. It has applications in game theory, telephone switching centres, facility locations, distributed computing, information retrieval, and communication networks.

The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length *d(u,v)* is called an $u - v$ geodesic. For subsets A and B of $V(G)$, the distance $d(A, B)$ is defined as $d(A, B) = min\{d(x, y) : x \in A, y \in B\}$. An $u - v$ path of length $d(A, B)$ is called an $A - B$ geodesic joining the sets A, B, where $u \in A$ and $v \in B$. A vertex x is said to lie on an $A - B$ geodesic if x is a vertex of an $A - B$ geodesic. For $A = \{u, v\}$ and $B = \{z, w\}$ with uv and zw edges, we write an $A - B$ geodesic as $uv - zw$ geodesic and $d(A, B)$ as $d(uv, zw)$. Let $G = (V, E)$ be a connected graph with at least three vertices. A set $S \subseteq E$ is called an edge-to-vertex geodetic set if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The edge-to-vertex geodetic number $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is an edge-to-vertex g_{ev} set of G. The edge-to-vertex geodetic number of a graph was studied in [5,9-11]. Let W be a subset of a set of vertices V of G. A Steiner tree for W (*Steiner* W - *tree*) is a connected subgraph of G with a minimum number of edges that contains all vertices of W. The number of edges in a Steiner W-tree is the *Steiner distance* $d(W)$ of W in G. The Steiner distance of a graph was studied in [2]. The Steiner interval $S(W)$ contains all the vertices that lie on some Steiner W - tree. If $S(W) = V$, we call W a *Steiner set* of G. A Steiner set of minimum cardinality is a minimum Steiner set or simply a *s-set* and its cardinality is the *Steiner number* $s(G)$ of G. The Steiner number of a graph was introduced in [3] and further studied in [6,7].

For a non-empty set W of edges in a connected graph in G, the edge-to-vertex Steiner distance $d_{ev}(W)$ of W is the minimum size of a tree containing $V(W)$ and is called an edge-to-vertex Steiner tree with respect to W or a *Steiner* W_{ev}-*tree* of G. For a given set $W \subseteq E(G)$, there may be more than one *Steiner* W_{ev}-*tree* in G. In fact,it may occur that T_1 and T_2 are *Steiner* W_{ev}-*trees* with $V(T_1) \neq V(T_2)$; however $V(W) \subseteq V(T_1) \cap V(T_2)$. For $W \subseteq E$, let $S_{ev}(W)$ denote the set of all vertices of G that lie on some *Steiner* W_{ev}-*tree*. If $S_{ev}(W) = V$, then W is called an *edge-to-vertex Steiner set* of G. The edge-to-vertex Steiner number $s_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex Steiner sets and any edge-to-vertex Steiner sets of cardinality $s_{ev}(G)$ is a minimum edge-to-vertex Steiner set of G or s_{ev}-*set* of G.For the graph G given in Figure 1.1, $W = \{v_1v_2, v_2v_{10}, v_4v_5, v_7v_8\}$ is a s_{ev}-set of G so that $s_{ev}(G) = 4$. The edge-to-vertex Steiner number of a graph was introduced in [8]. Steiner tree problem is used in

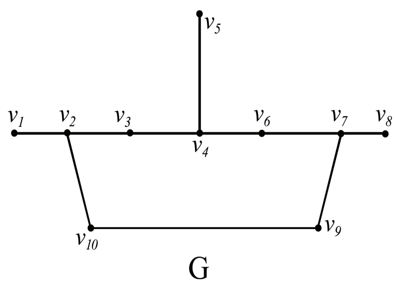
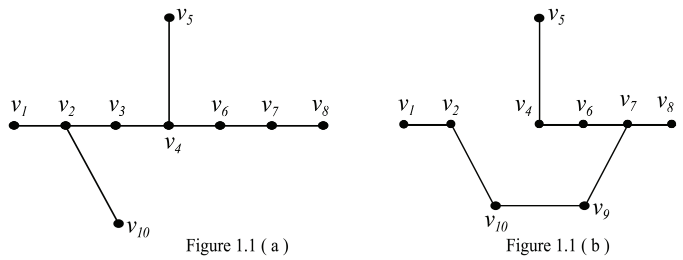


Figure 1.1



Two Steiner W_{ev}-trees of G

combinatorial optimization and computer science especially in design of computer circuits. They have numerous applications in industries. Applying the edge-to-vertex Steiner tree concept improves the effectiveness in networks.

