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$\{C_n, C_4\}$-DECOMPOSITION OF THE LINE GRAPH OF THE COMPLETE GRAPH

K. ARTHI[1], C. SANKARI[1], R. SANGEETHA[1*], §

ABSTRACT. For given positive integer $n \geq 4$, let C_n, K_n and $L(K_n)$ respectively denote a cycle with n edges, a complete graph on n vertices and the line graph of the complete graph K_n. For a given graph G, if $H_1, H_2, ..., H_l$ are the edge disjoint subgraphs such that $E(G) = E(H_1) \cup E(H_2) \cup ... \cup E(H_l)$, then we say that $H_1, H_2, ..., H_l$ *decompose* G. If G has a decomposition into copies of H_1 and H_2 using atleast one of each, then we say that G has a $\{H_1, H_2\}$-decomposition (or) G is $\{H_1, H_2\}$-decomposable. In this paper, it is proved that $L(K_n)$ is $\{C_n, C_4\}$-decomposable.

Keywords: Complete graph, Line graph, Hamilton Cycle, Perfect Matching, Decomposition of Graphs.

AMS Subject Classification: 05C38, 05C70, 05C76.

1. Introduction

In this paper, all the graphs are finite and simple. For given positive integers $n \geq 3$, a *cycle* with n edges is denoted by C_n and for given positive integers $n \geq 2$, a *path* with n edges is denoted by P_n, a *complete graph* on n vertices is denoted by K_n, a *complete bipartite graph* on m, n vertices is denoted by $K_{m,n}$ and a *star* with n edges is denoted by $S_n = K_{1,n}$. A *Hamilton cycle* of a graph G is a cycle that contains every vertex of G. A *perfect matching* of a graph G is a 1-regular spanning subgraph of G. Let G be a graph with vertex set V and $S \subset V$. The subgraph of G whose vertex set is S and whose edge set is the set of edges of G that have both ends in S is called the subgraph of G induced by S and is denoted by $\langle S \rangle$; we say that $\langle S \rangle$ is an *induced subgraph* of G. The *line graph* $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$ and $e_ie_j \in E(L(G))$ if and only if the edges e_i and e_j are incident with a common end vertex in G. The line graph of the complete graph K_n is denoted by $L(K_n)$. We note that $|V(L(K_n))|=\binom{n}{2}$. For a given graph G, if $H_1, H_2, ..., H_l$ are the edge disjoint subgraphs such that $E(G) = E(H_1) \cup E(H_2) \cup ... \cup E(H_l)$, then we say that $H_1, H_2, ..., H_l$ *decompose* G. If $H_i \cong H$ for $i = 1, 2, ..., l$, then we say that G admits a

[1] Department of Mathematics, A. V. V. M. Sri Pushpam College, (Affiliated to Bharathidasan University), Poondi, Thanjavur, Tamil Nadu, India.
e-mail: arthi1505@gmail.com; ORCID: https://orcid.org/0000-0002-0399-1343.
e-mail: sankari9791@gmail.com; ORCID: https://orcid.org/0000-0003-3325-7163.
e-mail: jaisangmaths@yahoo.com; ORCID: https://orcid.org/0000-0003-2726-0956.
[*] Corresponding author.
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H-*decomposition* and denote it by $H|G$. If G has a decomposition into copies of H_1 and H_2 using atleast one of each, then we say that G has a $\{H_1, H_2\}$-decomposition (or) G is $\{H_1, H_2\}$-decomposable. If G has a decomposition into α_1 copies of H_1,..., α_l copies of H_l, for all non-negative values of $\alpha_1, ..., \alpha_l$ satisfying trivial necessary conditions, then we say that G has a $\{{H_1}^{\alpha_1}, ..., {H_l}^{\alpha_l}\}$- decomposition or G is *fully* $\{H_1, ..., H_l\}$-decomposable. A *bowtie* is the union of two triangles with exactly one common vertex and we denote it by B.

In 1996, Cox and Rodger [3] raised the following question: For what values of m and n does there exists an m-cycle decomposition of $L(K_n)$? This problem has been answered, when $m \in \{3, 4, 5, 6, 2^l, \binom{n}{2}\}$ in [1–6,8]. In 2019, Ganesamurthy *et al.* [5] obtained the necessary and sufficient conditions for $\{{C_3}^{\alpha}, {P_4}^{\beta}, B^{\gamma}\}$ and $\{{C_3}^{\alpha}, {S_3}^{\beta}\}$-decompositions of $L(K_n)$. Recently, the authors [7] proved that if $n \geq 4$, then $L(K_n)$ is $\{C_n, S_4\}$-decomposable. In this paper, we prove that $L(K_n)$ is $\{C_n, C_4\}$-decomposable, for $n \geq 4$.

