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BARGMANN'S VERSUS FOR FRACTIONAL FOURIER TRANSFORMS AND APPLICATION TO THE QUATERNIONIC FRACTIONAL HANKEL TRANSFORM

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Abstract. We present a general formalism à la Bargmann for constructing fractional Fourier transform associated to specific class of integral transforms on separable Hilbert spaces. As concrete application, we consider the quaternionic fractional Fourier transform on the real half-line and associated to the hyperholomorphic second Bargmann transform for the slice Bergman space of second kind. This leads to an extended version of the well-known fractional Hankel transform. Basic properties are derived including inversion formula and Plancherel identity.

Keywords: Fractional Fourier transform; Fractional Hankel transform; Slice hyperholomorphic Bergman space; Second Bargmann transform; Laguerre polynomials; Bessel functions.

AMS Subject Classification: 30G35.

1. Introduction

The fractional Fourier transform (FrFT), which is special generalization of the Fourier integral transform, is a powerful tool in many fields of research including mathematics, physics and engineering sciences [1, 17, 14]. Its introduction goes back to 1929. In fact, it was considered implicitly in Wiener's work [20], when discussing the extension of certain results of H. Weyl and leading later to Fourier developments of fractional order. Mainly, Wiener sets out to find a one–parameter family of unitary integral operators

$$\mathcal{F}_\alpha\varphi(x) := \int_{-\infty}^{+\infty} K_\alpha(x,v)\varphi(v)dv$$

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on $L^2(\mathbb{R})$ for which the n–th Hermite function $h_n(x) = H_n(x)e^{-x^2/2}$ is an eigenfunction with $e^{in\alpha}$ as corresponding eigenvalue. The explicit Wiener formula for the kernel function K_α is a limiting case of Mehler's formula for the Hermite functions. This fact was rediscovered sixty years later in quantum mechanics by Namias [15], and showed earlier by Hörmander [12]. Recently, this has been extended to a FrFT related to the generalized Laguerre functions by exploiting their relation to Hermite polynomials [5].

Another and elegant way to define FrFT is given implicitly in Bargmann's seminal paper [3] (see also [4, 11, 19]). Indeed, associated to the classical Segal–Bargmann transform $\mathcal{B}$, mapping $L^2(\mathbb{R}^d)$ onto the Fock–Bargmann space, one considers $\mathcal{R}_\theta := \mathcal{B}^{-1} \circ T_\theta \circ \mathcal{B}$, with $T_\theta f(z) := f(\theta z)$, which defines a unitary homeomorphism transform on $L^2(\mathbb{R}^d)$ when $\theta = e^{i\alpha}$; $\alpha \in \mathbb{R}$, and satisfies

$$\mathcal{R}_{e^{i\alpha}} h_n(x) = e^{in\alpha} h_n(x).$$

In the present paper, we provide à la Bargmann a general abstract formalism for constructing fractional transform associated to given special invertible integral transform $\mathcal{S}_{X,Y} : \mathcal{H}_X \longrightarrow \mathcal{H}_Y$,

$$\mathcal{S}_{X,Y}\varphi(y) = \int_X R(x,y)\varphi(x)\omega_X(x)d\lambda(x),$$

on an arbitrary infinite separable functional Hilbert space $\mathcal{H}_X = L^2(X; \omega_X(x)dx)$. Namely, we deal with integral transforms of the form $\mathcal{S}^{-1}_{X,Y} \circ T_\theta \circ \mathcal{S}_{X,Y}$, where T_θ ia an appropriate action of a group G. We show that the performed fractional integral transform inherits numerous properties from the ones of $\mathcal{S}_{X,Y}$. The explicit computation shows that the kernel function of $\mathcal{S}^{-1}_{X,Y} \circ T_\theta \circ \mathcal{S}_{X,Y}$ can be expressed explicitly in terms of the kernel function $R(x,y)$ (see (10) below). As concrete application, we deal with a special quaternionic fractional Fourier transform (QFrFT) acting on the right quaternionic Hilbert space

$$L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+) = L^2_{\mathbb{H}}\left(\mathbb{R}^+, x^\alpha e^{-x}dx\right), \quad \alpha > 0,$$

and associated to the second Bargmann transform for hyperholomorphic Bergman space of second kind [7]. More precisely, they are the family of (left) integral transforms

$$\mathcal{L}^\alpha_\theta \varphi(y) := \int_0^\infty K^\alpha_\theta(x,y)\varphi(x)dx, \tag{1}$$

whose kernel function can be shown to be given in terms of the modified Bessel function I_α of first kind. They verify $\mathcal{L}^\alpha_\theta(\varphi^\alpha_n(x)) = \theta^n \varphi^\alpha_n(x)$. We also prove that $\mathcal{L}^\alpha_\theta$ is continuous, interpolates continuously the identity operator to the Fourier-Bessel transform and satisfies the index law (semi-group property) $\mathcal{L}^\alpha_\theta \circ \mathcal{L}^\alpha_\eta = \mathcal{L}^\alpha_{\theta\eta}$, so that the inverse of $\mathcal{L}^\alpha_\theta$ reads simply $\mathcal{L}^\alpha_{1/\theta}$. When $|\theta| = 1$, the constructed family of QFrFT for $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ appears embedded in a strongly continuous one-parameter group of unitary operators and coincides with the fractional Hankel transform [16, 13] with quaternionic parameter (QFrHT). The exposition of these ideas in the quaternionic setting add some technical difficulties which we overcome using tools from the theory of slice regular functions.