Throughout the following G denotes a connected graph with at least three vertices. The following theorems are used in the sequel.

Theorem 1.1. [8] *If v is an extreme vertex of a connected graph G, then every edge-to-vertex Steiner set contains at least one extreme edge that is incident with v.*

Theorem 1.2. [8] *Let G be a connected graph and W be a s_{ev}-set of G. Then no cut-edge of G which is not an end-edge of G belongs to W.*

2. The Edge-to-Vertex Steiner Domination Number of a Graph

In general edge-to vertex dominating set is not an edge-to-vertex Steiner set in a connected graph G. Also the converse is not valid in general. This has motivated us to study the new edge-to vertex domination conception of edge-to-vertex Steiner domination. In this section, some general properties satisfied by this concept are studied and also we determine the edge-to-vertex Steiner domination number of some standard graphs.

Definition 2.1. *A set $W \subseteq E$ is said to be an edge-to-vertex Steiner dominating set of G if W is both an edge-to-vertex dominating set and an edge-to-vertex Steiner set of G. The edge-to-vertex Steiner domination number γ_{sev} of G is the minimum cardinality of its edge-to-vertex Steiner dominating set of G and any edge-to-vertex Steiner dominating set of cardinality $\gamma_{sev}(G)$ is a γ_{sev}-set of G.*

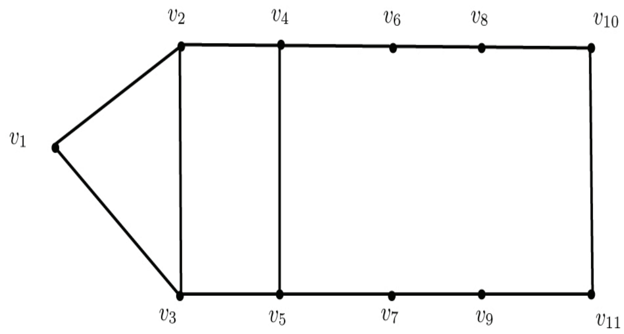


Figure 2.1

Example 2.1. *For the graph G in Figure 2.1,* $W = \{v_1v_2, v_1v_3, v_4v_5, v_{10}v_{11}\}$ *is a minimum edge-to-vertex Steiner dominating set of G so that* $\gamma_{sev}(G) = 4$..

Remark 2.1. *There can be more than one* γ_{sev}*-set of* G. *For the graph G in Figure 2.1,* $W_1 = \{v_1v_2, v_1v_3, v_8v_{10}, v_9v_{11}\}$ *is another* γ_{sev}*-set of* G.

Theorem 2.1. *For a connected graph G of size* $q \geq 2$, $2 \leq max(\gamma_{ev}(G), s_{ev}(G)) \leq \gamma_{sev}(G) \leq q$.

Proof: A γ_{sev}-set needs at least two edges and so $\gamma_{sev}(G) \geq 2$. Also the set of all edges of G is an edge-to-vertex Steiner dominating set of G so that $\gamma_{sev}(G) \leq q$. Thus $2 \leq max\{\gamma_{ev}(G), s_{ev}(G)\} \leq \gamma_{sev}(G) \leq q$. □

Remark 2.2. *The bounds in Theorem 2.1 are sharp. For* $G = C_4$, $\gamma_{sev}(G) = 2$. *For the star* $G = K_{1,q}$ $(q \geq 2)$, *it is clear that the set of all edges is the unique so that* $\gamma_{sev}(G) = q$. *Also the bound in Theorem 2.1 can be strict. For the graph G given in Figure 2.1,* $\gamma_{ev}(G) = S_ev(G) = 3, \gamma_{sev}(G) = 4$ *and* $q = 13$. *Thus* $2 < max\{\gamma_{ev}(G), s_{ev}(G)\} < \gamma_{sev}(G) < q$.

