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# WHOLE DOMINATION IN GRAPHS

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Abstract. In this paper, a new parameter of domination number in graphs is defined which is called whole domination number denoted by $\gamma_{wh}(G)$. Some bounds of whole domination number and the number of edges depend on it has been established. Furthermore, the effect of deletion vertex, edge, or add edge have been studied. Also, the effect of the contracting an edge is determined. Finally, some operations between the two graphs have been calculated.

Keywords: Whole dominating set, Whole domination number, Deletion, Contracting an edge.

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## 1. Introduction

We consider a graph $G = (V(G), E(G))$ for simplicity $G = (V, E)$as finite,simple and undirected, where $V$ mentioned to the nonempty vertex set and $E$ to the unordered pair of vertices may be empty which is called edge set. For all other terms and notions not provided in this paper, the reader can review [4] and [5]. The number of edges that incident with a vertex say $v$ is called the degree of $v$ and denoted by $\deg(v)$. The minimum and maximum degree $\delta(G)$ and $\Delta(G)$ , respectively. The notion of domination is important parameter in graph theory because it has the potential to solve many real-life. Furthermore, there are different kinds of fields of graph theory as a labeled graph [1]and [2], topological graph [7], and others. A subset $D$ of the vertex set is called a dominating set if each vertex in the set $V - D$ is adjacent to at least one vertex in $D$. The minimum cardinality of all dominating set is called the domination number of $G$ and denoted by $\gamma(G)$. The first introduced this notion is Berge in his book [3] and the first used by Ore in [9]. To date many papers have been written on domination in graphs like [6],[8],[10] and [11]. Here, a new definition is introduced called whole domination. Some fundamental results on whole domination are presented. Further several bounds for the whole domination number are stated. Whole domination for $G_1 \times G_2$ , $G_1 + G_2$ and $G_1 \cup G_2$ where, $G_1$

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and $G_2$ are two graphs have whole domination number is discussed. Also, the effects on whole domination parameter when a graph is modified by deleting a vertex or deleting or adding or contraction an edge are presented.

**Definition 1.1.** [4] *Let $G_1$ and $G_2$ be two disjoint graphs. The union $G_1 \cup G_2$ of $G_1$ and $G_2$ is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.*

**Definition 1.2.** [4] *The Cartesian product $G_1 \times G_2$ of $G_1$ and $G_2$ is the graph having vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $G$ are adjacent if and only if either $u_1 = v_1$ and $u_2\ v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$.*

**Definition 1.3.** [4] *The join (addition) $G_1 + G_2 of G_1 and G_2$ is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1)$ and $v \in V(G_2)\}$.*

## 2. Whole domination number

Throughout this section, a new parameter of dominating set (for simplicity DS) for in a graph is initiated , it is called a whole dominating set (WDS). Also, some properties of WDS are mentioned.

**Definition 2.1.** *Let $G$ be a simple connected graph, a proper subset $D \subset V$ is called WDS, if every vertex in set $V - D$ is adjacent to all vertices in set D. The set D is called minimal WDS ($MWDS$) if it has no proper WDS. The whole domination number denoted by $\gamma_{wh}(G)$ for simplicity $\gamma_{wh}$ is the minimum cardinality of a MWDS. The MWDS has $\gamma_{wh}$is called $\gamma_{wh}$- set.*

**Remark 2.1.** *i) If G is disconnected graph, then G has no WDS.*
*ii) A graph G has a WDS if and only if there is a spanning complete bipartite subgraph.*
*iii) If G has a WDS, then daim(G)= 2 .*
*iv) D is a MWDS iff D is a WDS and there is no vertex in D joining with the all others vertices in $D$.*
*v) If D is a MWDS ,then G[D] is not a complete graph.*