The paper is organized as follows. In Section 2, we recall the definition of FrHT due to Namias that we adapt it to the quaternionic setting. In Section 3, we present a general abstract formalism for constructing QFrHT by means of eigenvalue equation involving orthogonal basis of certain quaternionic Hilbert space. Section 4 is devoted to the reconstruction of QFrHT for $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ by Bargmann versus, and show how to derive in a simple way their basic properties such as the Plancherel and inversion formulas.

We conclude by noting that all needed notions and notations on hyperholomorphic Bergman spaces are those fixed in [7].

2. Quaternionic fractional Hankel transform (à la Namias)

In this section, we adopt the Namias' approach for constructing fractional Hankel transform for the quaternionic right Hilbert space $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$; $\alpha > 0$, of all quaternionic-valued functions on the half real line $\mathbb{R}^+$ that are square integrable with respect to the inner product

$$\langle \varphi, \psi \rangle_\alpha = \int_{\mathbb{R}^+} \overline{\varphi(x)} \psi(x) x^\alpha e^{-x} dx.$$

We denote by $\|\cdot\|_\alpha$ the associated norm. A complete orthonormal system in $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ is given by the functions

$$\varphi^\alpha_n(x) := \left(\frac{n!}{\Gamma(\alpha+n+1)} \right)^{1/2} L^{(\alpha)}_n(x), \tag{2}$$

where $L^{(\alpha)}_n(x)$ denotes the generalized Laguerre polynomials

$$L^{(\alpha)}_n(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} \left(x^{n+\alpha} e^{-x} \right). \tag{3}$$

The Hille–Hardy identity [2, (6.2.25) p. 288]

$$\begin{aligned} R^\alpha_\theta(x,y) &= \sum_{n=0}^{+\infty} \frac{n!}{\Gamma(\alpha+n+1)} \theta^n L^{(\alpha)}_n(x) L^{(\alpha)}_n(y) \\ &= \frac{1}{1-\theta} \left(\frac{1}{\theta xy} \right)^{\alpha/2} \exp\left(-\frac{\theta(x+y)}{1-\theta} \right) I_\alpha \left(\frac{2\sqrt{\theta}}{1-\theta} \sqrt{xy} \right) \end{aligned} \tag{4}$$

is valid for $|\theta| < 1$ and nonnegative integer α. Here $I_\alpha(\xi)$ denotes the modified Bessel function of first kind [2, p.222]

$$I_\alpha(\xi) = \left(\frac{\xi}{2} \right)^\alpha \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(\alpha+n+1)} \left(\frac{\xi}{2} \right)^{2n}. \tag{5}$$

Thus, we can rewrite the kernel function $K^\alpha_\theta(x,y) := x^\alpha e^{-x} R^\alpha_\theta(x,y)$ as

$$K^\alpha_\theta(x,y) = \frac{1}{1-\theta} \left(\frac{x}{\theta y} \right)^{\alpha/2} \exp\left(-\frac{x+\theta y}{1-\theta} \right) I_\alpha \left(\frac{2\sqrt{\theta}}{1-\theta} \sqrt{xy} \right), \tag{6}$$

so that the corresponding integral operator is well–defined on $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ by

$$\mathcal{L}^\alpha_\theta(\varphi)(y) = \int_0^{+\infty} K^\alpha_\theta(x,y) \varphi(x) dx \tag{7}$$

is what we call here quaternionic fractional Hankel transform (QFrHT). The Laguerre polynomial $\varphi^\alpha_n(x)$ in (2) is (left) eigenfunction of $\mathcal{L}^\alpha_\theta$ with θ^n as corresponding (right) eigenvalue,

$$\mathcal{L}^\alpha_\theta(\varphi^\alpha_n(x)) = \varphi^\alpha_n(x) \theta^n.$$

This readily follows from the definition of $\mathcal{L}^\alpha_\theta$.

Remark 2.1. *Such transform is closely connected to the fractional Hankel transform* [16, 13]. *In fact the last one appears as the limit case of $\mathcal{L}^\alpha_\theta$ when restricting θ to $|\theta| = 1$.*

We conclude this section by proving that for $|\theta| < 1$, the integral transform $\mathcal{L}^\alpha_\theta$ defines a continuous k–contraction from $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ into itself with $k = (1 - |\theta|^2)^{-1/2}$.