Theorem 2.2. *If v is an extreme vertex of a connected graph G, then every edge-to-vertex Steiner dominating set of G contains at least one extreme edge that is incident with v.*

Proof: Since every edge-to-vertex Steiner dominating set of G is an edge-to-vertex Steiner set of G, the result follows from Theorem 1.1. □

Corollary 2.1. *Every end edge of a connected graph G belongs to every edge-to-vertex Steiner dominating set of G.*

Proof: The follows from Theorem 2.2. □

Theorem 2.3. *If G is any connected graph of size q with number of end edges k, then* $max\{2, k\} \leq \gamma_{sev}(G) \leq q$.

Proof: This follows from Theorem 2.1 and Corollary 2.1. □

Theorem 2.4. *Let G be a connected graph with cut vertices and W an edge-to-vertex Steiner dominating set of G. Then every branch of G contains an element of W.*

Proof: Suppose that there is a branch B of G at a cut vertex v which has no element of W. By Corollary 2.1, B does not contain any end-edge of G. Therefore $|V(B)| \geq 2$. Let u be a vertex of G such that $u \neq v$. Since W is an edge to-vertex Steiner dominating set of

G, u lies on a Steiner W_{ev}-tree of G, say T. Since W contains no element of B and V is a cut-vertex of G,v lies on T. Which implies T contains a cycle, which is a contradiction to T is a tree. □

Theorem 2.5. *Let G be a connected graph with cut edges and W an edge to-vertex Steiner dominating set of G. Then for any cut-edge of G, which is not an end-edge, each of the two-components of $G - e$ contains an element of W.*

Proof: Let $e = uv$. Let G_1 and G_2 be the two components of $G - e$. Without loss of generality, let us assume that $u \in V(G_1)$ and $v \in V(G_2)$. Let B_1 be the branch at u and B_2 be the branch at v. The G_1 contains B_1 and G_2 contains B_2. Hence by Theorem 2.4, each of G_1 and G_2 contain an element of W. □

Theorem 2.6. *Let G be a connected graph and e be an end edge of G. Let W be a γ_{sev}-set of G. If f is a cut edge of G which is adjacent to e and not an end edge of G, then $f \notin W$.*

Proof: The proof is similar to the proof of Theorem 1.2. □

Corollary 2.2. *For any non-trivial $\gamma_{sev}(T) \geq k$, where k is the number of end edges of G.*

Proof: This follows from Corollary 2.1. □

Corollary 2.3. *For the star $G = K_{1,q}$ $(q \geq 2)$, $\gamma_{sev}(G) = q$.*

Proof: This follows from Corollary 2.2. □

Corollary 2.4. *If G is a double star, then $\gamma_{sev}(G) = q - 1$.*

Proof: This follows from Corollary 2.1 and Theorem 2.6. □

Theorem 2.7. *For p even, a set W of edges of $G = K_p$ $(p \geq 4)$ is a minimum edge-to-vertex Steiner dominating set of K_p if and only if W consists of $p/2$ independent edges.*

Proof: Let W be any set of $p/2$ independent edges of K_p. Since $V(W) = V$, the spanning tree of G is a Steiner W_{ev}-tree of G, so that W is a edge-to-vertex Steiner dominating set of G. It follows that $\gamma_{sev}(G) \leq p/2$. If $\gamma_{sev}(G) < p/2$, then there exists an edge-to-vertex Steiner dominating set W' of K_p such that $\mid W_1' \mid < p/2$. Therefore, there exists at least one vertex v of K_p such that v is not incident with any edge of W'. Hence v does not lie on any Steiner W_{ev}-tree of G, which is a contradiction. Thus, W is a minimum edge-to-vertex Steiner dominating set of K_p. Conversely, let W be a minimum edge-to-vertex Steiner dominating set of K_p. Let W' be any set of $p/2$ independent edges of K_p. Then by first part of this theorem, W' is the minimum edge-to-vertex Steiner dominating set of K_p. Therefore $\mid W_1' \mid = p/2$. Hence $\mid W \mid = p/2$. If W is not independent, then there exists a vertex v of K_p such that v is not incident with any edge of W. Therefore v does not lie on any Steiner W_{ev}-tree of G. Hence it follows that W is not an edge-to-vertex Steiner dominating set of G, which is a contradiction. Therefore, W consists of $p/2$ independent edges. □

Theorem 2.8. *A set W of edges of $G = K_{n,n}$ $(n \geq 2)$ is a minimum edge-to-vertex Steiner dominating set of G if and only if W consists of n independent edges.*