**Observation 2.1.** *i) $\gamma_{wh}(P_n) = 1$ if $2 \le n \le 3$ , otherwise $P_n$ has no WDS.*
*ii) $\gamma_{wh}(C_n) = \left\{ \begin{array}{l} 1,\ if\ n = 3 \\ 2,\ if\ n = 4 \end{array} \right\}$, otherwise $C_n$ has no WDS.*
*iii) $\gamma_{wh}(K_n) = \gamma_{wh}(W_n) = 1$.*
*iv) $\gamma_{wh}(K_{m,n}) = min\{m, n\}$.*

**Theorem 2.1.** *If G has a $\gamma_{wh} \neq 1$,then $\gamma_{wh}(n - \gamma_{wh}) \le m \le \lfloor \frac{(n-\gamma_{wh})(n-\gamma_{wh}-2)}{2} \rfloor + \lfloor \frac{\gamma_{wh}(\gamma_{wh}-2)}{2} \rfloor + \gamma_{wh}(n - \gamma_{wh})$, where m and n are size and order of G respectively.*

*Proof.* Let $D$ be a $\gamma_{wh} - set$ of a graph. The graph $G$ can be classified into three classes depend on it is edges as follows.
i) The lower bound of m happens when the induced subgraph $G[V - D]$is null graph while the upper bound happens when each vertex in $G[V - D]$ is adjacent to all other vertices in this $G[V - D]$ except one. Let $m_1$ be the number of edges in this case, so $0 \le m_1 \le \lfloor \frac{(n-\gamma_{wh})(n-\gamma_{wh}-2)}{2} \rfloor$, $\gamma_{wh} \neq 1$ ,since $|\, V - D \,| = n - \gamma_{wh}$.
ii) The edges that joins the vertices of induced subgraph $G[D]$, as same manner in previous case. One can calculate easily that $0 \le m_2 \le \lfloor \frac{\gamma_{wh}(\gamma_{wh}-2)}{2} \rfloor$, $\gamma_{wh} \neq 1$
iii)By definition of WDS, it is obvious that the number of edges that incedint to vertices one of them in $G[V - D]$ and the other in $G[D]$ is $m_3 = \gamma_{wh}(n - \gamma_{wh})$.
Therefore, by combing the results in i, ii, and iii, the proof is done.

□

**Corollary 2.1.** *If $G$ has a $\gamma_{wh} = 1$, then $n-1 \leq m \leq \frac{n(n-1)}{2}$, $n \geq 2$ where $m$ and $n$ are size and order of $G$ respectively.*

**Proposition 2.1.** *If $G$ has a WDS, then $\bar{G}$ has no WDS, where $\bar{G}$ is complement of $G$ .*

*Proof.* If $G$ has a WDS $D$, then $\bar{G}$ has at least two disjoint components, one of them contains $D$ and the other contains $V-D$. Thus, according to Remark 2.1($i$), $\bar{G}$ has no WDS. □

**Proposition 2.2.** *If $G$ has a whole domination number , then $\gamma_{wh}(n-\gamma_{wh}) \leq \sum_{i=1}^{\gamma_{wh}} deg(v_i)$ $\leq \gamma_{wh}(n-2)$ , where $v_i \in D$ and $n$ is the order of $G$ .*

*Proof.* For every vertex $v_i$ in $D$: $i = 1, 2, ..., \gamma_{wh}, (n-\gamma_{wh}) \leq deg(v_i) \leq (n-2)$ , since each vertex in this case is adjacent to all vertices in an induced subgraph $G[V-D]$. Furthermore, this vertex may be adjacent to at most $(\gamma_{wh}-2)$ vertices with the other vertices in $D$. (according to Remark 2.1($iv$)). Therefore, we get the result. □

**Proposition 2.3.** *If $G$ has a $\gamma_{wh} = 1$, then either $\gamma_{wh}(G-v) \geq \gamma_{wh}(G)$ or $G-v$ has no WDS.*

*Proof.* The graph $G$ can be classified into two classes as follows.
Case 1. If a graph $G$ has just one vertex (say $v$) such that this vertex $v$ is adjacent to all other vertices in $G$, then $G-v$ has a WDS if there are two vertices or more are adjacent to all other vertices in $G-v$ which mean that $\gamma_{wh}(G-v) > \gamma_{wh}(G)$. Otherwise, $G-v$ has no WDS.
Case 2. If there is a vertex say $s \neq v$ such that $deg(s) = n-1$, then $\gamma_{wh}(G-v) = \gamma_{wh}(G)$ $= 1$.
From the two cases above, the required result is obtained.