Proposition 2.1. *For* $|\theta| < 1$ *and every* $\varphi \in L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$*, we have*

$$\|\mathcal{L}^{\alpha}_{\theta}(\varphi)\|^2 \leq \left(\frac{1}{1-|\theta|^2}\right)\|\varphi\|^2.$$

Proof. Since $(\varphi^{\alpha}_n)_n$ in (2) is a complete orthonormal system in $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$, we can expand any $f \in L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ as $f(x) = \sum\limits_{n=0}^{+\infty}\varphi^{\alpha}_n c_n$ for some $c_n \in \mathbb{H}$. Hence, using the fact that $\mathcal{L}^{\alpha}_{\theta}(\varphi^{\alpha}_n)(x) = \varphi^{\alpha}_n(x)\theta^n$, we get

$$\mathcal{L}^{\alpha}_{\theta}(f)(x) = \sum_{n=0}^{+\infty}\varphi^{\alpha}_n(x)\theta^n c_n. \tag{8}$$

Using the orthogonality of φ^{α}_n, we obtain

$$\begin{aligned}\|\mathcal{L}^{\alpha}_{\theta}(\varphi)\|^2 &= \sum_{n=0}^{+\infty}|\theta|^{2n}|c_n|^2\\ &\leq \left(\sum_{n=0}^{+\infty}|\theta|^{2n}\right)\left(\sum_{n=0}^{+\infty}|c_n|^{2n}\right)\\ &\leq \left(\frac{1}{1-|\theta|^2}\right)\|\varphi\|^2\end{aligned}$$

which requires $|\theta| < 1$. ☐

3. Abstract Bargmann's formalism for fractional integral transform

This section is devoted to present a general formalism for constructing like fractional Fourier transform. For this, we explore Bargmann's idea related to Segal-Bargmann transform. It will be applied in Section 4 to recover the quaternionic fractional Hankel transform discussed in Section 2. Thus, let $\mathcal{H}_X$ and $\mathcal{H}_Y$ be two arbitrary infinite functional right quaternionic separable Hilbert spaces with orthonormal bases $\{\varphi_n; n = 0, 1, \cdots\}$ and $\{\psi_n; n = 0, 1, \cdots\}$ defined on given sets X and Y, respectively. The corresponding inner scalar products are given by

$$\langle\varphi, \phi\rangle_{\mathcal{H}_X} = \int_X \overline{\varphi}(x)\phi(x)\omega_X(x)dx$$

and

$$\langle\Psi, \Phi\rangle_{\mathcal{H}_Y} = \int_Y \overline{\Psi}(y)\Phi(y)\omega_Y(y)dy,$$

respectively, for some weight functions ω_X and ω_Y. Associated to the data $(X, \mathcal{H}_X, \varphi_n)$ and $(Y, \mathcal{H}_Y, \psi_n)$, we consider the integral transform $\mathcal{S}_{XY} : \mathcal{H}_X \longrightarrow \mathcal{H}_Y$ of the form

$$\mathcal{S}_{XY}(\varphi)(y) = \int_X \overline{R(x, y)}\varphi(x)\omega_X(x)dx.$$

We assume that $\mathcal{S}_{XY}$ is well defined on $\mathcal{H}_X$ such that $\mathcal{S}_{XY}(\varphi_n) = \psi_n$. This is equivalent to say that the kernel function $R(x, y)$ on $X \times Y$ can be expanded as

$$R(x, y) = \sum_{n=0}^{\infty}\varphi_n(x)\overline{\psi_n(y)}$$

whenever the series in the right-hand side is uniformly and absolutely convergent. Subsequently, $\mathcal{S}_{XY}$ is an invertible integral kernel transform, whose inverse is given by

$$\mathcal{S}_{XY}^{-1}\psi(x) = \int_Y R(x,y)\psi(y)\omega_Y(y)dy$$

for $x \in X$ and $\psi \in \mathcal{H}_Y$. We then perform the fractional transform associated to $\mathcal{S}_{XY}$ to be the $\mathcal{F}_g$ given by the commutative diagrams

$$\begin{tikzcd}
\mathcal{H}_X \arrow[r, "\mathcal{S}_{XY}"] \arrow[d, "\mathcal{F}_g"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{H}_Y \arrow[d, "\Gamma_g"] \\
\mathcal{H}_X & \mathcal{H}_Y \arrow[l, "\mathcal{S}_{XY}^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "\psi"] \arrow[d, "\widetilde{\Gamma}_g"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{H} \arrow[d, "\Gamma_g"] \\
Y \arrow[r, "\psi"'] & \mathbb{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] & \mathcal{G}(U) \\
\mathcal{F}(V) \arrow[r] \arrow[u] & \mathcal{G}(V) \arrow[u]
\end{tikzcd}$$

for every $g \in G$, $\psi \in \mathcal{H}_Y$, where $\widetilde{\Gamma} : G \times Y \longrightarrow Y$; $(g, y) \longmapsto \widetilde{\Gamma}(g, y) = \widetilde{\Gamma}_g(y) = g(y)$, is a special action of some group G on Y that we have extended to $\mathcal{H}_Y$ by considering $\Gamma : G \times \mathcal{H}_Y \longrightarrow \mathcal{H}_Y$ with $\Gamma(g,\psi)(y) = \Gamma_g(\psi)(y) = \psi(g(y))$ with $y \in Y$ and $\psi \in \mathcal{H}_Y$. Namely,

$$\mathcal{F}_g = \mathcal{S}_{XY}^{-1} \circ \Gamma_g \circ \mathcal{S}_{XY}; \quad g \in G.$$

Therefore, for every $\psi \in \mathcal{H}_Y$, we have

$$\mathcal{F}_g(\varphi)(x) = \int_Y R(x,y) \left(\int_X \overline{R(x', g(y))}\varphi(x')\omega_X(x')dx' \right) \omega_Y(y)dy. \tag{9}$$

