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ANALYSIS OF A DYNAMIC CONTACT PROBLEM FOR ELECTRO-VISCOELASTIC MATERIALS WITH TRESCA'S FRICTION

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Abstract. We consider a mathematical model which describes the dynamic process of contact between two electro-viscoelastic bodies with damage. The contact is bilateral and is modeled with Tresca's friction law. The damage of the materials caused by elastic deformations. We derive a variational formulation for the model which is in the form of a system involving the displacement field, the electric potential and the damage. Then we provide the existence of a unique weak solution to the model. We also study the finite element approximations of the problem and derive error estimates. Finally, we present numerical simulation results in the study of a two-dimensional example.

Keywords: Dynamic process, piezoelectric, monotone operator, fixed point, weak solution, damage.

AMS Subject Classification: 74M15, 74M10, 74F05.

1. Introduction

In this paper, we study a bilateral contact problem involves viscous friction of Tresca type described in [1]. A nonlinear electro-viscoelastic constitutive law is used to model the piezoelectric material. The piezoelectricity can be described as follows: when mechanical pressure is applied to a certain classes of crystalline materials (e.g ceramics $BaTiO_3$, $BiFeO_3$), the crystalline structure produces a voltage proportional to the pressure. Conversely, when an electric field is applied, the structure changes his shape producing dimensional modifications in the material. Different models have been developed to describe the interaction between the electrical and mechanical fields see for example [5, 15] and the

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references therein. Contact problems involving elasto-piezoelectric materials [6, 20]. Different models of viscoelastic piezoelectric problems in [3, 7, 16, 18, 19] have been studied, contact problems for electro-elasto-viscoplastic materials were studied in [2, 14].

The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [12] from the virtual power principle. The models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [10]. The new idea of [11, 12] was the introduction of the damage function $\alpha^\ell = \alpha^\ell(x,t)$, which is the ratio between the elastic moduli of the damage and damage-free materials. In an isotropic and homogeneous elastic material, let E_Y^ℓ be the Young modulus of the original material and E_{eff}^ℓ be the current modulus, then the damage function is defined by $\alpha^\ell = E_{eff}^\ell / E_Y^\ell$. Clearly, it follows from this definition that the damage function α^ℓ is restricted to have values between zero and one. When $\alpha^\ell = 1$, there is no damage in the material, when $\alpha^\ell = 0$, the material is completely damaged, when $0 < \alpha^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [21]. The differential inclusion used for the evolution of the damage field is

$$\dot{\alpha}^\ell - \Delta\alpha^\ell + k^\ell \partial\chi_{K^\ell}(\alpha^\ell) \ni S^\ell(\varepsilon(u^\ell), \alpha^\ell) \quad \text{in } \Omega^\ell \times (0,T), \tag{1}$$

where k^ℓ is a positive coefficient and K^ℓ the set of admissible damage defined by

$$K^\ell = \{\xi \in H^1(\Omega^\ell);\ 0 \le \xi \le 1, \text{ a.e. in } \Omega^\ell\}. \tag{2}$$

The paper is structured as follows. In Section 2, we present the physical setting and describe the mechanical problem. We derive a variational formulation, list the assumptions on the data, and give the variational formulation of the problem. In Section 3, we state our main existence and uniqueness result which is based on classical result of nonlinear first order evolution inequalities and equations with monotone operators and the fixed point arguments. In Section 4, we introduce a fully discrete scheme to solve the problem numerically. Under certain solution regularity assumptions, we derive an optimal order error estimate. Finally, we present the numerical solution together with simulation result son a two-dimensional test problem.

2. Problem statement and variational formulation

Let us consider two electro-viscoelastic bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2,3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be regular enough, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and on two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $meas\Gamma_1^\ell > 0$, $meas\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The Ω^ℓ body is submitted to f_0^ℓ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on Γ_1^ℓ. The surface tractions f_2^ℓ act on Γ_2^ℓ. We also assume that the electrical potential vanishes on Γ_a^ℓ and a surface electric charge of density q_2^ℓ is prescribed on Γ_b^ℓ. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The classical form of bilateral contact with Tresca's friction and damage between two electro-viscoelastic bodies is given by:

Problem P. For $\ell = 1, 2,$ find a displacement field $u^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}^d$, a stress field

$\sigma^\ell : \Omega^\ell \times (0,T) \to \mathbb{S}^d$, an electric potential $\varphi^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}$, an electric displacement field $D^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}^d$, and a damage $\alpha^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}$ such that

$$\sigma^\ell = \mathcal{A}^\ell \varepsilon(\dot{u}^\ell) + \mathcal{B}^\ell(\varepsilon(u^\ell), \alpha^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell \text{ in } \Omega^\ell \times (0,T), \tag{3}$$

$$D^\ell = \mathcal{E}^\ell \varepsilon(u^\ell) + \mathcal{C}^\ell E(\varphi^\ell) \quad \text{in } \Omega^\ell \times (0,T), \tag{4}$$

$$\dot{\alpha}^\ell - \Delta \alpha^\ell + k^\ell \partial \chi_{K^\ell}(\alpha^\ell) \ni S^\ell(\varepsilon(u^\ell), \alpha^\ell) \quad \text{in } \Omega^\ell \times (0,T), \tag{5}$$

$$\operatorname{Div} \sigma^\ell + f_0^\ell = \rho^\ell \ddot{u}^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{6}$$

$$\operatorname{div} D^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{7}$$

$$u^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0,T), \tag{8}$$

$$\sigma^\ell \nu^\ell = f_2^\ell \quad \text{on } \Gamma_2^\ell \times (0,T), \tag{9}$$

$$\left\{\begin{array}{l} [u_\nu] = 0\ ,\ \sigma_\tau^1 = -\sigma_\tau^2 \equiv \sigma_\tau, \quad \|\sigma_\tau\| \le g \\ \|\sigma_\tau\| < g \Rightarrow [\dot{u}_\tau] = 0 \\ \|\sigma_\tau\| = g \Rightarrow \exists \delta \ge 0 \text{ such that } \sigma_\tau = -\delta\,[\dot{u}_\tau] \end{array}\right. \quad \text{on } \Gamma_3 \times (0,T), \tag{10}$$

$$\frac{\partial \alpha^\ell}{\partial \nu^\ell} = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{11}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{12}$$

$$D^\ell \cdot \nu^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{13}$$

$$u^\ell(0) = u_0^\ell,\ \dot{u}^\ell(0) = v_0^\ell,\ \alpha^\ell(0) = \alpha_0^\ell \quad \text{in } \Omega^\ell. \tag{14}$$

Here, Eqs (3) and (4) represent the electro-viscoelastic constitutive law. The evolution of the damage field is governed by the inclusion given by the relation (5). Next, Eqs (6) and (7) are the equations of motion written for the stress field and of balance written for the electric displacement field, respectively, in which Div and div denote the divergence operators for tensor and vector valued functions. Conditions (8) and (9) are the displacement and traction boundary conditions, respectively. The relation (11) represents a homogeneous Neumann boundary condition, (12) and (13) represent the electric boundary conditions, and (14) are the initial conditions. Conditions (10) represent the bilateral contact condition with Tresca's friction law where $[u_\nu] = u_\nu^1 + u_\nu^2$ is the stands for the displacements in normal direction, and where the friction yield limit is g which is assumed to depend only on each point of Γ_3, where $[u_\tau] = u_\tau^1 - u_\tau^2$ stands for the jump of the displacements in tangential direction.