2. Preliminaries

Let $v_0, v_1, ..., v_{n-1}$ be the vertices of G. The notation $(v_0, v_1, ..., v_{n-1})$ denotes a cycle with n edges $\{v_0, v_1\}, \{v_1, v_2\}, ..., \{v_{n-1}, v_0\}$. Let X, Y be two disjoint subsets of $V(G)$. Then $E(X, Y)$ denotes the set of edges in G, whose one end vertex is in X and the other end vertex is in Y. The notation $\langle E(X, Y)\rangle$ denotes the graph induced by the edges of $E(X, Y)$. Let $V(K_n)$=$\{\infty\} \cup \{0, 1, 2, ..., n-2\}$.

Remark 2.1. *If n is odd, then the Walecki's construction of Hamilton cycles H_i of K_n is as follows: $H_i = (\infty, i, n-2+i, 1+i, n-3+i, ..., \lfloor\frac{n}{2}\rfloor + 1 + i, \lfloor\frac{n}{2}\rfloor - 1 + i, \lfloor\frac{n}{2}\rfloor + i)$, $0 \leq i \leq \lfloor\frac{n}{2}\rfloor - 1$, where addition is taken modulo $(n-1)$.*

Remark 2.2. *If n is even, then the Walecki's construction of Hamilton cycles H_i of K_n and a perfect matching F is as follows: $H_i = (\infty, i, n-2+i, 1+i, n-3+i, ..., \frac{n}{2} - 2 + i, \frac{n}{2} + i, \frac{n}{2} - 1 + i)$, $0 \leq i \leq \frac{n}{2} - 2$, where addition is taken modulo $(n-1)$ and a perfect matching F, where $E(F)$=$\{\{\infty, n-2\}, \{0, n-3\}, \{1, n-4\}, ..., \{\frac{n}{2} - 2, \frac{n}{2} - 1\}\}$.*

Let $\mathcal{P}_k(X)$ denote the set of all k-element subsets of an n-element set X. Then the vertex set of the line graph of K_n is $\mathcal{P}_2(\{\infty\} \cup \{0, 1, 2, ..., n-2\})$. Two vertices $\{u, v\}$ and $\{x, y\}$ are adjacent in $L(K_n)$ if any one of the following holds: (i) $u = x$, (ii) $u = y$, (iii) $v = x$, (iv) $v = y$.

3. Main Result

Theorem 3.1. *If $n \geq 4$, then $L(K_n)$ is $\{C_n, C_4\}$-decomposable.*

Proof. **Case 1:** Let $n \geq 5$ be odd. From Remark 2.1, K_n can be decomposed into Hamilton cycles H_i, $0 \leq i \leq \lfloor\frac{n}{2}\rfloor - 1$. *i.e.*, $E(K_n)$=$E(H_0) \cup E(H_1) \cup ... \cup E(H_{\lfloor\frac{n}{2}\rfloor - 1})$. Then we can write, $V(L(K_n)) = V(L(H_0)) \cup V(L(H_1)) \cup ... \cup V(L(H_{\lfloor\frac{n}{2}\rfloor-1}))$, *i.e.*, $V(L(K_n)) = \bigcup_{i=0}^{\lfloor\frac{n}{2}\rfloor-1} V(L(H_i))$, where $V(L(H_i))$ =$\{\{\infty, i\}, \{i, n-2+i\}, \{n-2+i, 1+i\}, \{1+i, n-3+i\}, ..., \{\lfloor\frac{n}{2}\rfloor+1+i, \lfloor\frac{n}{2}\rfloor-1+i\}, \{\lfloor\frac{n}{2}\rfloor-1+i, \lfloor\frac{n}{2}\rfloor+i\}, \{\lfloor\frac{n}{2}\rfloor+i, \infty\}\}$ and the addition is taken modulo $n-1$ with residues $0, 1, 2, ..., n-2$. Therefore, $|V(L(H_i))| = n$, $0 \leq i \leq \lfloor\frac{n}{2}\rfloor - 1$.