To change the order of the integrals, stronger conditions need to be imposed on the integrand so the requirements of Fubini's theorem are met. This holds true when for example $\omega_Y(y)dy$ is a finite measure on Y and the function $(x', y) \longmapsto R(x,y)\overline{R(x', g(y))}$ belongs to $L^2(X \times Y, \omega_X(x')\omega_Y(y)dx'dy)$ for every fixed $x \in X$ and $g \in G$. A sufficient condition, when $\omega_X(x')dx'$ and $\omega_Y(y)dy$ are finite measures on X and Y, respectively, is $|R(x,y)|^2 \in L^2(Y, \omega_Y(y)dx'dy)$ for every fixed $x \in X$ and $|R(x', g(y))|^2 \in L^2(X \times Y, \omega_X(x')\omega_Y(y)dx'dy)$ for every fixed $g \in G$. Thus, under such kind of conditions, we get

$$\mathcal{F}_g(\varphi)(x) \overset{Fubini}{=} \int_X \widetilde{R_g}(x', x)\varphi(x')\omega_X(x')dx',$$

where $\widetilde{R_g}(x', x)$ stands for

$$\widetilde{R_g}(x', x) = \left\langle R(x', g(y)), R(x,y) \right\rangle_{\mathcal{H}_Y}. \tag{10}$$

An expansion of $\widetilde{R_g}(x', x)$, at least formally, is the following

$$\begin{aligned}
\widetilde{R_g}(x', x) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varphi_n(x) \langle \psi_n, \Gamma_g \psi_m \rangle_{\mathcal{H}_Y} \overline{\varphi_m(x')} \\
&= \sum_{n=0}^{\infty} \varphi_n(x)\chi_n(g)\, \overline{\varphi_n(x')} =: R_g(x', x). \qquad (11)
\end{aligned}$$

The last equality follows under the additional assumption that

$$\Gamma_g \psi_m(y) = \psi_m(g(y)) = \chi_m(g)\psi_m(y). \tag{12}$$

According to the above discussion, we reformulate the following definitions.

Definition 3.1. *If the series in the right-hand side of* (11) *converges absolutely and uniformly to* $R_g(x', x)$*, then*

$$\mathcal{F}_g(\varphi)(x) := \int_X R_g(x', x)\varphi(x')\omega_X(x')dx'$$

defines a like-fractional Fourier transform for the data $(\mathcal{H}_X, \varphi_n, \chi_n)$.

Remark 3.1. *We have* $\mathcal{F}_g(\varphi_n) = \varphi_n\chi_n(g)$*. This gives an integral representation for* φ_n.

Definition 3.2. *We call fractional Fourier transform associated to* $\mathcal{S}_{XY}$ *and* Γ *the integral transform*

$$\widetilde{\mathcal{F}}_g(\varphi)(x) = \int_X \widetilde{R_g}(x', x)\varphi(x')\omega_X(x')dx'$$

with

$$\widetilde{R_g}(x', x) = \left\langle R(x', g(y)), R(x, y)\right\rangle_{\mathcal{H}_Y} \tag{13}$$

provided that (13) *exists.*

Remark 3.2. *The construction is valid for any arbitrary complex or quaternionic Hilbert spaces. Its description is more simpler when dealing with complex Hilbert spaces. Thus, when the quaternionic Hilbert spaces are considered, the fractional Fourier transforms in Definitions 3.1 and 3.2 are called quaternionic (QFrFT).*

Remark 3.3. *The equality* $\widetilde{\mathcal{F}}_g(\varphi)(x) = \mathcal{S}_{XY}^{-1}\Gamma_g\mathcal{S}_{XY}(\varphi)(x)$ *holds true under further assumptions on the kernel function allowing the application of Fubini's theorem to* (9)*. In this case, if* χ_m *in* (12) *is a character of the group* G*, i.e.,* Γ_g *satisfies* $\Gamma_{gg'} = \Gamma_g\Gamma_{g'}$*, then* $\widetilde{\mathcal{F}}_g$ *is invertible with inverse given by* $\widetilde{\mathcal{F}}_{g^{-1}}$.

Remark 3.4. *Possible description of other properties of the considered QFrFT, like its behavior with ordinary derivatives, with fractional derivatives, with fractional integrals, as well as the discussion of its eventual role in the resolution of ordinary and partial differential equations is closely connected to the initial transform* $T_{X,Y}$ *and its kernel function.*

4. Application: The QFrFT for $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$

The needed notions and the notations from the theory of slice regular functions as mentioned at the end of the introductory section are those used in [7].