We now proceed to obtain a variational formulation of Problem **P**. For this purpose, we introduce additional notation and assumptions on the problem data. Here and in what follows the indices i and j run between 1 and d, the summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. Let $E_0 = L^2(\Omega^1) \times L^2(\Omega^2)$, $H^\ell = [L^2(\Omega^\ell)]^d$, $H_1^\ell = [H^1(\Omega^\ell)]^d$, $\mathcal{H}^\ell = [L^2(\Omega^\ell)]_s^{d\times d}$, $E_1 = H^1(\Omega^1) \times H^1(\Omega^2)$, $H = H^1 \times H^2$, $\mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2$ and define the following spaces: $V^\ell = \{v^\ell \in [H^1(\Omega^\ell)]^d;\ v^\ell\big|_{\Gamma_1^\ell} = 0\}$, $W^\ell = \{\psi^\ell \in H^1(\Omega^\ell);\ \psi^\ell\big|_{\Gamma_a^\ell} = 0\}$, $\mathcal{W}^\ell = \{\mathbf{D}^\ell \in H^\ell;\ \operatorname{div} \mathbf{D}^\ell \in L^2(\Omega^\ell)\}$, $W = W^1 \times W^2$, $\mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2$, $\mathbf{V} = \{v \in V^1 \times V^2;\ [v_\nu]\big|_{\Gamma_3} = 0\}$.

Since $mes\Gamma_1^\ell > 0$, Korn's inequality holds and there exists a constant $C_K > 0$ depending only on Ω^ℓ and Γ_1^ℓ, such that

$$\|\varepsilon(v^\ell)\|_{\mathcal{H}^\ell} \ge C_K \|v^\ell\|_{H_1^\ell}, \quad \forall v^\ell \in V^\ell, \tag{15}$$

On the space V^ℓ, we consider the inner product and the associated norm given by

$$(u^\ell, v^\ell)_{V^\ell} = (\varepsilon(u^\ell), \varepsilon(v^\ell))_{\mathcal{H}^\ell}, \quad \forall u^\ell, v^\ell \in V^\ell, \tag{16}$$

and let $\left\|v^\ell\right\|_{V^\ell}$ the associated norm given by

$$\left\|v^\ell\right\|_{V^\ell} = \|\varepsilon(v^\ell)\|_{\mathcal{H}^\ell}, \quad \forall v^\ell \in V^\ell. \tag{17}$$

Notice also that, since $mes(\Gamma_a^\ell) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\zeta^\ell\|_{L^2(\Omega^\ell)^d} \geq C_F \, \|\zeta^\ell\|_{H^1(\Omega^\ell)}, \quad \forall\, \zeta^\ell \in W^\ell, \tag{18}$$

where $C_F > 0$ is a constant which depends only on Ω^ℓ and Γ_a^ℓ and $\nabla\xi^\ell = (\xi^\ell_{,i})$. A proof of Friedrichs-Poincaré inequality may be found in ([17, p.188]).
Further, we denote by X' the dual space of X, and we use the notation $\langle .,. \rangle_{X'\times X}$ to represent the duality pairing between X' and X.
For the convenience of the reader, we recall the following standard result for parabolic variational inequalities and the abstract result (see, e.g., [21, p.47-48]).

Theorem 2.1. *Let $X \subset Y = Y' \subset X'$ be a Gelfand triple. Let K be a nonempty, closed and convex set of X. Assume that $A : X \to X'$ is a continuous and symmetric linear operator which satisfies, there exists $C_2 \in \mathbb{R}$ and $C_3 > 0$ such that*

$$\langle Av, v\rangle_{X'\times X} + C_2\|v\|_Y^2 \geq C_3\|v\|_X^2, \quad \forall v \in X. \tag{19}$$

Then, for all $u_0 \in K$ et $f \in L^2(0,T;Y)$, there exists a unique function u which satisfies

$$u \in W^{1,2}(0,T;X) \cap L^2(0,T;Y) \tag{20}$$

$$u(t) \in K, \quad \forall t \in [0,T], \tag{21}$$

$$\begin{aligned} &\langle \dot{u}(t), v - u(t)\rangle_{V'\times V} + \langle Au(t), v - u(t)\rangle_{V'\times V} \\ &\geq \langle f(t), v - u(t)\rangle_{V'\times V}, \ \forall v \in K, \ a.e.t \in (0,T), \end{aligned} \tag{22}$$

$$u(0) = u_0. \tag{23}$$

Theorem 2.2. *Let $X \subset Y = Y' \subset X'$ be a Gelfand triple. Assume that $A : X \to X'$ is a hemicontinuous and monotone operator which satisfies*

$$\langle Av, v\rangle_{X'\times X} \geq \omega\|v\|_X^2 + \lambda, \quad \forall v \in X, \tag{24}$$

$$\|Av\|_{X'} \leq C_1(\|v\|_X + 1), \quad \forall v \in X, \tag{25}$$

for some constants $\omega > 0$, $C_1 > 0$, and $\lambda \in \mathbb{R}$. Then, for every $u_0 \in Y$ and $f \in L^2(0,T;X')$, there exists a unique function u which satisfies

$$\begin{aligned} &u \in L^2(0,T;X) \cap C([0,T];Y), \quad \dot{u} \in L^2(0,T;X'), \\ &\dot{u}(t) + Au(t) = f(t) \ a.e.t \in (0,T), \\ &\quad u(0) = u_0. \end{aligned}$$

We now list assumptions on the data. Assume the operators $\mathcal{A}^\ell, \mathcal{B}^\ell, \mathcal{E}^\ell, \mathcal{C}^\ell$ and S^ℓ satisfy the following conditions ($L_{\mathcal{A}^\ell}, m_{\mathcal{A}^\ell}, L_{\mathcal{B}^\ell}, m_{\mathcal{C}^\ell}$ and L_{S^ℓ} being positive constants).

$$\left.\begin{array}{l}\text{(a) } \mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) } \|\mathcal{A}^\ell(x,\varepsilon_1) - \mathcal{A}^\ell(x,\varepsilon_2)\| \le L_{\mathcal{A}^\ell}\|\varepsilon_1-\varepsilon_2\| \ \ \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega^\ell.\\ \text{(c) } (\mathcal{A}^\ell(x,\varepsilon_1) - \mathcal{A}^\ell(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2) \ge m_{\mathcal{A}^\ell}\|\varepsilon_1-\varepsilon_2\|^2\\ \qquad \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d \text{ a.e } x \in \Omega^\ell.\\ \text{(d) For any } \xi \in \mathbb{S}^d, x \mapsto \mathcal{A}^\ell(x,\xi) \text{ is measurable on } \Omega^\ell,\\ \text{(e) The mapping } x \mapsto \mathcal{A}^\ell(x,0) \in \mathcal{H}^\ell.\end{array}\right\} \tag{26}$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{B}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d.\\ \text{(b) } \|\mathcal{B}^\ell(x,\varepsilon_1,\alpha_1) - \mathcal{B}^\ell(x,\varepsilon_2,\alpha_2)\| \le L_{\mathcal{B}^\ell}(\|\varepsilon_1-\varepsilon_2\| + |\alpha_1-\alpha_2|)\\ \qquad \forall \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d, \forall \alpha_1,\ \alpha_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega^\ell.\\ \text{(c) The mapping } x \mapsto \mathcal{B}^\ell(x,\varepsilon,\alpha) \text{ is measurable in } \Omega^\ell \ \ \forall \varepsilon \in \mathbb{S}^d, \forall \alpha \in \mathbb{R}.\\ \text{(d) The mapping } x \mapsto \mathcal{B}^\ell(x,0,0) \in \mathcal{H}^\ell.\end{array}\right\} \tag{27}$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d.\\ \text{(b) } \mathcal{E}^\ell(x,\tau) = (e^\ell_{ijk}(x)\tau_{jk}) \quad \text{where } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell).\end{array}\right\} \tag{28}$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{C}^\ell : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d.\\ \text{(b) } \mathcal{C}^\ell(x,E) = (c^\ell_{ij}(x)E_j) \quad \forall E = (E_i) \in \mathbb{R}^d, \quad \text{a.e. } x \in \Omega^\ell.\\ \text{(c) } c^\ell_{ij} = c^\ell_{ji},\ c^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i,j \le d.\\ \text{(d) } \mathcal{C}^\ell E.E \ge m_{\mathcal{C}^\ell}|E|^2, \ \ \forall E = (E_i) \in \mathbb{R}^d, \text{ a.e. } x \in \Omega^\ell.\end{array}\right\} \tag{29}$$

$$\left.\begin{array}{l}\text{(a) } S^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}.\\ \text{(b) } |S^\ell(x,\xi_1,d_1) - S^\ell(x,\xi_2,d_2)| \le L_{S^\ell}\big(|\xi_1-\xi_2| + |d_1-d_2|\big),\\ \qquad \forall \xi_1,\xi_2 \in \mathbb{S}^d, \forall d_1,d_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega^\ell.\\ \text{(c) For any } \xi \in \mathbb{S}^d, d \in \mathbb{R}, \quad x \mapsto S^\ell(x,\xi,d) \text{ is measurable on } \Omega^\ell.\\ \text{(d) The mapping } x \mapsto S^\ell(x,0,0) \text{ belongs to } L^2(\Omega^\ell).\end{array}\right\} \tag{30}$$

