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NEIGHBORHUB NUMBER OF GRAPHS

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Abstract. Let G be a graph. A neighborhub set (n-hub set) S of G is a set of vertices with the property that for any pair of vertices outside of S, there is a path between them with all intermediate vertices in S and $G = \bigcup_{v\in S} < N[v] >$. The neighborhub number (n-hub number) $h_n(G)$ is then defined to be the size of a smallest neighborhub set of G. In this paper, the neighborhub number for several classes of graphs is computed, bounds in terms of other graph parameters are also determined.

Keywords: Neighborhood number, Connected neighborhood number, Hub number, Total hub number, Neighborhub number.

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1. Introduction

Consider a transportation network, for example a network of locations(streets) in a city. We want to identify minimum locations of this network such that there is an easy passage between other locations of this network that passes solely through these identified locations. Converting it into graph theoretic terms, let a graph $G = (V, E)$ represent this network. Now, our concern is with a set $S \subseteq V$ of minimum cardinality such that any $v_i, v_j \in V \setminus S$ are connected by a path having only elements of S. By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges, $\delta(G)$ denote the minimum degree among the vertices of G. For graph theoretic terminology, we refer to [2].

The open neighborhood $N(v)$ of a vertex v in G is the set of vertices adjacent to v and its closed neighborhood $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its open neighborhood $N(S) = \bigcup_{v\in S} N(v)$ and its closed neighborhood $N[S] = S \cup N(S)$. For any two graphs G_1 and G_2 having disjoint vertex set V_1 and V_2, and edge sets E_1 and E_2, respectively, their

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corona $G_1 \circ G_2$ is the graph obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. For every $v \in V_1$, denote by G_2^v, the copy of G_2 whose vertices are attached one by one to the vertex v [2].

A subset D of G is called a dominating set of G if each vertex of $V \setminus D$ is adjacent to at least one vertex of D. The domination number of G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [3].
A dominating set D of a connected graph G is called a connected dominating set if the induced subgraph $\langle D \rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$ [3].

A subset S of V is called a neighborhood set of G if $G = \bigcup\limits_{v \in S} < N[v] >$. A neighborhood set S is said to be minimal if no proper subset of S is a neighborhood set. The minimum cardinality of a minimal neighborhood set of G is called the neighborhood number of G and is denoted by $\eta(G)$ [16]. Various types of neighborhood numbers have been defined and studied by several authors. E. Sampathkumar and Prabha S. Neeralagi [17] introduced the concept of connected neighborhood number of graphs. A neighborhood set S of G is called a connected neighborhood set if the induced subgraph $< S >$ is connected. The minimum cardinality of a connected neighborhood set of G is called the connected neighborhood number of G and is denoted by $\eta_c(G)$.

For $v \in V(G)$, the contraction of v in G (denoted by G/v) is the graph obtained by deleting v and putting a clique on the (open) neighborhood of v. If two neighbors of v are already adjacent, then they remain simply adjacent [18]. Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H, (this includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$; call such an H-path trivial) [18].

A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) \setminus S$, there is an S-path in G between x and y [18]. The smallest size of a hub set in G is called a hub number of G and is denoted by $h(G)$ [18]. A hub set S of G is a restrained hub set of G if for any two vertices $u, v \notin S$, there is a path between them with all intermediate vertices in $V \setminus S$, the minimum cardinality of S in G is called a restrained hub number of G and is denoted by $h_r(G)$ [7]. Various types of hub numbers have been defined and studied by several authors [6, 8, 9, 10, 11, 12, 13, 14].