□

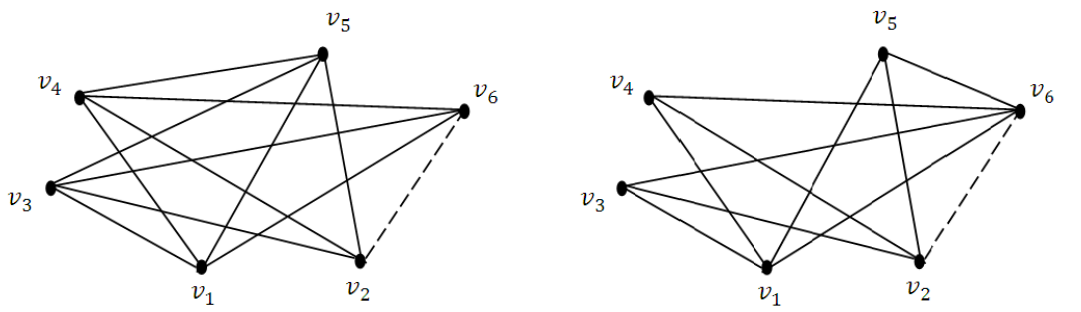
**Theorem 2.2.** *If $G$ has a $\gamma_{wh} \neq 1$,then $\gamma_{wh}(G-v) \leq \gamma_{wh}(G)$.*

*Proof.* Let $D$ be a $\gamma_{wh}-$ set of a graph $G$. By deleting a vertex from $G$, two different cases are obtained.
Case 1. If we delete a vertex $v \in V-D$, then two subcases are obtained as follows.
i) If $|D| = |V-D| = \gamma_{wh}$ , then $\gamma_{wh}(G-v) < \gamma_{wh}(G)$ , since in this case, the cardinal number of vertex set of induced subgraph $G[(V-D)-\{v\}]$ is less than the cardinal number of vertex set of induced subgraph. Also, the vertices in are adjacent to all vertices in $(V-D)-\{v\}$ are adjacent to all vertices in $D$.
ii) If $|D| < |V-D|$, then set $D$ remains the MWDS and $\gamma_{wh}(G-v) = \gamma_{wh}(G)$.
Case 2. If we delete the vertex $v$ where $v \in D$ . In this case the set $D-\{v\}$ remains the MWDS in $G$ and $\gamma_{wh}(G-v) < \gamma_{wh}(G)$. □

**Theorem 2.3.** *If $G$ has a $\gamma_{wh}$, then either $\gamma_{wh}(G-e) \geq \gamma_{wh}(G)$ or $G-e$ has no dominating set.*

*Proof.* Let $D$ be a $\gamma_{wh}-$ set of a graph $G$. By deleting an edge from a graph $G$, we got the following two cases.
Case 1. If an edge $e$ is deleted from $G$ where,$e$ is incident on two vertices in $V-D$ or of in $D$, then the minimum whole dominating set is not influenced by this deletion. Thus, $\gamma_{wh}(G-e) = \gamma_{wh}(G)$

Case 2. If an edge $e$ is incident on two vertices one of them from $V-D$ and the other from of $D$, then there are two different subcases.
i) If there is others whole dominating set say $D_1$ such that $D_1$ is not influenced when the edge $e$ is deleting, then $\gamma_{wh}(G-e) = |D_1| \geq \gamma_{wh}(G)$. (as an example, see Figure $2(A), D = \{v_1, v_2\}, D_1 = \{v_3, v_4\}, e = v_2v_6$, Figure $2(B), D = \{v_6\}$, $D_1 = \{v_3, v_4, v_5\}$, $e = v_2v_6$ ).