The mass density and the friction yield limit g satisfies

$$\rho^\ell \in L^\infty(\Omega^\ell), \min_{\ell=1,2} \inf_{x\in\Omega^\ell} \rho^\ell(x) = \rho^* > 0, \tag{31}$$

$$g \in L^\infty(\Gamma_3), \quad g \ge 0 \ \text{ on } \Gamma_3. \tag{32}$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0^\ell \in L^2(0,T;H^\ell), \quad f_2^\ell \in L^2(0,T;L^2(\Gamma_2^\ell)^d), \tag{33}$$

$$q_0^\ell \in C(0,T;L^2(\Omega^\ell)), \quad q_2^\ell \in C(0,T;L^2(\Gamma_b^\ell)), \tag{34}$$

$$q_2^\ell(t) = 0 \quad \text{on } \Gamma_3 \ \ \forall t \in [0,T]. \tag{35}$$

Finally, we assume that initial data satisfy the regularity

$$u_0^\ell \in V^\ell, \ \ v_0^\ell \in V^\ell, \ \ \alpha_0^\ell \in K^\ell. \tag{36}$$

We define the mappings $F : [0,T] \to \mathbf{V}'$, $q : [0,T] \to W$, $a : E_1 \times E_1 \to \mathbb{R}$ and $j : \mathbf{V} \to \mathbb{R}$ respectively, by

$$\left.\begin{array}{l}\langle F(t), v\rangle_{V'\times V} = \sum\limits_{\ell=1}^{2} \int\limits_{\Omega^\ell} f_0^\ell(t) v^\ell dx + \sum\limits_{\ell=1}^{2} \int\limits_{\Gamma_2^\ell} f_2^\ell(t) v^\ell da, \quad \forall v \in V,\\ (q(t),\phi)_W = \sum\limits_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t)\phi^\ell dx - \sum\limits_{\ell=1}^{2} \int_{\Gamma_b^\ell} q_2^\ell(t)\phi^\ell da, \ \ \forall \phi \in W,\\ a(\zeta,\xi) = \sum\limits_{\ell=1}^{2} k^\ell \int_{\Omega^\ell} \nabla\zeta^\ell.\nabla\xi^\ell dx, \qquad \text{and} \qquad j(v) = \int_{\Gamma_3} g\|\,[v_\tau]\,\|_{L^2(\Gamma_3)^d}\, da.\end{array}\right\} \tag{37}$$

From the assumptions (33) and (34) it follows that

$$F \in L^2(0,T;V'), \quad q \in C(0,T;W). \tag{38}$$

We use a modified inner product on H given by

$$((u,v))_H = \sum_{\ell=1}^{2} (\rho^\ell u^\ell, v^\ell)_{H^\ell}, \quad \forall u,v \in H, \tag{39}$$

that is, it is weighted with ρ^ℓ. We let $|||.|||_H$ be the associated norm, i.e.,

$$|||v|||_H = ((v,v))_H^{\frac{1}{2}}, \quad \forall v \in H. \tag{40}$$

It follows from assumption (31), that $|||.|||_H$ and $\|.\|_H$ are equivalent norms on H, and also the inclusion mapping of V into H is continuous and dense. Identifying H with its own dual, we can write the Gelfand triple $V \subset H \subset V'$, so we have

$$\langle u,v\rangle_{V'\times V} = ((u,v))_H, \quad \forall u \in H, \ \forall v \in V. \tag{41}$$

By a standard procedure based on integration by parts and Green's formula, we obtain the following weak formulation of the piezoelectric contact problem P.

Problem PV. Find $u : [0,T] \to V$, $\varphi : [0,T] \to W$ and $\alpha : [0,T] \to E_1$ such that

$$\left.\begin{array}{l} \langle \ddot{u}(t), w - \dot{u}(t)\rangle_{V'\times V} + \sum\limits_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\dot{u}^\ell) + \mathcal{B}^\ell(\varepsilon(u^\ell), \alpha^\ell), \varepsilon(w^\ell - \dot{u}^\ell(t)))_{\mathcal{H}^\ell} + \\ \sum\limits_{\ell=1}^{2} \left(\left(\mathcal{E}^\ell\right)^* \nabla\varphi^\ell, \varepsilon(w^\ell - \dot{u}^\ell(t))\right)_{\mathcal{H}^\ell} + j(w) - j(\dot{u}(t)) \geq \langle F(t), w - \dot{u}(t)\rangle_{V'\times V} \\ \forall w \in V, \quad \text{a.e. } t \in (0,T), \end{array}\right\} \tag{42}$$

$$\sum_{\ell=1}^{2} (\mathcal{C}^\ell \nabla\varphi^\ell(t) - \mathcal{E}^\ell \varepsilon(u^\ell(t)), \nabla\phi^\ell)_{H^\ell} = (q(t), \phi)_W, \quad \forall \phi \in W, \ \text{a.e. } t \in (0,T), \tag{43}$$

$$\left.\begin{array}{l} \alpha(t) \in K = K^1 \times K^2, \ \ \sum\limits_{\ell=1}^{2} (\dot{\alpha}^\ell(t), \xi^\ell - \alpha^\ell(t))_{L^2(\Omega^\ell)} + a(\alpha(t), \xi - \alpha(t)) \\ \geq \sum\limits_{\ell=1}^{2} (S^\ell(\varepsilon(u^\ell(t)), \alpha^\ell(t)), \xi^\ell - \alpha^\ell(t))_{L^2(\Omega^\ell)}, \quad \forall \xi \in K, \quad \text{a.e. } t \in (0,T), \end{array}\right\} \tag{44}$$

$$u(0) = u_0, \ \dot{u}(0) = w_0, \ \alpha(0) = \alpha_0. \tag{45}$$

The existence of a unique solution to Problem **PV** will be presented in the next section.

3. Main existence and uniqueness result

Now, we state and prove our existence and uniqueness result.

Theorem 3.1. *Under the assumptions (26)–(36). Then there exists a unique solution* $\{u, \varphi, \alpha\}$ *to problem* **PV**. *Moreover, the solution satisfies*

$$u \in W^{1,2}(0,T;V) \cap C^1(0,T;H) \cap W^{2,2}(0,T;V'), \tag{46}$$

$$\varphi \in C(0,T;W), \tag{47}$$

$$\alpha \in W^{1,2}(0,T;E_0) \cap L^2(0,T;E_1). \tag{48}$$

The proof of Theorem 3.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 3.1 hold, and let a $\eta \in L^2(0,T;V')$ be given. In the first step, we consider the following variational problem.