Proof: Let W be any set of n independent edges of $G = K_{n,n}$ $(n \geq 2)$. Since $V(W) = V$, the spanning tree of G is a Steiner W_{ev} tree of G, it follows that $s_{ev}(G) \leq n$. If $s_{ev}(G) < n$, then there exists an edge-to-vertex Steiner dominating set W' of $K_{n,n}$ such that $|W'| < n$. Therefore, there exists at least one vertex v of $K_{n,n}$ such that v is not incident with any edge of W'. Hence v does not lying on any Steiner W_{ev}-tree of G, which is a contradiction. Hence W is a minimum edge-to-vertex Steiner dominating set of $K_{n,n}$. Conversely, let W be a minimum edge-to-vertex Steiner dominating set of G. Let W' be any set of n independent edges of G. Then as in the first part of this theorem, W' is a minimum edge-to-vertex Steiner dominating set of G. Therefore, $|W'| = n$. Hence $|W| = n$. If W is not independent, then there exists a vertex v of G such that v is not incident with any edge of W and also v does not lie on any Steiner W_{ev}-tree of G. Hence W is not an edge-to-vertex Steiner dominating set of G, which is a contradiction. Thus W consists of n independent edges. □

Corollary 2.5. *For the complete graph K_p ($p \geq 4$) with p even, $\gamma_{sev}(K_p) = p/2$.*

Theorem 2.9. *For the complete graph $G = K_p$ ($p \geq 5$) with p odd, $\gamma_{sev}(K_p) = \frac{p+1}{2}$.*

Proof: Let S consist of any set of $\frac{p-3}{2}$ independent edges of K_p and S' consist of *2* adjacent edges of K_p, each of which is independent with the edges of S. Let $W = S \cup S'$. Then $V(W) = V$. Therefore the spanning tree of G is an edge-to- vertex Steiner W_{ev}-tree of G. It follows that $s_{ev}(G) \leq \frac{p+1}{2} + 2 = \frac{p+1}{2}$. If $s_{ev}(G) < \frac{p+1}{2}$, then there exists an edge-to-vertex Steiner dominating set W' of K_p such that $|\ W'\ | < \frac{p+1}{2}$. Therefore there exists at least one vertex v of K_p such that v is not incident with any edge of W'. Hence the vertex v does not lie any Steiner W_{ev}-tree of G, which is a contradiction. Thus W is a minimum edge-to-vertex Steiner dominating set of K_p. Hence $s_{ev}(G) = \frac{p+1}{2}$. □

Corollary 2.6. *For the complete bipartite graph $G = K_{n,n}$ ($n \geq 2$), $\gamma_{sev}(G) = n$.*

Theorem 2.10. *For the complete bipartite graph $G = K_{m,n}$ ($2 \leq m \leq n$), $\gamma_{sev}(G) = n$.*

Proof: Let $X = \{x_1, x_2, \ldots, x_m\}$, and $Y = \{y_1, y_2, .., y_n\}$ be the bipartition of G. Let T consist of the set of $m-1$ independent edges $x_1y_1, x_2y_2, ..., x_{m-1}y_{m-1}$ and T' consist of the $n-m+1$ adjacent edges $x_my_m, x_my_{m+1}, ..., x_my_n$. Let $W = T \cup T'$. Then $S(W) = V$. Hence it follows that $\gamma_{sev}(G) = m-1+n-m+1 = n$. If $\gamma_{sev} < n$, then there exists an edge-to-vertex Steiner set W' of G such that $|\ W'\ | < n$. Therefore there exists at least one vertex v of G such that v is not incident with any edge of W'. Hence v does not lie any Steiner W_{ev}-tree of G, which is a contradiction. Therefore $\gamma_{sev}(G) = n$. □

Theorem 2.11. *For the cycle $G = C_p, (p \geq 3)$,*

$$\gamma_{sev}(C_p) = \begin{cases} 2 & \text{if } p = 3,4,6,8 \\ 3 & \text{if } p = 5,7 \\ \left\lceil \frac{p}{4} \right\rceil & \text{if } p \geq 9 \end{cases}$$

Proof: Let us prove this result by the method of mathematical induction. Clearly this result is true for $p = 3,4,5,6$ and 7. Assume that it is true for $p = k$.

i.e., $\gamma_{sev}(C_k) = \left\lceil \frac{k}{4} \right\rceil, k \geq 9$.