In view of the explicit expression of the kernel function in (6), we see that we can consider the limit case of the Hille–Hardy formula which corresponds to $|\theta| = 1$ with $\theta \neq 1$. We show below that this can recovered by the formalism presented in Definition 3.2 and specified for $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$, so that for $\theta = 1$ the considered transform reduces further to the identity operator of $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$. To this end, we begin by recalling that the hyperholomorphic second Bargmann integral transform [7], defined by

$$[\mathcal{A}^\alpha_{slice}\varphi](q) = \frac{1}{\sqrt{\pi\Gamma(\alpha)}\,(1-q)^{\alpha+1}} \int_0^{+\infty} \exp\left(\frac{tq}{q-1}\right)\varphi(t)t^\alpha e^{-t}dt, \tag{14}$$

is the quaternionic analogue of the complex second Bargmann transform introduced by Bargmann himself in [3, p.203]. It establishes a unitary isometry from $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ onto the slice hyperholomorphic Bergman space (of second kind) on the unit ball $\mathbb{B}$ in $\mathbb{R}^4$,

$$A^{2,\alpha}_{slice}(\mathbb{B}) := \mathcal{SR}(\mathbb{B}) \cap L^{2,\alpha}(\mathbb{B}_I), \tag{15}$$

where $I \in \mathbb{S} = \{q \in \mathbb{H}; q^2 = -1\}$; $\mathbb{B}_I = \mathbb{B} \cap \mathbb{C}_I$ is the unit disc in the slice $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$, and

$$L^{2,\alpha}(\mathbb{B}_I) := \left\{ f : \mathbb{B} \longrightarrow \mathbb{H};\ \int_{\mathbb{B}_I} |f(z)|^2 d\lambda^\alpha_I(z) < +\infty \right\}.$$

Here $d\lambda^\alpha_I$ denotes the Bergman measure on the unit disc $\mathbb{B}_I$ in $\mathbb{R}^2$ given by

$$d\lambda^\alpha_I(z = x + Iy) = \left(1 - x^2 - y^2\right)^{\alpha-1} dxdy.$$

Consequently, we have

$$A^{2,\alpha}_{slice}(\mathbb{B}) = \left\{ f(q) = \sum_{n=0}^{\infty} q^n c_n;\ c_n \in \mathbb{H},\ \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\alpha+n+1)} |c_n|^2 < +\infty \right\},$$

so that the restriction to $\mathbb{B}_I$ is the classical Bergman space on the unit disc of $\mathbb{C}_I$. It should be mentioned here that the scalar product defining $L^{2,\alpha}(\mathbb{B}_I)$,

$$\langle f, g\rangle_I := \int_{\mathbb{B}_I} \overline{f(z)} g(z) d\lambda^\alpha_I(z),$$

is independent of I when acting on $A^{2,\alpha}_{slice}(\mathbb{B}) \times A^{2,\alpha}_{slice}(\mathbb{B})$, i.e., $\langle f, g\rangle_I = \langle f, g\rangle_J$ for any $f, g \in A^{2,\alpha}_{slice}(\mathbb{B})$ and any I, J such that $I^2 = J^2 = -1$.

The inverse of the second Bargmann transform $\mathcal{A}^\alpha_{slice}$ is well–defined from $A^{2,\alpha}_{slice}(\mathbb{B})$ onto $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$, and is given by [7]

$$[\mathcal{A}^\alpha_{slice}]^{-1} f(t) = \frac{1}{\sqrt{\pi\Gamma(\alpha)}} \int_{B_I} \exp\left(\frac{t\overline{z}}{\overline{z}-1}\right) \frac{(1-|z|^2)^{\alpha-1}}{(1-z)^{\alpha+1}} f|_{B_I}(z) dxdy. \tag{16}$$

Notice for instance that the definition of $A^{2,\alpha}_{slice}(\mathbb{B})$ is based on the classical one on a given disc $\mathbb{B}_I$. This was possible by extending the complex holomorphic functions to the whole $\mathbb{B}$ by the representation formula (see for example [6]). While the transform $\mathcal{A}^\alpha_{slice}$ in (14) is associated to the kernel function

$$A^\alpha_{slice}(x; q) := \frac{1}{\sqrt{\pi\Gamma(\alpha)}\,(1-q)^{\alpha+1}} \exp\left(\frac{xq}{q-1}\right) \tag{17}$$

on $\mathbb{R}^+ \times \mathbb{B}$, and obtained as bilinear generating function involving the functions $(\varphi^\alpha_n)_n$ in (2) and the orthonormal basis of $A^{2,\alpha}_{slice}(\mathbb{B})$ given by the functions

$$f_n(q) = \left(\frac{\Gamma(n+\alpha+1)}{\pi\Gamma(\alpha)n!}\right)^{1/2} q^n. \tag{18}$$

Now, by means of $\mathcal{A}^\alpha_{slice}$, its inverse $[\mathcal{A}^\alpha_{slice}]^{-1}$ and the angular unitary operator $\Gamma_\theta(f)(q) = f(q\theta)$, we perform the transform

$$\widetilde{\mathcal{L}^\alpha_\theta} := [\mathcal{A}^\alpha_{slice}]^{-1} \Gamma_\theta \mathcal{A}^\alpha_{slice} \tag{19}$$

on $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$. According to Definition 3.2 and Remark 3.2, this transform is the quaternionic fractional Fourier transform associated to $\mathcal{A}^\alpha_{slice}$. Here we consider the $U_{\mathbb{H}}(1)$–action