(A) $\gamma_{wh}(G-e) = \gamma_{wh}(G)$ (B) $\gamma_{wh}(G-e) > \gamma_{wh}(G)$

FIGURE 1. $\gamma_{wh}(G-e) \geq \gamma_{wh}(G)$

ii) If there is no other whole dominating set, then $G-e$ has no whole dominating set. Therefore, we get the result.

□

**Theorem 2.4.** *If $G$ has a $\gamma_{wh}$, then $\gamma_{wh}(G+e) \leq \gamma_{wh}(G)$, where $e \in \bar{G}$.*

*Proof.* Let $D$ be a MWDS with $|D| = \gamma_{wh}$ . By adding an edge $e$ to a graph $G$ , then there are two different cases are gotten.
Case 1. If the edge $e$ incident to two vertices from $V-D$, then the whole domination is not influenced by this addition . Thus, $\gamma_{wh}(G+e) = \gamma_{wh}(G)$.
Case 2. If $e$ joins two vertices in $D$, then if at least one vertex which is incident on this edge becomes adjacent to all vertices in $D$, then $\gamma_{wh}(G+e) = 1$. Thus, $\gamma_{wh}(G+e) < \gamma_{wh}(G)$. Otherwise,$\gamma_{wh}(G+e) = \gamma_{wh}(G)$. We cannot add an edge $e$ to a graph $G$ if one of its vertices belongs to the set $D$ and the other belongs to the set $V-D$, since $e \notin \bar{G}$ in this case according to definition of WDS. Therefore, the result is obtained. □

**Theorem 2.5.** *If $G$ has a $\gamma_{wh}$, then $\gamma_{wh}(G \setminus e) \leq \gamma_{wh}(G)$.*

*Proof.* Let $D$ be a $\gamma_{wh}-$ set of a graph $G$. By contracting an edge $e$ of graph $G$, three different cases are obtained.
Case 1. If we contract the edge $e$ which is incident on two vertices of $V-D$, then there are two different subcases .
i) If $|D| < |V-D|$, then the MWDS is not influenced by this contraction. Thus,$\gamma_{wh}(G \setminus e) = \gamma_{wh}(G)$.
ii) If $|D| = |V-D|$, then $V-D$ becomes the MWDS. Thus, $\gamma_{wh}(G \setminus e) < \gamma_{wh}(G)$.
Case 2. If $e$ is incident on two vertices one of them from $V-D$ and the other from $D$, then we have two subcases as follows.
i) If $\gamma_{wh} \neq 1$ , then the new vertex obtained by this contraction belongs to set $V-D$ , since this vertex joins with all vertices in set $D$ except the vertex which is incident on the contracted edge. Thus, $\gamma_{wh}(G \setminus e) \leq \gamma_{wh}(G)$.

ii) If $\gamma_{wh} = 1$, then the new vertex obtained by this contraction belongs to set $D$. $\gamma_{wh}(G \setminus e) = \gamma_{wh}(G)$

Case 3. If contracting an edge $e$ incident on two vertices from $D$, then it is clear that the set $D$ is decreasing by one vertex and will still minimum whole dominating set. Thus, $\gamma_{wh}(G \setminus e) < \gamma_{wh}(G)$. Therefore, from all cases above, we obtain $\gamma_{wh}(G \setminus e) \leq \gamma_{wh}(G)$. □

**Proposition 2.4.** *If the graphs $G_1$ and $G_2$ have whole dominating sets, then $G_1 \times G_2$ has whole dominating set if and only if $G_1 \cong G_2 \cong P_2$.*

*Proof.* If $G_1 \times G_2$ has a WDS, then the $daim(G_1 \times G_2)$= 2 according to Remark 2.1(iii), thus each of $G_1$ and $G_2$ must save only two vertices. These vertices must be joined, since both $G_1$ *and* $G_2$ have a whole dominating set by hypotheses. Therefore, $G_1 \cong G_2 \cong P_2$. Conversely, it is obvious. □