To prove $\gamma_{sev}(C_{k+1}) = \left\lceil \frac{k+1}{4} \right\rceil$.

Clearly $\gamma_{sev}(C_k) \leq \gamma_{sev}(C_{k+1}) \leq \gamma_{sev}(C_{k+1}) + 1$.

$$\begin{aligned}
\gamma_{sev}(C_{k+1}) + 1 \geq \gamma_{sev}(C_k) \\
&= \left\lceil \frac{k}{4} \right\rceil = \left\lceil \frac{(k+1)-1}{4} \right\rceil \\
&= -\left\lfloor \frac{(k+1)-1}{4} \right\rfloor \text{ since } \lceil -x \rceil = \lfloor -x \rfloor \\
&= -\left\lfloor -\frac{(k+1)}{4} + \frac{1}{4} \right\rfloor \\
&\geq -\left\{ \left\lfloor -\frac{(k+1)}{4} \right\rfloor + \left\lfloor \frac{1}{4} \right\rfloor \right\} \text{ since } \lceil x \rceil + \lceil y \rceil \leq \lceil x+y \rceil \\
&\geq -\left\{ \left\lfloor -\frac{(k+1)}{4} \right\rfloor + \left\lfloor \frac{-1}{4} \right\rfloor \right\} \text{ since } \lfloor x \rfloor \leq \lfloor -x \rfloor \\
&= \left\lceil \frac{(k+1)}{4} \right\rceil + \left\lceil \frac{1}{4} \right\rceil \text{ since } \lfloor -x \rfloor = -\lfloor -x \rfloor \\
&= \left\lceil \frac{(k+1)}{4} \right\rceil + 1 \\
\gamma_{sev}(C_{k+1}) &\geq \left\lceil \frac{(k+1)}{4} \right\rceil .....(1)
\end{aligned}$$

Also in a cycle, an edge can dominate almost 4 vertices and so

$\gamma_{sev}(C_{k+1}) \leq \left\lceil \frac{k+1}{4} \right\rceil$ .....(2)

From (1) and (2)

$\gamma_{sev}(C_{k+1}) = \left\lceil \frac{k+1}{4} \right\rceil$.

Hence by mathematical induction , $\gamma_{sev}(C_p) = \left\lceil \frac{p}{4} \right\rceil$,for $k \geq 9$. □

3. Some results on the Edge-to-Vertex Steiner Dominating Number of a Graph

In this section, we characterized connected graphs G of size q with $\gamma_{sev} = q$ or $q-1$. Also we give some realization results concerning the edge-to-vertex Steiner domination number of G.

Theorem 3.1. *Let G be a connected graph with $\gamma_{ev}(G) = 2$. Then $\gamma_{sev}(G) \leq 3$.*

Proof: Let $S = \{e, f\}$ be a γ_{ev}-set. If $d(e, f) = 2$, then S is a γ_{sev}-set of G so that $\gamma_{sev}(G) = 2$. Suppose that $d(e, f) = 1$. Then there exists $u \in G$ such that u is not incident with e and u is not incident with f. Let x be a vertex which is incident with either e or f. Then $\{e, f, xu\}$ is an edge-to-vertex steiner dominating set of G so that $\gamma_{sev}(G) \leq 3$. □

Theorem 3.2. *Let G be a connected graph of size $q \geq 4$ which is not a tree. Then $\gamma_{sev}(G) \leq q-2$.*

Proof: If the graph G is a cycle C_p $(p \geq 4)$, then by Theorem 2.11, $\gamma_{sev}(G) \leq q-2$. If the graph G is not a cycle, let $C : v_1, v_2, v_3, ..., v_k, v_1$ $(k \geq 3)$ be a smallest cycle in G and let v be a vertex such that v is not on C and v be adjacent to v_1(say). Now $W = E(G) - \{v_1v_2, v_1v_k\}$ is an edge-to-vertex Steiner dominating set of G so that $\gamma_{sev}(G) \leq q-2$. □