Problem P_{u_η}. Find a displacement field $u_\eta : [0,T] \to V$ such that

$$\left.\begin{array}{l}\langle \ddot{u}_\eta(t), w - \dot{u}_\eta(t)\rangle_{V'\times V} + \sum\limits_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\dot{u}_\eta^\ell(t)), \varepsilon(w^\ell - \dot{u}_\eta^\ell(t)))_{\mathcal{H}^\ell} + \\ j(w) - j(\dot{u}_\eta(t)) \geq \langle F(t) - \eta(t), w - \dot{u}_\eta(t)\rangle_{V'\times V},\ \forall w \in V,\ a.e.\ t \in (0,T). \\ u_\eta(0) = u_0, \quad \dot{u}_\eta(0) = v_0.\end{array}\right\} \tag{49}$$

We define the function $A : V \to V'$ by

$$\langle Au, v\rangle_{V'\times V} = \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(u^\ell), \varepsilon(v^\ell))_{\mathcal{H}^\ell}, \quad \forall u, v \in V, \tag{50}$$

Using variable velocity $v_\eta = \dot{u}_\eta$, the Problem P_{u_η} is written for a.e. $t \in (0,T)$, as follows

Problem P_{v_η}. Find the velocity field $v_\eta : [0,T] \to V$ such that

$$\left.\begin{array}{l}\langle \dot{v}_\eta(t), w - v_\eta(t)\rangle_{V'\times V} + \langle Av_\eta(t), w - v_\eta(t)\rangle_{V'\times V} + j(w) - j(v_\eta(t)) \\ \geq \langle F_\eta(t), w - v_\eta(t)\rangle_{V'\times V},\ \forall w \in V. \\ v_\eta(0) = v_0.\end{array}\right\} \tag{51}$$

Where $F_\eta = F - \eta$.

Lemma 3.1. *Assume that (26) and (32) hold, then the operator A and the functional j defined respectively by (50) and (37) satisfy*

$$\left\{\begin{array}{l}(a)\ A : V \to V' \textit{ is hemicontinuous and strongly monotone}, \\ (b)\ \exists\ C_1 \geq 0,\ \exists\ C_2 \geq 0, \forall v \in V\ \|Av\|_{V'} \leq C_1\|v\|_V + C_2, \\ (c) \textit{ for all sequence } (u_n) \textit{ and } u \textit{ in } L^2(0,T;V) \textit{ such that} \\ \ u_n \rightharpoonup u \textit{ weakly in } L^2(0,T;V) \textit{ then } Au_n \rightharpoonup Au \textit{ star weakly in } L^2(0,T;V') \\ \quad \textit{and} \quad \liminf\limits_{n\to+\infty} \int_0^T \langle Au_n(t), u_n(t)\rangle_{V'\times V} dt \geq \int_0^T \langle Au(t), u(t)\rangle_{V'\times V} dt\end{array}\right.$$

$$\left\{\begin{array}{l}(a')\ j : V \to \mathbb{R} \textit{ is convex and lower semi-continuous functional}, \\ \ \textit{There exists a sequence of } C^1 \textit{ convex functions } (j_n) : V \to \mathbb{R} \textit{ such that} \\ (b')\ \exists\ d_1 \geq 0,\ \exists\ d_2 \geq 0,\ \forall n \in \mathbb{N}, \quad \|j_n'(v)\|_{V'} \leq d_1\|v\|_V + d_2, \\ (c')\ \forall v \in L^2(0,T;V), \quad \lim\limits_{n\to+\infty} \int_0^T j_n(v(t))dt = \int_0^T j(v(t))dt, \\ (d')\ \textit{There exists a sequence } (v_n) \textit{ and } v \textit{ in } L^2(0,T;V) \textit{ such that} \\ \ v_n \rightharpoonup v \textit{ weakly in } L^2(0,T;V) \textit{ then } \liminf\limits_{n\to+\infty}\ \inf \int_0^T j_n(v_n(t))dt \geq \int_0^T j(v(t))dt.\end{array}\right.$$

$j_n'(v)$ denotes the Fréchet derivative of j_n at v.

Proof. From the definition (50)and assumption (26), we can verify that A satisfies the conditions (a)-(b), and applying Lebesgue theorem, we deduce the condition (c).
On the other hand, by using the continuous embedding from V into $L^2(\Gamma_3)^d$, we find that j is continuous and convex. To approximate the function j, we use the following functional $(j_n) : V \to \mathbb{R}$ defined by

$$j_n(v) = \int_{\Gamma_3} g\sqrt{\|v_\tau\|^2_{L^2(\Gamma_3)^d} + exp(-n)}\ da, \quad \forall v \in V_1, \quad \forall n \in \mathbb{N}.$$

We verify that the Frechet derivative of j_n at v is given by

$$\langle j_n'(v), h\rangle_{V'\times V} = \int_{\Gamma_3} g\frac{(v_\tau, h_\tau)}{\sqrt{\|v_\tau\|^2_{L^2(\Gamma_3)^d} + e^{-n}}} da, \quad \forall h \in V. \tag{52}$$

We note that j_n is continuously differentiable. One may show that for all $a \geq 0$, $b \geq 0$, such $a+b=1$, for all reals $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\sqrt{(ax+by)^2+e^{-n}} \leq a\sqrt{x^2+e^{-n}}+b\sqrt{y^2+e^{-n}}. \tag{53}$$

Then j_n is convex for all $n \in \mathbb{N}$. Using (52) it follows that

$$\exists\, C \geq 0,\ \forall v \in V,\ \|j_n'(v)\|_{V'} \leq C\|g\|_{L^\infty(\Gamma_3)},$$

therefore (b') is satisfied. From the definition of j_n we have $\lim\limits_{n\to+\infty} j_n(v)=j(v)$ and as j_n is continuous on V, applying the Lebesgue theorem, we deduce the property (c'). Finally, (d') is a consequence of the fact that

$$\forall v \in V,\ \forall n \in \mathbb{N},\quad j_n(v) \geq j(v).$$

We conclude that the operator A, the functional j and j_n satisfy the conditions of the lemma 52. $\square$

Lemma 3.2. *Under assumptions (26) and (32), for all $\eta \in L^2(0,T;V')$, the Problem P_{v_η} has a unique solution with the regularity*

$$v_\eta \in C(0,T;H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V').$$

Proof. Using the conditions of the lemma 3.1, it is deduced that the operator $A+j_n'$ is hemicontinu and strongly monotone. Thus, (see, e.g., [21, p.48]), for any $n \in \mathbb{N}$ exists a unique function $v_\eta^n \in L^2(0,T;V) \cap C(0,T;H) \cap W^{1,2}(0,T;V')$ such as

$$\begin{cases} \dot{v}_\eta^n(t)+Av_\eta^n(t)+j_n'(v_\eta^n(t))=F_\eta(t), \quad \text{a.e. } t \in (0,T), \\ v_\eta^n(0)=v_0. \end{cases}$$

Then, we obtain

$$\begin{aligned} &\langle \dot{v}_\eta^n(t), w-v_\eta^n(t)\rangle_{V'\times V}+\langle Av_\eta^n(t), w-v_\eta^n(t)\rangle_{V'\times V}+j(w)-j(v_\eta^n(t)) \\ &\geq \langle F_\eta(t), w-v_\eta^n(t)\rangle_{V'\times V},\ \forall w \in V, \quad \text{a.e. } t \in (0,T). \end{aligned}$$

We have

$$\begin{aligned} &\langle \dot{v}_\eta^n(t), v_\eta^n(t)\rangle_{V'\times V}+\langle Av_\eta^n(t), v_\eta^n(t)\rangle_{V'\times V}+\langle j_n'(v_\eta^n(t)),(v_\eta^n(t))\rangle_{V'\times V} \\ &=\langle F_\eta(t), v_\eta^n(t)\rangle_{V'\times V}, \quad \text{a.e. } t \in (0,T). \end{aligned}$$

Integrating the latest equality on $[0,t],\ t \in [0,T]$, using (26) and monotony of j_n' to infer that

$$\exists\, C>0,\quad \forall n \in \mathbb{N},\quad \forall t \in [0,T],\ \|v_\eta^n(t)\|_H \leq C,\quad \int_0^T \|v_\eta^n(t)\|_V^2 dt \leq C.$$