$\theta(q) := q\theta$ of $G = U_{\mathbb{H}}(1)$ on $\mathbb{B}$, that we extend to the hyperholomorphic Bergman space $A^{2,\alpha}_{slice}(\mathbb{B})$ by considering

$$\Gamma_\theta(f)(q) = f_\star(q\theta) := \sum_{n=0}^{\infty} q^n\theta^n c_n \tag{20}$$

for given $f(q) = \sum_{n=0}^{\infty} q^n c_n \in A^{2,\alpha}_{slice}(\mathbb{B})$. The function $q \longmapsto f_\star(q\theta)$ is in fact the slice regularization of $q \longmapsto f(q\theta)$ obtained by making use of the left $\star_s^L$-product for left slice regular functions $f(q) = \displaystyle\sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \displaystyle\sum_{n=0}^{\infty} q^n b_n$ on $\mathbb{H}$ defined by [9]

$$(f \star_s^L g)(q) = \sum_{n=0}^{\infty} q^n \left(\sum_{k=0}^{n} a_k b_{n-k}\right). \tag{21}$$

In particular, we have

$$(f_n)_\star(q\theta) := f_n(q)\theta^n, \tag{22}$$

and therefore we may prove the following.

Proposition 4.1. *For $\theta \in \mathbb{H}$ with $|\theta| \leq 1$, the transform $\widetilde{\mathcal{L}^\alpha_\theta}$ in* (19) *defines a continuous integral transform from $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ onto $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ with norm not exceed* 1. *For $|\theta| = 1$, we have*

$$\left\langle \widetilde{\mathcal{L}^\alpha_\theta}\varphi, \widetilde{\mathcal{L}^\alpha_\theta}\psi \right\rangle = \langle \varphi, \psi\rangle.$$

Proof. The operator $\widetilde{\mathcal{L}^\alpha_\theta}$ in (19) is well–defined from $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ into itself if and only if the action Γ_θ leaves the space $A^{2,\alpha}_{slice}(\mathbb{B})$ invariant, which is clear from the definition of Γ_θ given through (20). Moreover, using the fact $\mathcal{A}^\alpha_{slice}\varphi^\alpha_n = f_n$ as well as (22), we get

$$\widetilde{\mathcal{L}^\alpha_\theta}(\varphi^\alpha_n(y)) = [\mathcal{A}^\alpha_{slice}]^{-1}\left(f_n(\cdot)\theta^n\right)(y) = \varphi^\alpha_n(y)\theta^n.$$

In addition, under the condition that $|\theta| = 1$, it is clear that Γ_θ preserves the scalar product in $A^{2,\alpha}_{slice}(\mathbb{B})$. Indeed, for every $f = \sum_{n=0}^{\infty} f_n c_n$ and $g = \sum_{n=0}^{\infty} f_n d_n \in A^{2,\alpha}_{slice}(\mathbb{B})$, we have

$$\begin{aligned}\langle \Gamma_\theta f, \Gamma_\theta g\rangle_{A^{2,\alpha}_{slice}(\mathbb{B})} &= \sum_{n,m=0}^{\infty} \overline{c_n}\,\overline{\theta^n}\langle f_n, f_m\rangle_{A^{2,\alpha}_{slice}(\mathbb{B})}\theta^m d_m \\ &= \sum_{n=0}^{\infty} \overline{c_n}|\theta^n|^2 d_n \\ &= \langle f, g\rangle_{A^{2,\alpha}_{slice}(\mathbb{B})}.\end{aligned}$$

Accordingly, the identity $\left\langle \widetilde{\mathcal{L}^\alpha_\theta}\varphi, \widetilde{\mathcal{L}^\alpha_\theta}\psi \right\rangle = \langle \varphi, \psi\rangle$ follows as composition of operators preserving scalar product. □

Remark 4.1. *As particular case, we have the Plancherel formula $\left\|\widetilde{\mathcal{L}^\alpha_\theta}\psi\right\| = \|\psi\|$ when $|\theta| = 1$. This can be recovered directly from the definition of $\widetilde{\mathcal{L}^\alpha_\theta}$, since in this case Γ_θ is un isometry like the Bargmann transform and its inverse.*

Corollary 4.1. *If $|\theta| = 1$, then the QFrFT in* (19) *defines a unitary transform from $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ into $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$.*

Remark 4.2. *The family of one–parameter transforms $\mathcal{L}^{\alpha}_{\theta}$ verifies the semi-group property $\mathcal{L}^{\alpha}_{\theta}\circ\mathcal{L}^{\alpha}_{\eta} = \mathcal{L}^{\alpha}_{\theta\eta}$, so that its inverse is $\mathcal{L}^{\alpha}_{1/\theta}$ when $\theta \neq 0$. But we do not have $\mathcal{L}^{\alpha}_{\theta}\circ\mathcal{L}^{\alpha}_{\eta} = \mathcal{L}^{\alpha}_{\eta}\circ\mathcal{L}^{\alpha}_{\theta}$ in general, for lack of commutativity in $\mathbb{H}$. However, $\mathcal{L}^{\alpha}_{\theta} \circ \mathcal{L}^{\alpha}_{\eta} = \mathcal{L}^{\alpha}_{\theta\eta} = \mathcal{L}^{\alpha}_{\eta} \circ \mathcal{L}^{\alpha}_{\theta}$ holds only when θ and ψ belongs to the same slice $\mathbb{C}_I := \mathbb{R} + I\mathbb{R} \subset \mathbb{H}$; $I^2 = -1$.*

The next result gives the explicit expression of the inverse of $\widetilde{\mathcal{L}^{\alpha}_{\theta}}$.