We denote $V(L(H_i)) = V_i$, $0 \le i \le \lfloor \frac{n}{2} \rfloor - 1$. We write, $L(K_n) = \bigcup_{i=1}^{2} G_i$, where

$$V(G_1) = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} V_i \quad ; \quad E(G_1) = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} E(L(H_i))$$

$$V(G_2) = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} V_i \quad ; \quad E(G_2) = \bigcup_{\substack{i=0 \\ i<j}}^{\lfloor \frac{n}{2} \rfloor - 1} E(V_i, V_j)$$

Obviously, each $L(H_i)$, $0 \le i \le \lfloor \frac{n}{2} \rfloor - 1$ is a C_n. Hence G_1 is C_n-decomposable.

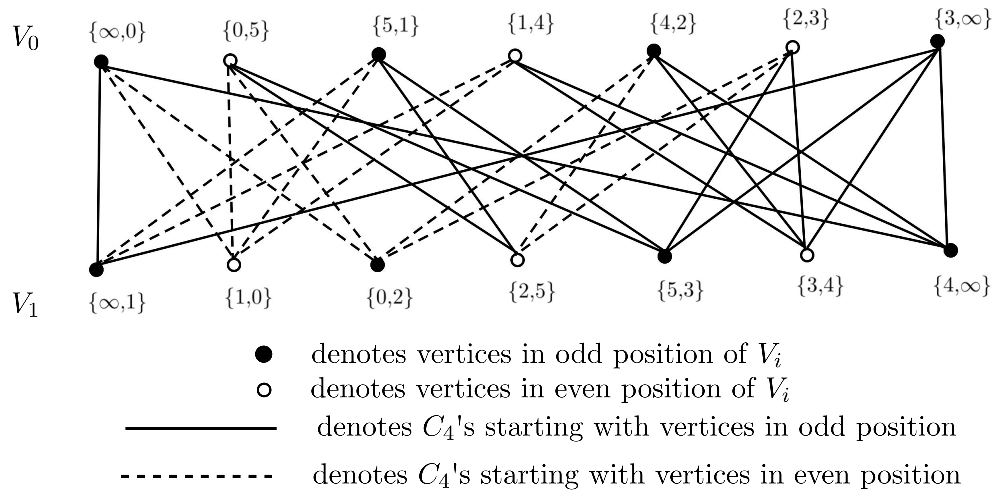

FIGURE 3.1. C_4-decomposition of $\langle E(V_0, V_1) \rangle$ in $L(K_7)$

Now, each induced subgraph $\langle E(V_i, V_j) \rangle$, $0 \le i < j \le \lfloor \frac{n}{2} \rfloor - 1$ is a 4-regular bipartite graph. We say the vertices $\{\infty, i\}, \{n-2+i, 1+i\}, ..., \{\lfloor \frac{n}{2} \rfloor + 1 + i, \lfloor \frac{n}{2} \rfloor - 1 + i\}$ and $\{\lfloor \frac{n}{2} \rfloor + i, \infty\}$ of V_i are in odd position and $\{i, n-2+i\}, \{1+i, n-3+i\}, ..., \{\lfloor \frac{n}{2} \rfloor - 2 + i, \lfloor \frac{n}{2} \rfloor + 1 + i\}$ and $\{\lfloor \frac{n}{2} \rfloor - 1 + i, \lfloor \frac{n}{2} \rfloor + i\}$ of V_i are in even position. Now, we will construct $\lfloor \frac{n}{2} \rfloor$ copies of C_4's in the even position vertices of V_0 and $\lfloor \frac{n}{2} \rfloor + 1$ copies of C_4's in the odd position vertices of V_0. So, totally we can construct n copies of C_4's in $\langle E(V_0, V_1) \rangle$ as follows: In $\langle E(V_0, V_1) \rangle$, we first construct C_4's starting with the vertices in even position of V_0. Now, consider the vertex $\{0, n-2\}$ of V_0. Since the first coordinate of this vertex is zero, we look for vertices of V_1 whose one of the coordinates is zero. There are exactly two such vertices namely, $\{1, 0\}$ and $\{0, 2\}$. Choose the vertex of V_0 which is adjacent to both $\{1, 0\}$ and $\{0, 2\}$, other than $\{0, n-2\}$. This is exactly $\{\infty, 0\}$. Now $(\{0, n-2\}, \{1, 0\}, \{\infty, 0\}, \{0, 2\})$ is a C_4. Similarly, we construct C_4 starting with the next vertex in the even position of V_0. In this way we construct C_4 starting with each vertex in the even position of V_0.