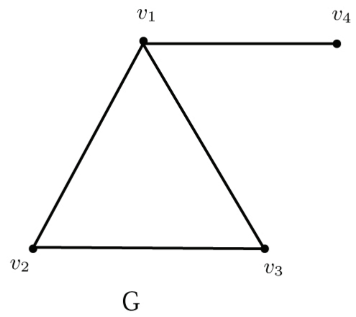


Figure 3.1

Remark 3.1. *The bound in Theorem 3.2 is sharp. For the graph G given in Figure 3.1, $W = \{v_1v_4, v_2v_3\}$ is a γ_{sev}-set of G so that $\gamma_{sev}(G) = 2 = q - 2$.*

Theorem 3.3. *For any connected graph G, with size $q \geq 3$, $\gamma_{sev}(G) = q$ if and only if G is a star.*

Proof: Let G be a star. Then by Corollary 2.3, $\gamma_{sev}(G) = q$. Conversely, let $\gamma_{sev}(G) = q$. Suppose that G is not a star. Then G contains at least one edge e, which is not an end edge of G. Then $W = E(G) - e$ is an edge-to-vertex Steiner dominating set of G so that $\gamma_{sev}(G) \leq q - 1$, which is a contradiction. Therefore G is the star $K_{1,q}$. □

Theorem 3.4. *For any connected graph G with $q \geq 3$, $\gamma_{sev}(G) = q - 1$ if and only if G is either C_3 or a double star.*

Proof: Let $q = 3$. If $G = C_3$ then we have done. If $G = P_4$, then G is a double star. So we have done. If $G = K_{1,3}$, then $\gamma_{sev}(G) = 3 = q$, which is not so. Let us assume that $q \geq 4$. If G is not a tree, then by Theorem 3.2, $\gamma_{sev}(G) \leq q - 2$, which is a contradiction. Therefore G is a tree. If G is a double star, then we have done. Suppose that G is not a double star. If G is a star, then by Theorem 3.3, $\gamma_{sev}(G) = q$, which is not so. If G is neither a star nor a double star, then G contains at least two internal edges. Which implies, $\gamma_{sev}(G) \leq q - 2$, which is a contradiction. Therefore G is either C_3 or a double star. Converse is clear. □

In view of Theorem 2.1, we have the following realization results.

Theorem 3.5. *For every positive integers a and b such that $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_{ev}(G) = a$ and $\gamma_{sev}(G) = b$.*

Proof: Let $P_i : u_i, v_i, w_i, x_i$ $(1 \leq i \leq a - 1)$ be a copy of path on four vertices. Let H be a graph obtained from P_i $(1 \leq i \leq a - 1)$ by introducing a new vertex y and joining y with each u_i $(1 \leq i \leq a - 1)$, and joining with each x_i $(1 \leq i \leq a - 1)$. Let G be a graph obtained from H by adding the new vertices $z_1, z_2, ..., z_{b-a+1}$ and joining each z_i $(1 \leq i \leq b - a + 1)$ with y. The graph G is shown in Figure 3.2.

First we show that $\gamma_{ev}(G) = a$. Let W be a γ_{ev}-set of G. It is easily observed that W contains at least one edge from $G - y$. Therefore $\gamma_{ev}(G) \geq a - 1$. Let $W = \{v_1w_1, v_1w_2, ..., v_{a-1}w_{a-1}\}$. Then W is not a γ_{ev}-set of G and so $\gamma_{ev}(G) \geq a$ on the other hand $W \cup \{yu_1\}$ is a γ_{ev}-set of G so that $\gamma_{ev}(G) = a$.

Next we prove that $\gamma_{sev}(G) = b$. Let $Z = \{yz_1, yz_2, ..., yz_{b-a+1}\}$ be the set of all end

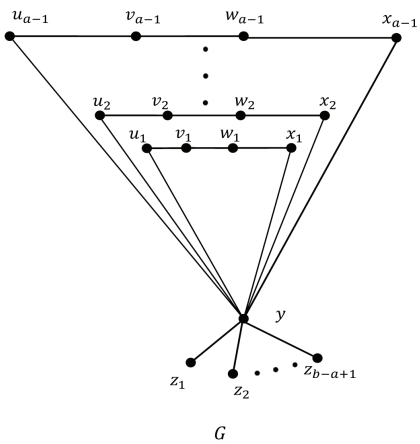


Figure 3.2

edges of G. By Corollary 2.1, Z is a subset of every γ_{ev}-set of G. It is easily observed that every γ_{sev}-set of G contains at least one edge from $G-y$ and so $\gamma_{sev}(G) = b-a+1+a-1 = b$. Now $W_1 = W \cup Z$ is a γ_{sev}-set of G so that $\gamma_{sev}(G) = b$. □