Proposition 4.2. *For any quaternionic $\theta \neq 0$, the inverse of $\widetilde{\mathcal{L}^{\alpha}_{\theta}}$ is given by*

$$(\widetilde{\mathcal{L}^{\alpha}_{\theta}})^{-1} = \mathcal{A}^{-1}\Gamma_{\theta}^{-1}\mathcal{A} = \widetilde{\mathcal{L}^{\alpha}_{1/\theta}}.$$

Proof. It is immediate form the definition of $\widetilde{\mathcal{L}^{\alpha}_{\theta}}$ and the fact that $\Gamma_{\theta} \circ \Gamma_{\eta} = \Gamma_{\theta\eta}$. □

The following result identifies the kernel function given by (13),

$$\widetilde{R^{\alpha}_{\theta}}(x, y) := \langle A^{\alpha}_{slice}(x; \theta(\cdot)), A^{\alpha}_{slice}(y; \cdot)\rangle_{L^{2,\alpha}(\mathbb{B}_I)} \tag{23}$$

of the QFrFT transform

$$[\widetilde{\mathcal{L}^{\alpha}_{\theta}}(\varphi)](y) = \left\langle \widetilde{R^{\alpha}_{\theta}}(\cdot, y), \varphi \right\rangle_{L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)}.$$

Theorem 4.1. *The kernel function $\widetilde{R^{\alpha}_{\theta}}(x, y)$ is a left slice regular and coincides with the kernel function of the fractional Hankel transform on the quaternionic unit ball. Moreover, the explicit expression of $\widetilde{\mathcal{L}^{\alpha}_{\theta}}$ is given by*

$$\widetilde{\mathcal{L}^{\alpha}_{\theta}}\varphi(y) = \frac{e^{\frac{\theta y}{\theta-1}}}{(1-\theta)(\theta y)^{\alpha/2}} \int_0^{\infty} x^{\alpha/2} I_{\alpha}\left(\frac{2\sqrt{\theta}}{(1-\theta)}\sqrt{xy}\right) e^{\frac{x}{\theta-1}}\varphi(x)dx \tag{24}$$

for any $\theta \in \mathbb{H}$ with $|\theta| \leq 1$ and $\theta \neq 1$, where I_{α} is as in (5).

Proof. Notice first that for $\theta = 1$ there is nothing to prove since in this case, the operator $\widetilde{\mathcal{L}^{\alpha}_{\theta}}$ reduces further to the identity operator of the Hilbert space $L^{2,\alpha}_{\mathbb{H}}(\mathbb{R}^+)$ and the $R^{\alpha}_{\theta}(x, y)$ in (23) can be considered as the Dirac delta function. To identify the closed expression of the kernel $\widetilde{R^{\alpha}_{\theta}}(x, y)$, we should notice that the Γ_{θ}-action reads

$$\Gamma_{\theta}(q \longmapsto A^{\alpha}_{slice}(x; q)) = (1 - q\theta)^{-\alpha-1} \star \exp_{\star}\left(xq\theta, [q\theta - 1]^{-1}\right),$$

where

$$\exp_{\star}\left(f(q), g(q)\right) = \sum_{n=0}^{\infty} \frac{f^{n\star}(q) \star g^{n\star}(q)}{n!}.$$

For θ being a non real quaternionic number, there exists a unique imaginary unit I_{θ}; $I_{\theta}^2 = -1$, such that $\theta \in \mathbb{C}_{I_{\theta}} \cap S^3$. By means of (23) and the independence of the scalar product $\langle f, g\rangle_I$ in I when acting on $A^{2,\alpha}_{slice}(\mathbb{B})$, we may write

$$\begin{aligned}\widetilde{R^{\alpha}_{\theta}}(x, y) &:= \langle \Gamma_{\theta}A^{\alpha}_{slice}(x; \cdot), A^{\alpha}_{slice}(y; \cdot)\rangle_{L^{2,\alpha}(\mathbb{B}_{I_{\theta}})} \\ &= \frac{1}{\pi\Gamma(\alpha)} \int_{B_I} \frac{\exp\left(\frac{xz\theta}{z\theta-1}\right)\exp\left(\frac{y\overline{z}}{\overline{z}-1}\right)}{(1 - z\theta)^{\alpha+1}(1 - \overline{z})^{\alpha+1}} \left(1 - |z|^2\right)^{\alpha-1} d\lambda_I(z)\end{aligned}$$

in view of the explicit expression of the kernel function A^{α}_{slice} in (17). Using the generating function for generalized Laguerre polynomials [2, p.288]

$$(1 - z)^{-\alpha-1} \exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L^{(\alpha)}_n(x)\, z^n,$$

provided $|z| < 1$, as well as Fejer's formula [18, Theorem 8.22.1, p. 198], it is not hard to see that the involved z-function series are uniformly convergent on any compact set contained in unit disk. Therefore, direct computation yields