Theorem 1.1. [18] *Let S be a subset of $V(G)$. Then G/S is complete if and only if S is a hub set of G.*

Theorem 1.2. [1] *For any connected graphs G and F such that $|V(G)| \geq 2$, $h(G \circ F) = |V(G)|$.*

2. Neighborhub number and Connected Neighborhub number

A set $S \subseteq V(G)$ is a neighborhub set (n-hub set) of G if it has the property that, for any $x, y \in V(G) \setminus S$, there is an S-path in G between x and y and $G = \bigcup\limits_{v \in S} < N[v] >$. The smallest size of an n-hub set in G is called a neighborhub number of G, and is denoted by $h_n(G)$. A neighborhub set S of G is called a connected neighborhub set (cn-set) if $< S >$ is connected. The minimum cardinality of a connected neighborhub set is called the connected neighborhub number of G and is denoted by $h_{cn}(G)$.

We now proceed to compute $h_n(G)$ for some standard graphs.

Proposition 2.1.

(1) For any path P_n,
$$h_n(P_n) = \begin{cases} n-2, & \text{if } n \geq 3 \ ; \\ 1, & \text{if } n = 2 \ . \end{cases}$$
.

(2) For any cycle C_n,
$$h_n(C_n) = \begin{cases} n-2, & \text{if } n \leq 5 \ ; \\ n-3, & \text{if } n \geq 6 \ . \end{cases}$$
.

(3) For any complete graph K_n, $h_n(K_n) = 1$, for all $n \geq 1$.
(4) For the complete bipartite graph $K_{m,n}$, $h_n(K_{m,n}) = min\{m, n\}$.
(5) For the wheel $W_{n+1} = C_n + K_1$, $h_n(W_{n+1}) = 1$.
(6) For a disconnected graph G having k components $G_1, G_2, \cdots, G_k$ of orders $n_1, n_2, \cdots, n_k$, respectively such that $n_1 \leq n_2 \leq \cdots \leq n_k$, $h_n(G) = n_1 + n_2 + \cdots + n_{k-1} + h_n(G_k)$.

Proposition 2.2. *For any connected graph G, $h(G) \leq h_n(G) \leq \eta_c(G)$.*

Proof. Every n-hub set is a hub set. So, $h(G) \leq h_n(G)$. Also, since every connected neighborhood set of G is a n-hub set, we have $h_n(G) \leq \eta_c(G)$. □

Remark 2.1.

(1) A hub set of G need not be a n-hub set of G. For example, consider a cycle on five vertices, $C_5 = (v_1, v_2, v_3, v_4, v_5, v_1)$. $S = \{v_1, v_3\}$ is a hub set of C_5, but it is not a n-hub set because $C_5 \neq \bigcup\limits_{v_i \in S} < N[v_i] >$.
(2) A n-hub set need not be a connected neighborhood set. For example, consider a cycle on six vertices $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$. Here $S = \{v_1, v_3, v_5\}$ is a n-hub set of C_6, but it is not a connected neighborhood set.

Proposition 2.3. *For any connected graph G, $\gamma(G) \leq h_n(G)$.*

Proof. Since every n-hub set is a dominating set, we have $\gamma(G) \leq h_n(G)$. □

Remark 2.2. *A dominating set of G need not be a n-hub set of G. For example, in a cycle $C_5 = (v_1, v_2, v_3, v_4, v_5, v_1), \{v_1, v_3\}$ is a dominating set but not a n-hub set.*

Theorem 2.1. *Let S be a n-hub set of G. Then G/S is complete graph.*

Proof. Let S be a n-hub set of G. Since every n-hub set is a hub set, it follows that S is a hub set of G. By Theorem 1.1 it follows that G/S is complete graph. □

The converse of above Theorem is not true. For example, consider a cycle on four vertices $C_4 = (v_1, v_2, v_3, v_4, v_1)$ and let $S = \{v_1\}$. Then $G/S \cong K_3$, the complete graph. But S is not a n-hub set of G.

Theorem 2.2. *For any connected graph G, $h_n(G) \leq \alpha_1(G)$, where α_1 is an edge covering number of G.*

Proof. Let $L = \{x_1, x_2, \cdots x_k\}$ be an edge cover for G. Then we can select a vertex v_i incident with each edge x_i such that $\{v_1, v_2, \cdots v_k\}$ is a n-hub set of G. Hence, $h_n(G) \leq \alpha_1(G)$. □

Equality in above Theorem holds for $G = K_2$.