We have

$$\exists\, C>0,\quad \forall n \in \mathbb{N}\quad \int_0^T \|\dot{v}_\eta^n(t)\|_{V'}^2 dt \leq C.$$

We can therefore extract a subsequence still denoted (v_η^n) to find that

$$\begin{cases} v_\eta^n \rightharpoonup v_\eta \text{ weakly in } L^2(0,T;V) \text{ and star weakly in } L^\infty(0,T;H), \\ \dot{v}_\eta^n \rightharpoonup \dot{v}_\eta \text{ star weakly in } L^2(0,T;V'). \end{cases}$$

It follows that

$$v_\eta \in C([0,T];H) \text{ and } v_\eta^n(t) \rightharpoonup v_\eta(t) \text{ star weakly in } H,\ \forall t \in [0,T]. \tag{54}$$

Then, we obtain

$$\begin{aligned}&\int_0^T \langle \dot{v}_\eta^n(t), w\rangle_{V'\times V} dt + \int_0^T \langle A v_\eta^n(t), w\rangle_{V'\times V} dt + \int_0^T j_n(w)dt\\ &\geq \int_0^T \langle \dot{v}_\eta^n(t), v_\eta^n(t)\rangle_{V'\times V} dt + \int_0^T \langle A v_\eta^n(t), v_\eta^n(t)\rangle_{V'\times V} dt + \int_0^T j_n(v_\eta^n(t))dt\\ &\quad + \int_0^T \langle F_\eta(t), w - v_\eta^n(t)\rangle_{V'\times V} dt, \ \forall w \in L^2(0,T;V)\end{aligned}$$

and we find

$$\begin{aligned}&\int_0^T \langle \dot{v}_\eta^n(t), w\rangle_{V'\times V} dt + \int_0^T \langle A v_\eta^n(t), w\rangle_{V'\times V} dt + \int_0^T j_n(w)dt\\ &\geq \tfrac{1}{2}\|v_\eta^n(T)\|_H^2 - \tfrac{1}{2}\|v_\eta^n(0)\|_H^2 + \int_0^T \langle A v_\eta^n(t), v_\eta^n(t)\rangle_{V'\times V} dt\\ &\quad + \int_0^T j_n(v_\eta^n(t))dt + \int_0^T \langle F_\eta(t), w - v_\eta^n(t)\rangle_{V'\times V} dt,\end{aligned} \tag{55}$$

using the assumptions of the lemma 3.2 and the weak lower-semi-continuity, we obtain that

$$\begin{aligned}&\int_0^T \langle \dot{v}_\eta, w - v_\eta\rangle_{V'\times V} dt + \int_0^T \langle A v_\eta, w - v_\eta\rangle_{V'\times V} dt + \int_0^T (j(w) - j(v_\eta))dt\\ &\geq \int_0^T \langle F_\eta, w - v_\eta\rangle_{V'\times V} dt, \quad \forall w \in L^2(0,T;V_1).\end{aligned} \tag{56}$$

The above inequality implies that

$$\begin{aligned}&\langle \dot{v}_\eta(t), w - v_\eta(t)\rangle_{V'\times V} + \langle A v_\eta(t), w - v_\eta(t)\rangle_{V'\times V} + j(w) - j(v_\eta(t))\\ &\geq \langle F_\eta(t), w - v_\eta(t)\rangle_{V'\times V} \ \ \forall w \in V, \ \ \text{a.e. } t \in (0,T).\end{aligned}$$

We conclude that P_{v_η} has at least a solution $v_\eta \in C(0,T;H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V')$. For the uniqueness, let $v_{1\eta}, v_{2\eta}$ be two solutions of P_{v_η}. We use (51) to obtain

$$\langle \dot{v}_{2\eta}(t) - \dot{v}_{1\eta}(t), v_{2\eta}(t) - v_{1\eta}(t)\rangle_{V'\times V} + \langle A v_{2\eta}(t) - A v_{1\eta}(t), v_{2\eta}(t) - v_{1\eta}(t)\rangle_{V'\times V} \geq 0.$$

Integrating the previous inequality, using (50) and (26), we find

$$\frac{1}{2}\|v_{2\eta}(t) - v_{1\eta}(t)\|_H^2 + \int_0^t \|v_{2\eta}(s) - v_{1\eta}(s)\|_V^2 ds \leq 0, \ \forall t \in [0,T],$$

which implies $v_{1\eta} = v_{2\eta}$. □

Let now $u_\eta : [0,T] \to V$ be the function defined by

$$u_\eta(t) = \int_0^t v_\eta(s)ds + u_0, \quad \forall t \in [0,T]. \tag{57}$$

In the study of Problem P_{u_η}, we have the following result.

Lemma 3.3. *P_{u_η} has a unique solution satisfying the regularity expressed in (46).*

Proof. The proof of lemma 3.3 is a consequence of lemma 3.2 and the relation (57). □

In the second step, let $(\eta, \mu) \in L^2(0,T;V' \times E_0)$ be given, we use the displacement field u_η obtained in Lemma 3.3 to consider the following variational problem.

Problem $P_{\eta\mu}$. Find a $\varphi_\eta : [0,T] \to W$ and $\alpha_\mu : [0,T] \to E_1$ such that

$$\left.\begin{aligned}&\sum_{\ell=1}^{2} (\mathcal{C}^\ell \nabla \varphi_\eta^\ell(t), \nabla \phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(u_\eta^\ell(t)), \nabla \phi^\ell)_{H^\ell} = (q(t), \phi)_W,\\ &\forall \phi \in W, \ \text{a.e. } t \in (0,T)\\ &\alpha_\mu(t) \in K, \ \sum_{\ell=1}^{2} (\dot{\alpha}_\mu^\ell(t) - \mu^\ell(t), \xi^\ell - \alpha_\mu^\ell)_{L^2(\Omega^\ell)} + a(\alpha_\mu(t), \xi - \alpha_\mu(t)) \geq 0\ ,\\ &\quad \forall \xi \in K \ \ \text{a.e.} t \in (0,T),\end{aligned}\right\} \tag{58}$$

where $K = K^1 \times K^2$. We have the following result for the problem.

Lemma 3.4. *There exists a unique solution of the Problem $P_{\eta\mu}$ and satisfies the regularity (47)-(48).*

Proof. For more details about the proof of this lemma, see lemmas 4.3. and 4.6. in [13].
Since $meas(\Gamma_a) > 0$, it follows from (29) and the Friedrichs-Poincaré inequality (18) that the bilinear form : $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\varphi, \phi) = \sum_{\ell=1}^{2} (\mathcal{C}^\ell \nabla \varphi^\ell(t), \nabla \phi^\ell)_{H^\ell} \quad \forall \varphi, \phi \in W. \tag{59}$$

is continuous, symmetric and coercive on W. Moreover, keeping in mind the regularity of q in (38), assumption (28) on the piezoelectric tensor $\mathcal{E}$ and the regularity $u_\eta \in C^1(0,T;H)$ obtained in lemma 3.3, we obtain that the function $L_\eta : [0,T] \to W$ given by

$$(L_\eta(t), \phi)_W = (q(t), \phi)_W + \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(u_\eta^\ell(t)), \nabla \phi^\ell)_{H^\ell} \quad \forall \phi \in W \tag{60}$$

is continuous. The existence and uniqueness part in lemma 3.4 is now a straight consequence of the well-known Lax-Milgram theorem combined with the equalities (59)- (60).