In the similar way we construct C_4's starting with each vertex in the odd position of V_0 as follows: Now consider the vertex $\{\infty, 0\}$ of V_0. Since the first coordinate of this vertex is ∞, we look for vertices of V_1 whose one of the coordinates is ∞. There are exactly two such vertices namely, $\{\infty, 1\}$ and $\{\lfloor \frac{n}{2} \rfloor + 1, \infty\}$. Choose the vertex of V_0 which is adjacent to both $\{\infty, 1\}$ and $\{\lfloor \frac{n}{2} \rfloor + 1, \infty\}$, other than $\{\infty, 0\}$. This is exactly $\{\lfloor \frac{n}{2} \rfloor, \infty\}$. Now $(\{\infty, 0\}, \{\infty, 1\}, \{\lfloor \frac{n}{2} \rfloor, \infty\}, \{\lfloor \frac{n}{2} \rfloor + 1, \infty\})$ is a C_4. Similarly, we construct C_4 starting with the next vertex in the odd position of V_0. In this way we construct C_4

starting with each vertex in the odd position of V_0. See Figure 3.1 for a C_4-decomposition of $\langle E(V_0, V_1)\rangle$ in $L(K_7)$. In the same way each induced subgraph $\langle E(V_i, V_j)\rangle$, $0 \leq i < j \leq \lfloor \frac{n}{2} \rfloor - 1$ is C_4-decomposable. Therefore, G_2 is C_4-decomposable. Hence $L(K_n)$ is $\{C_n, C_4\}$-decomposable.

Case 2: Let $n \geq 4$ be even. From Remark 2.2, K_n can be decomposed into Hamilton cycles H_i, $0 \leq i \leq \frac{n}{2} - 2$ and a perfect matching F, *i.e.*, $E(K_n)$=$E(H_0) \cup E(H_1) \cup ... \cup E(H_{\frac{n}{2}-2}) \cup E(F)$. Then we can write, $V(L(K_n)) = V(L(H_0)) \cup V(L(H_1)) \cup ... \cup V(L(H_{\frac{n}{2}-2})) \cup V(L(F))$, *i.e.*, $V(L(K_n)) = \bigcup\limits_{i=0}^{\frac{n}{2}-2} V(L(H_i)) \cup V(L(F))$, where $V(L(H_i))$ $=\{\{\infty, i\}, \{i, n-2+i\}, \{n-2+i, 1+i\}, \{1+i, n-3+i\}, ..., \{\frac{n}{2}-2+i, \frac{n}{2}+i\}, \{\frac{n}{2}+i, \frac{n}{2}-1+i\}, \{\frac{n}{2}-1+i, \infty\}\}$, the addition is taken modulo $n-1$ with residues $0, 1, 2, ..., n-2$ and $V(L(F))$=$\{\{\infty, n-2\}, \{0, n-3\}, \{1, n-4\}, ..., \{\frac{n}{2}-2, \frac{n}{2}-1\}\}$. Therefore, $|V(L(H_i))| = n$, $0 \leq i \leq \frac{n}{2} - 2$ and $|V(L(F))| = \frac{n}{2}$. We denote $V(L(H_i)) = V_i$, $0 \leq i \leq \frac{n}{2} - 2$ and $V(L(F))$ $= V_F$.