Theorem 2.3. *For any connected graph G, $\eta(G) \leq h_n(G)$.*

Proof. Let S be a n-hub set of G. Let W be a set of vertices such that $W \cap S = \{v\}$ and no vertex of $W \setminus \{v\}$ is adjacent to any vertex of $S \setminus \{v\}$ (nor in S themselves). If there exist $u, w \in W$, $u \neq v, w \neq v$ such that u and w are connected only by a trivial path in G, then either v is not adjacent to u or w or both, and $G \neq \bigcup_{x \in S} < N[x] >$, a contradiction to the fact that S is a n-hub set. So, v must be adjacent to all vertices of W. Then $G = \bigcup_{x \in S} < N[x] >$, so that S is a neighborhood set of G. Hence, $\eta(G) \leq h_n(G)$. □

Theorem 2.4. *For any connected graph G, $h_{cn}(G) \leq \eta_c(G)$.*

Proof. Let S be a connected neighborhood set of G. Then for any $x, y \in V \setminus S$, there exists an S-path between them and $G = \bigcup_{v \in S} < N[v] >$. This is stronger condition than that for a connected n-hub set. So, any connected neighborhood set is also a connected n-hub set. □

Theorem 2.5. *For any connected graph G with diameter $d(G)$, $h_n(G) \geq d(G) - 1$ and the inequality is sharp.*

Proof. Since $h(G) \leq h_n(G)$, and $h(G) \geq d(G) - 1$, we have $d(G) - 1 \leq h_n(G)$. The bound is sharp for path $P_n, n \geq 3$ for which $d = n - 1$ and $h_n(P_n) = n - 2$. □

Theorem 2.6. *Let $G = K_{m_1, m_2, \ldots, m_k}$ be a complete $k-$partite graph. Then*

(1) *If $m_i \geq 2$, $1 \leq i \leq k$, then $h_n(G) = 2$.*
(2) *If $m_i = 1$, for some i, $1 \leq i \leq k$, then $h_n(G) = 1$.*

Proof. (1) Let $S = \{u, v\}$, where u, v are any vertices of G belonging to different parts. Then, there is a $\{u, v\}$-path between any two vertices of $V(G) \setminus S$. So, S is a hub set of G. By the definition of complete k-partite graph, $N[u] \cup N[v] = V(G)$ and also, $< N[u] > \cup < N[v] >= G$. So, S is a n-hub set of G. Since $\{w\}$ is not a n-hub set of G for any $w \in V(G)$, it follows that S is a minimum n-hub set of G. Hence, $h_n(G) = 2$.

(2) Since $m_i = 1$, for some i, $1 \leq i \leq k$, let v be the vertex in the part of size one, then by the definition of complete $k-$partite graph $N[v] = V(G)$ and $< N[v] >= G$. Hence $\{v\}$ is a minimum n-hub set of G, therefore $h_n(G) = 1$.

□

Theorem 2.7. *Let T be a tree with order $n \geq 3$ and l leaves. Then*

$$h_n(T) = n - l.$$

Proof. Let S be the set of all nonleaf vertices and let $v \in V(T) \setminus S$, then v is a leaf of T and v should be adjacent to a vertex of S. Therefore $N[S] = V(T)$ and so $T = \bigcup_{u \in S} < N[u] >$. Since $< S >$ is connected and every vertex not in S is adjacent to a vertex of S, S is a n-hub set of T. Suppose that S is not minimum, let $S' \subset S$ be a n-hub set of T, there is a nonleaf vertex w such that $w \notin S'$, so there is a leaf vertex x adjacent to w only. Now $x \in V(T)$ and $x \notin N[S']$, hence S' is not a n-hub set of T. Therefore S is a minimum n-hub set of T. □