$$\begin{aligned}
\widetilde{R_\theta^\alpha}(x,y) &= \frac{1}{\pi\Gamma(\alpha)}\int_{\mathbb{D}}\left(\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)\theta^n z^n\right)\left(\sum_{m=0}^{\infty}L_m^{(\alpha)}(y)\overline{z}^m\left(1-|z|^2\right)^{\alpha-1}\right)d\lambda(z)\\
&= \frac{1}{\pi\Gamma(\alpha)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\theta^n L_n^{(\alpha)}(x)L_m^{(\alpha)}(y)\int_{\mathbb{D}}z^n\overline{z}^m\left(1-|z|^2\right)^{\alpha-1}d\lambda(z)\\
&= \sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+1+\alpha)}\theta^n L_n^{(\alpha)}(x)L_n^{(\alpha)}(y) \qquad (25)
\end{aligned}$$

for $|\theta z| < 1$ which holds true when $|\theta| \leq 1$ and $|z| < 1$. This provides the expansion series of the restriction of $\widetilde{R_\theta^\alpha}(x,y)$ to any $\mathbb{B}_I$. For $|\theta| < 1$, we recognize the Hille–Hardy identity (4) for Laguerre polynomials. Thus, we have

$$\widetilde{R_\theta^\alpha}(x,y) = \frac{1}{(1-\theta)}\left(\frac{1}{xy\theta}\right)^{\alpha/2} I_\alpha\left(\frac{2\theta^{1/2}}{1-\theta}\sqrt{xy}\right)\exp\left(-\frac{\theta(x+y)}{1-\theta}\right) \qquad (26)$$

for $|\theta| < 1$ and $\theta \notin \mathbb{R}$. This leads to (7) by considering the kernel function $\widetilde{R_\theta^\alpha}(x,y)x^\alpha e^{-x}$. The right-hand side in (26) is clearly a slice regular function in $\theta \in \mathbb{B}$ for x, y being reals. The extension of (26) to the whole unit open ball $\mathbb{B}$ relies on the Identity Principle for left slice regular functions [9], since both sides of (26) are left slice regular and coincide at least on the upper half unit ball. To conclude, we need only to examine the validity of the closed expression in the right-hand side of (26) for the expansion of $\widetilde{R_\theta^\alpha}(x,y)$ which remains valid when $|\theta| = 1$ with $\theta \neq 1$. This can be handled by fixing θ and let $\varepsilon \in (0,1)$, so that (26) holds true for $|\varepsilon\theta| < 1$, and next sending ε to 1^-, at least formally. This can be rigorously justified making use of test functions and classical argument from the Schwartz theory of distributions. □

Remark 4.3. *By taking $\theta = -1$ with $\sqrt{\theta} = i$ in* (24)*, we recover the classical Fourier-Bessel transform* [16, 13]

$$\left(\mathcal{H}_\alpha\psi\right)(y) := \int_0^\infty uJ_\alpha(yu)\,\psi(u)du$$

for $\psi \in L^2(\mathbb{R}^+)$, where J_α is the Bessel function of first kind associated to I_α in (5) *by $I_\alpha(x) = i^{-\alpha}J_\alpha(ix)$. Indeed, by setting $\widetilde{\mathcal{L}^\alpha} = \widetilde{\mathcal{L}^\alpha_{-1}}$ and making the change of variable $u^2 = x$ and the function $\psi(u) = x^{\alpha/2}e^{-x/2}\varphi(x) = u^\alpha e^{-u^2/2}\varphi(u^2)$ we get*

$$\widetilde{\mathcal{L}^\alpha}\varphi(y^2) = \frac{e^{y^2/2}}{i^\alpha y^\alpha}\int_0^\infty u^{\alpha+1}I_\alpha(iyu)\,e^{-u^2/2}\varphi(u^2)du = \frac{e^{y^2/2}}{y^\alpha}\left(\mathcal{H}_\alpha\psi\right)(y).$$

Remark 4.4. *The considered family of QFrFT on the real half-line appears embedded in a strongly continuous one-parameter group of unitary operators the quaternionic context. Moreover, it is continuous and interpolates continuously the identity operator ($\theta = 1$) to the Hankel transform* [2, p. 216] *corresponding to $\theta = -1$.*

Remark 4.5. *The considered transform can be used to reintroduce the hyperholomorphic Bergman space $A^{2,\alpha}_{slice}(\mathbb{B})$ in* (15) *as well as some of their specific generalization in the context of slice regular functions on the unit quaternionic ball by considering the dual transform of $\theta \longmapsto \widetilde{\mathcal{L}^\alpha_\theta}\varphi(y)$, for fixed $y \in (0,+\infty)$. For the limit case of $y = 0$, the last* *transform is nothing than the Bargmann transform in* (14)*. The concrete study of the spectral properties of these dual transforms is studied in* [10]*.*

5. Conclusion

By exploring Bargmann's idea related to Segal-Bargmann transform, we have been able to present a general abstract formalism for constructing like fractional Fourier transform associated to given specific integral transform between complex or quaternionc Hilbert spaces. This formalism is next applied for the hyperholomorphic second Bargmann transform to re-derive the quaternionic fractional Hankel transform constructed à la Namias and to derive their basic properties. As mentioned by one of the referees, the approach used in this paper improves some results established (in this paper and its ArXiv version [8]) an in and can be applied to some Hamiltonians characterizing the Field Reggeons Theory. Moreover, the obtained results can be applied in several areas. Especially, to solve certain classes of ordinary and partial differential equations.

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