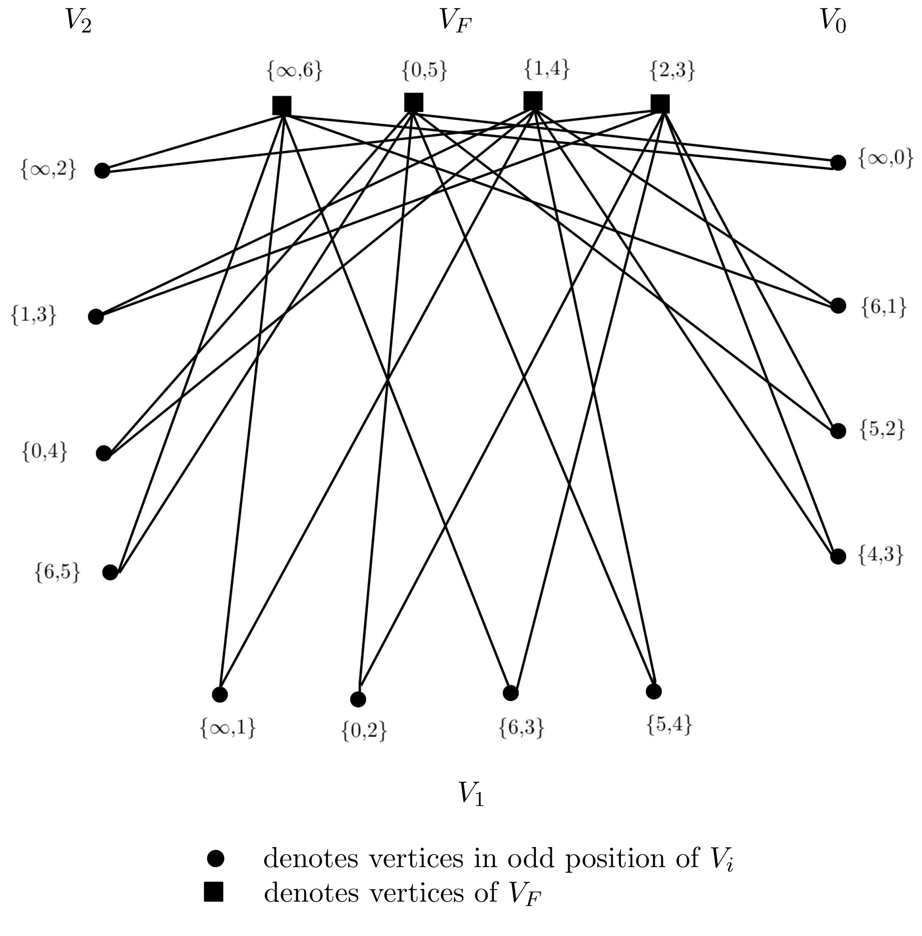

FIGURE 3.2. C_4-decomposition of G_3^o in $L(K_8)$

We write, $L(K_n) = \bigcup\limits_{i=1}^{3} G_i$, where

$$V(G_1) = \bigcup_{i=0}^{\frac{n}{2}-2} V_i \quad ; \quad E(G_1) = \bigcup_{i=0}^{\frac{n}{2}-2} E(L(H_i))$$
$$V(G_2) = \bigcup_{i=0}^{\frac{n}{2}-2} V_i \quad ; \quad E(G_2) = \bigcup_{\substack{i=0 \\ i<j}}^{\frac{n}{2}-2} E(V_i, V_j)$$
$$V(G_3) = \bigcup_{i=0}^{\frac{n}{2}-2} V_i \cup V_F \quad ; \quad E(G_3) = \bigcup_{i=0}^{\frac{n}{2}-2} E(V_i, V_F)$$