Corollary 2.1. *For the double star $S_{n,m}$, $h_n(S_{n,m}) = 2$.*

Corollary 2.2. *For the star $K_{1,n-1}, n \geq 3$, $h_n(K_{1,n-1}) = 1$.*

Theorem 2.8. *For a minimum n-hub set S of G, if there exists a vertex $v \in V(G) \setminus S$, such that $N(v) \subseteq S$, then*

$$h_n(\overline{G}) \leq h(G) + 1.$$

Proof. Let S be a minimum n-hub set of G, and $v \in V(G) \setminus S$ be a vertex with $N_G(v) \subseteq S$. Then in $\overline{G}$, $[V(G) \setminus S] \subseteq N_{\overline{G}}[v]$, and so $N_{\overline{G}}[S \cup \{v\}] = V(\overline{G})$, where $S \cup \{v\}$ is a hub set of $\overline{G}$ and $\overline{G} = \bigcup_{x \in S \cup \{v\}} < N[x] >$. Therefore $h_n(\overline{G}) \leq h(G) + 1$. □

Proposition 2.4. *Let S be any hub set of a graph G, if there exists a vertex $u \in S$, such that $N(u) \subseteq S$. Then S is a n-hub set of $\overline{G}$.*

Proof. Let S be any hub set of a graph G, assume that there exists a vertex $u \in S$ with $N_G(u) \subseteq S$. Then, in $\overline{G}$, $(V(G) \setminus S) \subseteq N_{\overline{G}}[u]$. Since $u \in S$ and $\overline{G} = \bigcup_{x \in S} < N_{\overline{G}}[x] >$, S is a n-hub set of $\overline{G}$. Hence, we get the result. □

Theorem 2.9. *For any graph G, $h_n(G) \leq h_r(G) + 1$.*

Proof. Let G be a graph, and S be a minimum restrained hub set of G. Let $C = N[S]$ and $J = V(G) \setminus C$. Since $J = V(G) \setminus N[S]$ and by the definition of the restrained hub set of G, any vertex $v \in J$, must be adjacent to every vertex of J, it follows that $< J >$ is a complete graph. Therefore $N[S] \cup N[v] = V(G)$ and $G = (\bigcup_{x \in S} < N[x] >) \cup < N[v] >$. So $S \cup \{v\}$ is a n-hub set of G. □

Theorem 2.10. *Every nontrivial connected graph G of order n satisfies $h_n(G) \leq n - 1$ with equality if and only if $G = K_2$.*

Proof. Let G be a connected nontrivial graph of order n. Clearly, for any $v \in V(G)$, $V(G) \setminus \{v\}$ is a n-hub set but is not minimal. So, $h_n(G) \leq n - 1$. Now, let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and without loss of generality, $S = \{v_2, v_3, \cdots, v_n\}$ be a minimum n-hub set of G. Then $V \setminus S = \{v_1\}$ and $v_1 \in N(v_i)$ for some $v_i \in S$. Contrarily suppose $n > 2$, we consider the following cases.
Case 1: $N(v_i) = \{v_1\}$. Since G is connected and $n \geq 3$, there exists $v_j \in S \setminus \{v_i\}$ such that $v_1 \in N(v_j)$ and $N(v_j) \setminus \{v_1\} \subseteq S$. Then $(S \setminus \{v_i, v_j\}) \cup \{v_1\}$ is a n-hub set of G, contradicting the minimality of S.
Case 2: $N(v_i) \neq \{v_1\}$. Then there exists $v_j \in S$ such that $v_j \in N(v_i)$. Since $N(v_i) \setminus \{v_1\} \subseteq S$ and $N(v_j) \setminus \{v_1, v_i\} \subseteq S$, it folows that $(S \setminus \{v_i, v_j\}) \cup \{v_1\}$ is a n-hub set, a contradiction to the minimality of S.
Hence $n = 2$. □

Corollary 2.3. *If G is a (n, m) graph with $\delta(G) \geq 1$ and k components $G_1, G_2, \cdots, G_k$ of orders $n_1 \leq n_2, \leq \cdots \leq n_k$, respectively, then $h_n(G) \leq n - 1$, with equality if and only if every component of G is K_2.*

Proof. Follows from Proposition 2.1 and above theorem. □

Corollary 2.4. *For any nontrivial graph G of order n, $h_n(G) + h_n(\overline{G}) \leq 2(n - 1)$ and $h_n(G) \cdot h_n(\overline{G}) \leq (n - 1)^2$.*

3. Neighborhub number of join and corona of graphs

Here we determine the neighborhub number of the join and corona of two graphs.