On the other hand, the inclusion of $(E_1, \|.\|_{E_1})$ into $(E_0, \|.\|_{E_0})$ is continuous and its range is dense. We denote by E_1' the dual space of E_1 and, identifying the dual of E_0 with itself, we can write the Gelfand triple

$$E_1 \subset E_0 = E_0' \subset E_1'.$$

Frome (41) we write

$$(\xi, \zeta)_{E_1' \times E_1} = (\xi, \zeta)_{E_0}, \quad \forall \xi \in E_0,\ \zeta \in E_1,$$

and we note that K is closed convex set in E_1. Then, using the definition (37) of the bilinear form a and the fact that $\alpha_0 \in K$ in (36). Thus using the Theorem 2.1, we deduce that there exists a unique function α_μ solution of the second relation in (58), which concludes the proof of the lemma. □

We consider the element $\Lambda(\eta(t), \mu(t)) = (\Lambda_1(\eta(t), \mu(t)), \Lambda_2(\eta(t), \mu(t))) \in V' \times E_0$ defined by the equations

$$\langle \Lambda_1(\eta(t), \mu(t)), v \rangle_{V' \times V} = \sum_{\ell=1}^{2} \big(\mathcal{B}^\ell(\varepsilon(u_\eta^\ell(t)), \alpha_\mu^\ell(t)) + (\mathcal{E}^\ell)^* \nabla \varphi_\eta^\ell(t),\ \varepsilon(v^\ell)\big)_{\mathcal{H}^\ell} \tag{61}$$

$$\Lambda_2(\eta(t), \mu(t)) = \sum_{\ell=1}^{2} S^\ell(\varepsilon(u_\eta^\ell(t)), \alpha_\mu^\ell(t)). \tag{62}$$

We have the following result.

Lemma 3.5. *The operator $\Lambda : L^2(0,T; V' \times E_0) \to L^2(0,T; V' \times E_0)$ has a unique fixed point (η^*, μ^*).*

Proof. By using arguments similar to those in the proof of lemma 4.9 in [14].
Let now $(\eta_1, \theta_1), (\eta_2, \theta_2) \in L^2(0,T; V' \times E_0)$. For simplicity, we use the notation $u_{\eta_i} = u_i,\ \dot{u}_{\eta_i} = \dot{u}_i = v_i,\ \varphi_{\eta_i} = \varphi_i$ and $\alpha_{\mu_i} = \alpha_i$, for $i = 1, 2$, we have

$$\|\Lambda_1(\eta_1(t), \mu_1(t)) - \Lambda_1(\eta_2(t), \mu_2(t))\|_{V'}^2 \le \sum_{\ell=1}^{2} \| \left(\mathcal{E}^\ell\right)^* \nabla \varphi_1^\ell(t) - \left(\mathcal{E}^\ell\right)^* \nabla \varphi_2^\ell(t) \|_{\mathcal{H}^\ell}^2$$

$$+ \sum_{\ell=1}^{2} \|\mathcal{B}^\ell(\varepsilon(u_1^\ell(t)), \alpha_1^\ell(t)) - \mathcal{B}^\ell(\varepsilon(u_2^\ell(t)), \alpha_1^\ell(t))\|_{\mathcal{H}^\ell}^2.$$

From the definition (61) combined with the assumptions (17), (27) on $\mathcal{B}$ and (28) on $\mathcal{E}$, we conclude that there is $C>0$ such that

$$\|\Lambda_1(\eta_1(t),\mu_1(t))-\Lambda_1(\eta_2(t),\mu_2(t))\|^2_{V'}\leq C(\|u_1(t)-u_2(t)\|^2_V+ \\ \|\alpha_1(t)-\alpha_2(t)\|^2_{E_0}+\|\varphi_1(t)-\varphi_2(t)\|^2_W). \tag{63}$$

Moreover, from (57), we have

$$\|u_1(t)-u_2(t)\|_V\leq\int_0^t\|v_1(s)-v_2(s)\|_V ds,\quad \forall t\in[0,T]. \tag{64}$$

Substituting $\eta=\eta_1, v=v_2$ and $\eta=\eta_2, v=v_1$ in (49), keeping in mind (50) and combining the resulting inequalities, we find

$$\langle\dot v_1-\dot v_2,v_1-v_2\rangle_{V'\times V}+\sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(v_1^\ell)-\mathcal{A}^\ell\varepsilon(v_2^\ell),\varepsilon(v_1^\ell-v_2^\ell))_{\mathcal{H}^\ell}+ \\ \langle\eta_1-\eta_2,v_1-v_2\rangle_{V'\times V}\leq 0$$

We integrate this inequality with respect to time. We use the initial conditions $v_1(0)-v_2(0)=v_0$ and the relation (26) to find that,

$$\min(m_{\mathcal{A}^1},m_{\mathcal{A}^2})\int_0^t\|v_1(s)-v_2(s)\|^2_V ds\leq-\int_0^t\langle\eta_1(s)-\eta_2(s),v_1(s)-v_2(s)\rangle_{V'\times V}ds.$$

Then, using the inequality $2ab\leq\frac{a^2}{m}+mb^2$, we obtain

$$\int_0^t\|v_1(s)-v_2(s)\|^2_V ds\leq C\int_0^t\|\eta_1(s)-\eta_2(s)\|^2_{V'}ds. \tag{65}$$

From (64) and (65), we deduce

$$\|u_1(t)-u_2(t)\|^2_V\leq C\int_0^t\|\eta_1(s)-\eta_2(s)\|^2_{V'}ds. \tag{66}$$

We use assumptions (28) and (29) on the piezoelectric and permittivity tensors respectively with the inequality of Friedrichs-Poincaré (18), it follows from (58) than

$$\|\varphi_1(t)-\varphi_2(t)\|^2_W\leq C\|u_1(t)-u_2(t)\|^2_V,\quad\forall t\in[0,T]. \tag{67}$$

Furthermore, by substituting $\mu=\mu_1,\ \xi=\alpha_1$ and $\mu=\mu_2,\ \xi=\alpha_2$ in (58) and subtracting the two inequalities obtained, we find

$$(\dot\alpha_1(t)-\dot\alpha_2(t),\alpha_1(t)-\alpha_2(t))_{E_0}+a(\alpha_1(t)-\alpha_2(t),\alpha_1(t)-\alpha_2(t)) \\ \leq(\mu_1(t)-\mu_2(t),\alpha_1(t)-\alpha_2(t))_{E_0},\quad\text{a.e. } t\in(0,T).$$

We integrate the previous inequality and applying the inequality of Hölder and Young with Gronwall's lemma, we deduce that

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{E_0}\leq C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_{E_0}ds. \tag{68}$$

We substitute (66)-(68) in (63) we obtain

$$\begin{aligned}&\textstyle\int_0^T\|\Lambda_1(\eta_1(s),\mu_1(s))-\Lambda_1(\eta_2(s),\mu_2(s))\|^2_{V'\times E_0}ds\\ &\leq C\textstyle\int_0^T(\|\eta_1(s)-\eta_2(s)\|^2_{V'}+\|\mu_1(s)-\mu_2(s)\|^2_{E_0})ds.\end{aligned} \tag{69}$$

Using the assumption (30), equality (17) with estimates (66) and (68) we obtain the estimate of Λ_1

$$\begin{aligned}&\int_0^T \|\Lambda_2(\eta_1(s),\mu_1(s)) - \Lambda_2(\eta_2(s),\mu_2(s))\|^2_{V'\times E_0} ds\\ &= \int_0^T \|S(\varepsilon(u_1(s)),\alpha_1(s)) - S(\varepsilon(u_2(s)),\alpha_2(s))\|^2_{V'\times E_0} ds\\ &\leq C\int_0^T (\|\eta_1(s)-\eta_2(s)\|^2_{V'} + \|\mu_1(s)-\mu_2(s)\|^2_{E_0})ds.\end{aligned} \tag{70}$$

Combining the inequalities (69) and (70) to obtain

$$\|\Lambda(\eta_1,\mu_1) - \Lambda(\eta_2,\mu_2)\|^2_{L^2(0,T;V'\times E_0)} \leq C\|\eta_1-\mu_1,\eta_2-\mu_2\|^2_{L^2(0,T;V'\times E_0)}. \tag{71}$$

Reiterating the inequality (71) n times leads to

$$\|\Lambda^n(\eta_1,\mu_1) - \Lambda^n(\eta_2,\mu_2)\|^2_{L^2(0,T;V'\times E_0)} \leq \frac{C^n}{n!}\|\eta_1-\mu_1,\eta_2-\mu_2\|^2_{L^2(0,T;V'\times E_0)}, \tag{72}$$

Thus, for n sufficiently large, Λ^n is a contraction on the Banach space $L^2(0,T;V'\times E_0)$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 3.1.