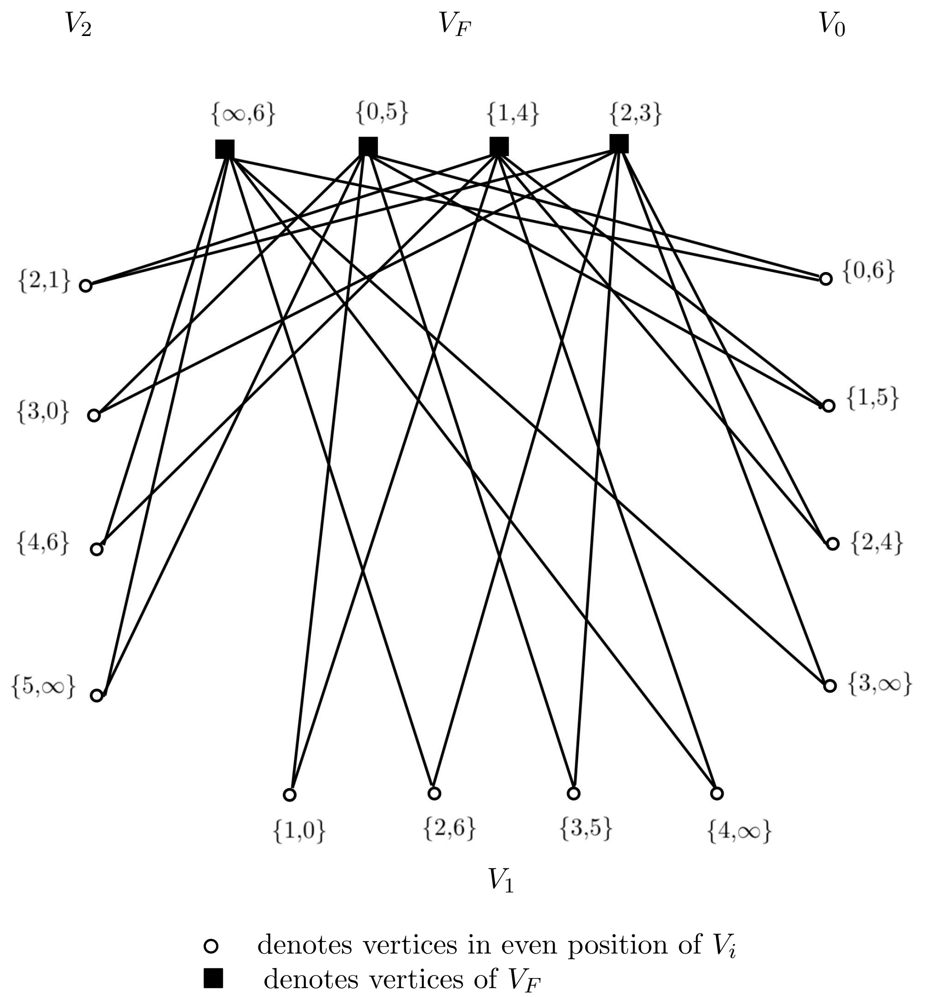

FIGURE 3.3. C_4-decomposition of G_3^e in $L(K_8)$

Obviously, each $L(H_i)$, $0 \leq i \leq \frac{n}{2} - 2$ is a C_n. Hence G_1 is C_n-decomposable and G_2 is C_4-decomposable as in Case 1.

Now we prove that G_3 is C_4-decomposable. The degree of a vertex of V_i in G_3 is 2 and the degree of a vertex of V_F in G_3 is $2(n-2)$. Let V_i^o denote the set of all vertices of V_i which are in odd positions and V_i^e denote the set of all vertices of V_i which are in even positions. Then $V_i = V_i^o \cup V_i^e$, $0 \leq i \leq \frac{n}{2} - 2$. We define two new graphs G_3^o and G_3^e as follows:

$$V(G_3^o) = \bigcup_{i=0}^{\frac{n}{2}-2} (V_i^o \cup V_F) \quad ; \quad E(G_3^o) = \bigcup_{i=0}^{\frac{n}{2}-2} E(V_i^o, V_F)$$
$$V(G_3^e) = \bigcup_{i=0}^{\frac{n}{2}-2} (V_i^e \cup V_F) \quad ; \quad E(G_3^e) = \bigcup_{i=0}^{\frac{n}{2}-2} E(V_i^e, V_F)$$

Obviously $G_3 = G_3^o \cup G_3^e$. Now we prove that G_3^o is C_4-decomposable. Consider pairs of vertices in V_F. Since the total number of vertices in V_F is $\frac{n}{2}$, we have $\binom{\frac{n}{2}}{2}$ pairs of V_F. For each pair of vertices in V_F we have 4 adjacent vertices in G_3 of which two vertices are in G_3^o and two vertices in G_3^e. For example, consider the pair of vertices $\{\infty, n-2\}$ and $\{0, n-3\}$ in V_F. The vertices in G_3^o adjacent to this pair are $\{\infty, 0\}$, $\{n-2, n-3\}$. These 4 vertices together forms a C_4. Similarly, the vertices in G_3^e adjacent to the given pair are $\{0, n-2\}$ and $\{n-3, \infty\}$. Obviously these four vertices forms a C_4. Thus we have 2 C_4's corresponding to each pair of vertices in V_F. Thus, we have 2 $\binom{\frac{n}{2}}{2}$ number of C_4's in G_3 and hence G_3 is C_4-decomposable. For example see Figure 3.2 and 3.3 for a C_4-decomposition of $L(K_8)$. Hence $L(K_n)$ is $\{C_n, C_4\}$-decomposable.

□

4. Conclusions

In this paper, we have proved that $L(K_n)$, the line graph of the complete graph K_n is $\{C_n, C_4\}$-decomposable, for $n \geq 4$.

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K. Arthi is currently doing her doctoral research in the Department of Mathematics, A. V. V. M. Sri Pushpam College (Affiliated to Bharathidasan University), Poondi, Thanjavur, (D.T), Tamil Nadu, India. She received her M.Sc. degree from the same College. Her research interest is graph decomposition problems.

C. Sankari is currently doing her doctoral research in the Department of Mathematics, A. V. V. M. Sri Pushpam College (Affiliated to Bharathidasan University), Poondi, Thanjavur, (D.T), Tamil Nadu, India. She received her M.Sc degree from S.B.K. College (Affiliated to Madurai Kamaraj University), Aruppukottai. Her research interest is graph decomposition problems.

R. Sangeetha for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.12, N.1.