Theorem 3.1. *Let G_1 and G_2 be two graphs such that $|V(G_1)| = 1$. Then $h_n(G_1 + G_2) = 1$.*

Proof. Let $V(G_1) = \{v\}$, then by the definition of $G_1 + G_2$, v is adjacent to every vertex of G_2, $< v >= G_1 + G_2$. Also v is the only vertex in a path between any two vertices of G_2, hence $\{v\}$ is a minimum neighborhub set of $G_1 + G_2$. □

Corollary 3.1. *Let G_1 be a graph with $|V(G_1)| = 1$ and G_2 be any graph. Then $h_n(G_1 \circ G_2) = 1$.*

Proof. Suppose $|V(G)| = 1$, then $G_1 \circ G_2$ is just $G_1 + G_2$, therefore by the previous theorem $h_n(G_1 \circ G_2) = 1$. □

Proposition 3.1. *Let G_1 and G_2 be any two non complete graphs such that $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$. Then $h_n(G_1 + G_2) = 2$.*

Proof. Let $v \in V(G_1)$, $u \in V(G_2)$. Let $S = \{u, v\}$, then v is adjacent to u in $G_1 + G_2$, let $w, z \in V(G_1 + G_2)$, we discuss the following cases:
Case 1: Let $w, z \in V(G_1)$. By definition of $G_1 + G_2$ the vertices w, z are adjacent to u. So, there is a $\{u\}$-path between w and z in $G_1 + G_2$.
Case 2: Let $w, z \in V(G_2)$. Using similar argument as in Case 1, there is a $\{v\}$-path between w and z in $G_1 + G_2$.
Case 3: Let $w \in V(G_1)$ and $z \in V(G_2)$. Then we observe by the definition of $G_1 + G_2$, that w is adjacent to u, and z is adjacent to v. Since v is adjacent to u, $wuvz$ is an S-path in $G_1 + G_2$. Note that, by the definition of $G_1 + G_2$, v is adjacent to every vertex of G_2 and u is adjacent to every vertex of G_1. So, $N[v] \cup N[u] = V(G_1 + G_2)$ and $< N[v] > \cup < N[u] >= G_1 + G_2$, this implies that S is a n-hub set of $G_1 + G_2$.
From all the above cases, we have $h_n(G_1 + G_2) \leq 2$. Since G_1 and G_2 are noncomplete graphs, neither $\{u\}$ nor $\{v\}$ is a n-hub set of $G_1 + G_2$. Hence $h_n(G_1 + G_2) \leq 2$. □

Theorem 3.2. *Let G_1 be a connected graph of order $n \geq 2$, and let G_2 be any graph. Then $h_n(G_1 \circ G_2) = n$.*

Proof. Let G_1 be a connected graph with order $n \geq 2$, and let $V(G_1) = \{v_1, v_2, ..., v_n\}$, then for a vertex $v_i \in V(G_1)$, by the definition of corona of G_1 and G_2 we conclude that $G_2^{v_i} \subseteq N[v_i]$, for every i, $1 \leq i \leq n$. So, $N[V(G)] = V(G_1 \circ G_2)$, now let $S = V(G_1)$, then by Theorem 1.2, S is a minimum hub set of $G_1 \circ G_2$ and $\bigcup\limits_{x \in S} < N[x] >= G_1 \circ G_2$, the result follows. □

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Veena Mathad for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.12, N.1.

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