Existence. Let $(\eta^*,\mu^*) \in L^2(0,T;V'\times E_0)$ be the fixed point Λ and let u and $\{\varphi,\alpha\}$ denote the solutions of problems P_{u_η} ,$P_{\eta\mu}$ respectively, for $\eta=\eta^*$ and $\mu=\mu^*$. The equalities $\Lambda_1(\eta^*,\mu^*)=\eta^*$ and $\Lambda_2(\eta^*,\mu^*)=\mu^*$ show that (42)–(75) are satisfied. Next, (45) and the regularity (46)–(48) follow from lemmas 3.3, and 3.4.

Uniqueness. Uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ and the unique solvability of the problems P_{u_η} and $P_{\eta\mu}$.

4. Discrete approximations

We now introduce a finite element method to approximate solutions of Problem $\mathbf{P}^{\mathcal{V}}$ and derive an error estimate on them. We consider finite dimensional spaces $\boldsymbol{V}^h \subset \boldsymbol{V}$, $W^h \subset W$, and $E^h \subset E_1$, approximating the spaces $\boldsymbol{V}$, W and E_1, respectively. Here, $h>0$ denotes the spacial discretization parameter. Secondly, the time derivatives are discretized by using a uniform partition of $[0,T]$, denoted by $0=t_0<t_1<...<t_N=T$. Let k be the time step size, $k=T/N$, and for a continuous function $F(t)$ let $\mathbf{f}_n = F(t_n)$. Finally, for a sequence $\{w_n\}_{n=0}^N$, we denote by $\delta w_n = (w_n - w_{n-1})/k$ the finite differences. In this section, no summation is assumed over a repeated index and c denotes a positive constant which depends on the problem data, but is independent of the discretization parameters, h and k. Thus, using the backward Euler scheme, the fully discrete approximation of Problem $\mathbf{P}^{\mathcal{V}}$ is the following.

Problem $\mathbf{P}^{\mathcal{V},hk}$. Find a discrete velocity field $\boldsymbol{v}^{hk} = \{(\boldsymbol{v}_n^{1,hk},\boldsymbol{v}_n^{2,hk})\}_{n=0}^N \subset \boldsymbol{V}^h$, a discrete electric potential $\varphi^{hk} = \{(\varphi_n^{1,hk},\varphi_n^{2,hk})\}_{n=0}^N \subset W^h$ and a discrete damage field $\alpha^{hk} = \{(\alpha_n^{1,hk},\alpha_n^{2,hk})\}_{n=0}^N \subset E^h \cap K = K^h$, such that $\boldsymbol{v}_0^{hk} = \boldsymbol{w}_0^h$, $\alpha_0^{hk} = \alpha_0^h$, and for all $n=1,...,N$

$$\begin{aligned}&(\delta\boldsymbol{v}_n^{\ell,hk}, \boldsymbol{w}_n^{\ell,h} - \boldsymbol{v}^{\ell,h})_{V'\times V} + \sum_{\ell=1}^2 (\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_n^{\ell,hk}) + \mathcal{B}^\ell\varepsilon(\boldsymbol{u}_n^{\ell,hk},\alpha_n^{\ell,hk}), \varepsilon(\boldsymbol{v}_n^{\ell,h} - \boldsymbol{w}^{\ell,h}))_{\mathcal{H}^\ell} +\\ &\sum_{\ell=1}^2 ((\mathcal{E}^\ell)^*\nabla\varphi_n^{\ell,hk}, \varepsilon(\boldsymbol{v}^{\ell,h} - \boldsymbol{w}^{\ell,h}))_{\mathcal{H}^\ell} + j(\boldsymbol{w}^{hk}) - j(\boldsymbol{v}_n^{hk}) \geq (\mathbf{f}_n, \boldsymbol{v}_n^h - \boldsymbol{w}^h)_{\boldsymbol{V}} \quad \forall \boldsymbol{w}^h \in \boldsymbol{V}^h,\end{aligned} \tag{73}$$

$$\sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi_n^{\ell,hk},\nabla\phi^{\ell,h})_{H^\ell}-\sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}_n^{\ell,hk}),\nabla\phi^{\ell,h})_{H^\ell}=(q_n,\phi^h)_W\ \forall\phi^h\in W^h, \tag{74}$$

$$\left.\begin{array}{l}\sum\limits_{\ell=1}^{2}(\delta\alpha_n^{\ell,hk}(t),\xi^{h,\ell}-\alpha_n^{\ell,hk}(t))_{L^2(\Omega^\ell)}+a(\alpha_n^{hk},\xi^h-\alpha_n^{hk}(t))\\ \geq\sum\limits_{\ell=1}^{2}(S^\ell(\varepsilon(u_n^{\ell,hk}),\alpha_n^{\ell,hk}),\xi^{h,\ell}-\alpha_n^{\ell,hk})_{L^2(\Omega^\ell)},\quad\forall\xi^h\in K^h,\ \text{ a.e. } t\in(0,T),\end{array}\right\} \tag{75}$$

where the discrete displacement field $\boldsymbol{u}^{hk}=\{(\boldsymbol{u}_n^{1,hk},\boldsymbol{u}_n^{2,hk})\}_{n=0}^N\subset\boldsymbol{V}^h$, is given by

$$\boldsymbol{u}_n^{\ell,hk}=k\sum_{j=1}^{n}\boldsymbol{v}_j^{\ell,hk}+\boldsymbol{u}_0^{\ell,h}.$$

Here $u_0^{\ell,h}$, $v_0^{\ell,h}$ and $\alpha_0^{\ell,h}$ are appropriate approximation of the initial condition u_0^ℓ, v_0^ℓ and α_0^ℓ, respectively, and $\varphi_0^{\ell,hk}$ is the unique solution of the seconde quation in Problem $\mathbf{P}^{\mathcal{V},hk}$ for $n=0$.

We notice that the fully discrete Problem $\mathbf{P}^{\mathcal{V},hk}$ can be seen as a coupled system of variational inequations. Using classical results of nonlinear variational inequations (see [8]), we obtain that Problem $\mathbf{P}^{\mathcal{V},hk}$ admits a unique solution in $\boldsymbol{V}^h\times W^h\times K^h$. Our interest in this section lies in estimating the numerical errors $\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}$, $\|\varphi_n-\varphi_n^{hk}\|_W$ and $\|\beta_n-\beta_n^{hk}\|_{L^\infty(\Gamma_3)}$. Let $V_h(\Omega^\ell)$, $W_h(\Omega^\ell)$ and B_h^ℓ consist of continuous and piecewise affine functions; that is,

$$V_h(\Omega^\ell)=\{\boldsymbol{v}_h^\ell\in[C(\overline{\Omega^\ell})]^d;\ v_h^\ell|_K\in[P_1(K)]^d,\forall K\in\mathcal{T}_h^\ell;\ v_h^\ell|_{\Gamma_1^\ell}\equiv0\},$$
$$W_h(\Omega^\ell)=\{\varphi_h^\ell\in C(\overline{\Omega^\ell});\ \varphi_h^\ell|_K\in P_1(K),\ \forall K\in\mathcal{T}_h^\ell;\ v_h^\ell|_{\Gamma_a^\ell}\equiv0\},$$
$$E_h^\ell=\{\alpha_h^\ell\in C(\overline{\Omega^\ell});\ \alpha_h^\ell|_K\in P_1(K),\ \forall K\in\mathcal{T}_h^\ell\},$$

where $C(\overline{\Omega^\ell})$ and $P_1(K)$ denote the space of continuous functions on $\overline{\Omega^\ell}$ and the space of the polynomials with the global degree one on K, respectively, and $\xi_h^\ell=\{c_1=x_0^\ell,x_1^\ell,...,x_{N^\ell-1}^\ell,x_{N^\ell}^\ell=c_2\}$ the set of nodes on Γ_3 belonging to triangulation $\mathcal{T}_h^\ell$. We define the spaces

$$\boldsymbol{V}^h=V_h(\Omega^1)\times V_h(\Omega^2),\quad W^h=W_h(\Omega^1)\times W_h(\Omega^2),\quad E^h=E_h^1\times E_h^2,$$
$$H^2=H^2(\Omega^1)\times H^2(\Omega^2),\quad K^h=E^h\cap K.$$

Then, we have the following corollary which states the linear convergence of the algorithm under suitable regularity condition.