Theorem 3.6. *For every positive integers a and b such that $2 \leq a \leq b$, there exists a connected graph G such that $s_{ev}(G) = a$ and $\gamma_{sev}(G) = b$.*

Proof: Let $P_i : u_i, v_i, w_i$ $(1 \leq i \leq b-a)$ be a copy of path on three vertices and $P : z, w_1, v_1, u_1, x, y$ be a path of order 6. Let H be a graph obtained from P_i $(1 \leq i \leq b-a)$ and P by joining each u_i $(2 \leq i \leq b-a)$ with y, and $w_i (2 \leq i \leq b-a)$ with z. Let G be a graph obtained from H by adding vertices $z_1, z_2, ..., z_{a-1}$ and joining each z_i $(1 \leq i \leq a-1)$ with z. The graph G is shown in Figure 3.3.

First we prove that $s_{ev}(G) = a$. Let $Z = \{zz_1, zz_2, ..., zz_{a-1}\}$ be the set of end edges of G. By Theorem 1.1, Z is a subset of every edge-to-vertex Steiner set of G. It is clear that Z is not an edge-to-vertex Steiner set of G and so $s_{ev}(G) \geq a$. Now $Z \cup \{xy\}$ is an edge-to-vertex Steiner set of G so that $s_{ev}(G) = a$.

Next we prove that $\gamma_{sev}(G) = b$. By Corollary 2.1, Z is a subset of every edge-to-vertex Steiner dominating set of G. Also it is easily observed that every edge-to-vertex Steiner dominating set of G contains each $v_i w_i$ $(1 \leq i \leq b-a)$ and so $\gamma_{sev}(G) \geq a-1+b-a = b-1$. Let $S = Z \cup \{v_1w_1, v_2w_2, ..., v_{b-a}w_{b-a}\}$. Then S is not an edge-to-vertex Steiner dominating set of G and $\gamma_{sev}(G) \geq b$. However $S \cup \{xy\}$ is an edge-to-vertex Steiner dominating set of G so that $\gamma_{sev}(G) = b$. □

4. Conclusion

In this article, we introduce and studied the concept of the edge-to-vertex Steiner domination number of a graph. In general, every edge-to-vertex Steiner set of G need not be an edge-to-vertex geodetic set of G. By the similar way, every edge-to-vertex Steiner dominating set of G need not be an edge-to-vertex geodetic dominating set of G. Hence it can be further investigated to find out under which condition the inequality $\gamma_{gev}(G) \leq \gamma_{sev}(G)$ or $\gamma_{sev}(G) \leq \gamma_{gev}(G)$ holds true.

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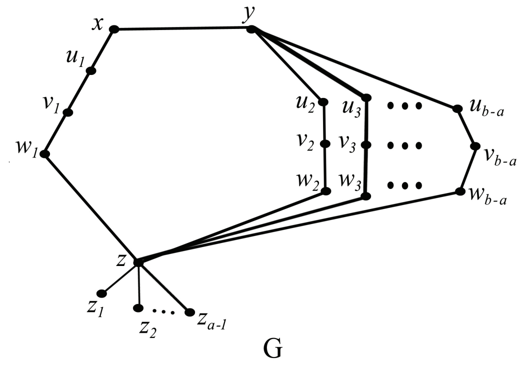

Figure 3.3

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Dr. J. John received his Ph.D. in Mathematics from Manonmaniam Sundaranar University, Tirunelveli, India. Since 2008, he has been at Government College of Engineering, Tirunelveli, 627 007, India. His research interests include geodetic, Steiner, detour, monophonic, geo colouring and decomposition concepts in graphs.

Dr. S. Ancymary received her Ph.D. in Mathematics from Manonmaniam Sundaranar University, Tirunelveli. India. Since 2020, she has been at St. John's college of Arts and Science, Nagercoil, India. Her research interests include geodetic and Steiner concepts in graphs.