**Proposition 2.5.** *If the graphs $G_1$ and $G_2$ have $\gamma_{wh}(G_1)$ and $\gamma_{wh}(G_2)$respectively, then* $\gamma_{wh}(G_1 + G_2) = min\{\gamma_{wh}(G_1), \gamma_{wh}(G_2)\}$

*Proof.* Since $G_1$ *and* $G_2$ have whole dominating sets then, let $D_1$ *and* $D_2$ be the whole dominating sets with minimal cardinality in the two graphs $G_1$ *and* $G_2$, respectively. It is clear that each vertex that belongs to $D_1$ or to $D_2$ joins with all vertices of $G_1 + G_2$. Therefore, $\gamma_{wh}(G_1 + G_2) = min\ \{\gamma_{wh}(G_1), \gamma_{wh}(G_2)\}$ □

**Observation 2.2.** *If the graphs $G_1$ and $G_2$ have whole domination numbers respectively, then $G_1 \cup G_2$ has no whole dominating set.*

**Proposition 2.6.** *Tree has WDS if and only if it is a star.*

*Proof.* If $G$ is a tree and it has a whole dominating set, then it is a whole dominating set $D$ with minimum cardinality. If $|D| > 1$, then $G$ has a cycle of order greater than or equal to four, and this is a contradiction with our assumption. Therefore, $|D| = 1$. Now, if $G[V - D]$ is not a null graph which mean it contains at least one edge, then the vertices incident on this edge with the vertex belongs to $D$ form a cycle and again this is a contradiction with our assumption. Thus, $|D| = 1$ and $G[V - D]$ is a null graph, then $G$ is star.

Conversely, the assertion is clear. □

**Theorem 2.6.** *If a graph $G$ has a $\gamma_{wh}$, then $n - \triangle(G) \leq \gamma_{wh}(G) \leq \delta(G)$. where $n$ is the order of $G$*

*Proof.* If a graph $G$ has a $\gamma_{wh}(G)$ which means there is a WDS with minimum cardinality $\gamma_{wh}(G)$. To prove the lower bound; $G$ has spanning complete bipartite subgraph $K_{\gamma_{wh}, n-\gamma_{wh}}$ . Now, if vertex $v$ belongs to set $D$, then $deg(v) \geq |V - D| = n - \gamma_{wh}$ and $deg(v) \geq |D| = \gamma_{wh}$ if $v$ belongs to the set $V - D$, Thus, the minimum degree of vertex $v$ is obtained when $v$ is an isoleted vertex in $G[V - D]$ . It is clear that in this case $deg(v) = \gamma_{wh}$ .

Now, to prove the upper bound, the minimum value of $\triangle(G)$ is obtained then the induced subgraphs $G[D]$ and $G[V - D]$ are null graphs. In this case $\triangle(G) = |V - D| = n - \gamma_{wh}$ which means $\gamma_{wh} = n - \triangle(G)$ , otherwise, $\gamma_{wh}(G) > n - \triangle(G)$, Thus, we get the result. □

**Proposition 2.7.** *If $G$ has a $\gamma_{wh}$, then $\beta_0(G) \leq n - \gamma_{wh}(G)$.*

*Proof.* If $G$ has a whole dominating set $D$ with cardinal number equal to $\gamma_{wh}$, then the maximum cardinality of independent set $\beta_0(G)$ can be obtained when the induced subgraph $G[V-D]$ is a null graph. In this case $\beta_0(G) = |V-D| = n - \gamma_{wh}(G)$, therefore $\beta_0(G) \leq n - \gamma_{wh}(G)$. □

## 3. Conclusions

Throughout this work, a new parameter of domination is been introduced. Many properties and boundaries are been discussed. Also, the effect of deletion, addition, or contraction of an edge or deletion of a vertex is been studied. Furthermore, some operations for the two graphs are determined.

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