Theorem 4.1. *Assume that (26)–(36) hold. Let $\{\boldsymbol{u},\varphi,\alpha\}$ and $\{\boldsymbol{u}_n^{hk},\varphi_n^{hk},\alpha_n^{hk}\}$ denote the solution to Problems $\boldsymbol{P}^{\mathcal{V}}$ and $\boldsymbol{P}^{\mathcal{V},hk}$, respectively. Under the following regularity conditions:*

$$\boldsymbol{u}\in C(0,T;[H^2]^d),\ \varphi\in C(0,T;H^2),\ \alpha\in C^1(0,T;H^2).$$

Then we obtain the estimate

$$\max_{1\leq n\leq N}\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}+\|\varphi_n-\varphi_n^{hk}\|_W+\|\alpha_n-\alpha_n^{hk}\|_{E_0}\}\leq C(h+k), \tag{76}$$

where positive constant C independent of the discretization parameters h and k.

Proof. We have the following approximation properties of the finite element spaces $\boldsymbol{V}^h$, W^h and E^h (see [9]),

$$\max_{1\le n\le N} \inf_{\boldsymbol{w}_n^h\in \boldsymbol{V}^h} \|\boldsymbol{u}_n-\boldsymbol{w}_n^h\|_{\boldsymbol{V}} \le ch\|\boldsymbol{u}\|_{C(0,T;[H^2]^d)},$$

$$\max_{1\le n\le N} \inf_{\phi_n^h\in W^h} \|\varphi_n-\phi_n^h\|_W \le ch\|\varphi_n\|_{C(0,T;H^2)},$$

$$\max_{1\le n\le N} \inf_{\lambda_n^h\in W^h} \|\alpha_n-\lambda_n^h\|_W \le ch\|\alpha_n\|_{C(0,T;H^2)}.$$

Then, using arguments similar to those used in [4, Theorem 2], we deduce (76). □

Numerical example. For the numerical simulations, we consider the next special case: "A dynamic process of contact between two viscoelastic bodies without damage" of two-dimensional test problem. There, the following notation are used: $\Omega^1=[0,1]\times[0,1]$, $\Omega^2=[0,1]\times[-1,0]$, $\Gamma_1^1=[0,1]\times\{1\}$, $\Gamma_2^1=\{0,1\}\times[0,1]$, $\Gamma_1^2=[0,1]\times\{-1\}$, $\Gamma_2^2=\{0,1\}\times[-1,0]$, $\Gamma_3=[0,1]\times\{0\}$. We model the material's behaviour with a constitutive law of the form (3), in which the two bodies have the same properties and the functions $\mathcal{E}^\ell$ vanishes. The compressible material response is governed by a linearly viscoelastic constitutive law in which the viscosity tensor $\mathcal{A}^\ell$ and the elasticity tensor $\mathcal{B}^\ell$, are given by

$$(\mathcal{A}^\ell(\tau))_{ij}=\frac{10^{-2}Er}{1-r^2}(\tau_{11}+\tau_{22})\delta_{ij}+\frac{10^{-2}E}{1+r}\tau_{ij},\quad 1\le i,j\le 2,\quad \ell=1,2,\quad \forall\tau\in\mathbb{S}^2,$$

$$(\mathcal{B}^\ell(\tau))_{ij}=\frac{Er}{1-r^2}(\tau_{11}+\tau_{22})\delta_{ij}+\frac{E}{1+r}\tau_{ij},\quad 1\le i,j\le 2,\quad \ell=1,2,\quad \forall\tau\in\mathbb{S}^2,$$

where $E=20000N/m^2$ is the Young's modulus, $r=0.3$, is the Poisson's ratio of the materials and δ_{ij} is the Kronecker delta. For computation we use the following data:

$$T=1s,\quad \boldsymbol{u}_0=(0,0)m,\quad \boldsymbol{v}_0=(0,0)m/s,\quad g=0N/m^3,\quad \rho^1=\rho^2=1kg/m^3,$$

$$f_0^1=(0.5,-0.5)N/m^3,\quad f_0^2=(0.5,0)N/m^3,\quad f_2^\ell=(f_{2,1}^\ell,f_{2,2}^\ell),\ \ \ell=1,2,\quad \text{where}$$

$$f_{2,1}^1=\begin{cases} 3\times10^{-2}N/m & \text{on}\ \ \{0\}\times[0,1],\\ -10^{-1}N/m & \text{on}\ \ \{1\}\times[0,1],\end{cases}\qquad f_{2,2}^1=10^{-1}N/m,$$

$$f_{2,1}^2=\begin{cases} 5\times10^{-2}N/m & \text{on}\ \ \{0\}\times[-1,0],\\ -10^{-2}N/m & \text{on}\ \ \{1\}\times[-1,0],\end{cases}\qquad f_{2,2}^2=-10^{-2}N/m.$$

To see the convergence behaviour of the fully discrete scheme, we compute a sequence of numerical solutions based on uniform partitions of the time interval $[0,1]$, and uniform triangulations of the domain $[0,1]\times[-1,1]$ of the type shown in Figure 1 which represents a coarse discretization ($h=k=1/4$). The results at the end of the simulation are illustrated Figure 2, which represent the approximation of displacements at $t=1s$.

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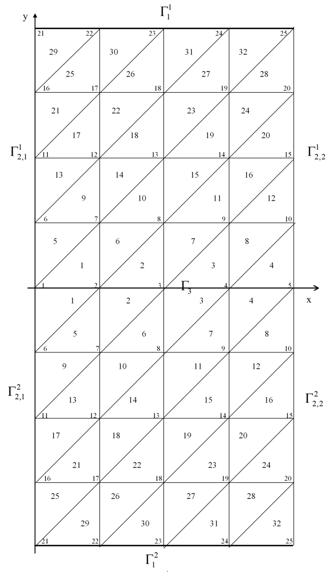
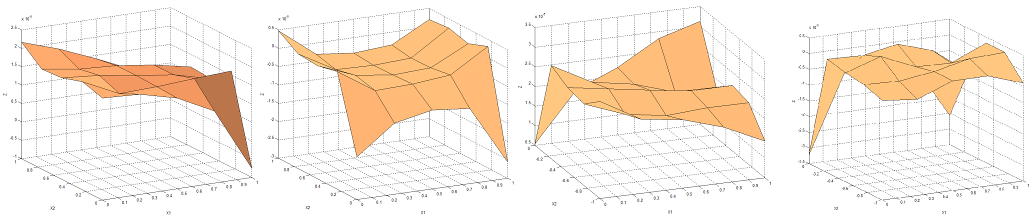


FIGURE 1. Discretization of the bodies Ω^1, Ω^2.



The approximation of $\boldsymbol{u}_1^1$. The approximation of $\boldsymbol{u}_2^1$. The approximation of $\boldsymbol{u}_1^2$. The approximation of $\boldsymbol{u}_2^2$.

FIGURE 2. The approximation of displacements $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ at $t = 1s$.

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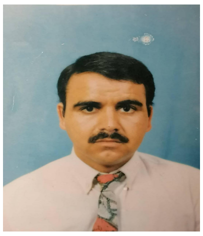

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