TWMS J. App. and Eng. Math. V.12, N.4, 2022, pp. 1526-1535

SOLVING EXISTENCE PROBLEMS VIA F-CONTRACTION IN MODIFIED b-METRIC SPACES

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Abstract. In this paper, we introduce a new abstract structure, expanded b-metric, as an natural extension of b-metric. We also define basic topological notions in expanded b-metric to able to investigate the existence of fixed point for such mappings under various F-contractive conditions. We provide example to illustrate the results presented herein.

Keywords: b-metric, expanded b-metric, fixed point, contraction.

AMS Subject Classification: 46T99, 47H10, 54H25.

1. Introduction and preliminaries

In the last decades, one of the attractive subject is theory fixed point. In particular, "changing the abstract structure so that a considered mappings possed a fixed point" has been densely studied. The concept of a metric space has been refining, extending and generalizing in several ways to guarantee the existence of a fixed point of certain mappings that are defined on these new structures. Among all, we recall the most interesting and more general notion, b-metric space. It was considered by several mathematician with different names (such as quasi-metric [5], general metric), but it has been famous by the publications of Bakhtin [4] and Czerwik in [7].

Recall (see, e.g., [4, 7]) that a b-metric d on a set X is a generalization of standard metric, where the triangular inequality is replaced by

$$d(x,z) \leq s[d(x,y) + d(y,z)],$$

for all $x, y, z \in X$, for some fixed $s \geq 1$.

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§ Manuscript received: September 10, 2020; accepted: October 08, 2020.

Let Ψ denote a family of mappings such that for each $\psi \in \Psi$, $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ and ψ is onto,

(1) $t \leq \psi(t)$ for every $t \in [0, +\infty)$,
(2) ψ' is increasing for every $t \in [0, +\infty)$ where $\frac{d\psi}{dt} = \psi'$ is derivative.

Remark 1.1. *Let $f, g : [0, +\infty) \longrightarrow \mathbb{R}$ be two functions such that $f(0) = g(0)$ and $\frac{df(x)}{dx} = f'(x) \leq g'(x) = \frac{dg(x)}{dx}$ for $x \in [0, +\infty)$, then we have $f(x) \leq g(x)$.*

Remark 1.2. *For every $\psi \in \Psi$ and for every $t \in [0, +\infty)$ we have $\psi^{-1}(t) \leq t \leq \psi(t)$ and $\psi^{-1}(0) = 0 = \psi(0)$.*

For example, if $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ defined by $\psi(t) = e^t - 1$, $\psi(t) = te^t$ and $\psi(t) = t^2 + 2t$ for every $t \in [0, +\infty)$, then it is easy to see that $\psi \in \Psi$.

Lemma 1.1. *Let $\psi \in \Psi$, then for every $x, y \in [0, +\infty)$, $r \in (0, 1)$ and for every $n \in \mathbb{N}$ we have:*

(1) $\psi(x + y) \geq \psi(x) + \psi(y)$,
(2) *ψ is continuous and is strictly increasing,*
(3) $r\psi(x) \geq \psi(rx)$,
(4) $\psi^{-1}(x + y) \leq \psi^{-1}(x) + \psi^{-1}(y)$,
(5) $\psi^{-1}(rx) \geq r\psi^{-1}(x)$,
(6) $\psi^n(x + y) \geq \psi^n(x) + \psi^n(y)$,
(7) $\psi^{-n}(x + y) \leq \psi^{-n}(x) + \psi^{-n}(y)$,
(8) $|\psi^{-1}(x) - \psi^{-1}(y)| \leq \psi^{-1}(|x - y|)$.

Proof. (1) If define $g(x) = \psi(x + b)$ and $f(x) = \psi(x) + \psi(b)$, then $f(0) = g(0)$ and $f'(x) = \psi'(x) \leq \psi'(x + b) = g'(x)$. Therefore, for every $x \in [0, +\infty)$ we have $f(x) \leq g(x)$ that is $\psi(x + y) \geq \psi(x) + \psi(y)$.

(2) It is clear that ψ is continuous. Since

$$\psi'(x) = \lim_{h \to 0} \frac{\psi(x + h) - \psi(x)}{h} \geq \lim_{h \to 0} \frac{\psi(h)}{h} \geq \lim_{h \to 0} \frac{h}{h} = 1.$$

Hence, ψ is strictly increasing.

(3) If define $f(x) = \psi(rx)$ and $g(x) = r\psi(x)$ for every $r \in (0, 1)$, then $f(0) = g(0) = 0$ and $f'(x) = r\psi'(rx) \leq r\psi'(x) = g'(x)$. Therefore, for every $x \in [0, +\infty)$ we have $f(x) \leq g(x)$ that is $\psi(rx) \leq r\psi(x)$.

(4) By part (2) we know first that the inverse ψ that is ψ^{-1} there exist and it is strictly increasing. Hence, if set $x = \psi^{-1}(x)$ and $y = \psi^{-1}(y)$ in (1), we get

$$\psi(\psi^{-1}(x) + \psi^{-1}(y)) \geq \psi(\psi^{-1}(x)) + \psi(\psi^{-1}(y)) = x + y.$$

That is $\psi^{-1}(x + y) \leq \psi^{-1}(x) + \psi^{-1}(y)$.

(5) If set $x = \psi^{-1}(x)$ in (3), we get $\psi(r\psi^{-1}(x)) \leq r\psi(\psi^{-1}(x))$. That is $r\psi^{-1}(x) \leq \psi^{-1}(rx))$.

(6) For $n = 1$ it is obvious. Suppose that (5) holds for some $n \geq 2$. Since

$$\begin{aligned} \psi^{n+1}(x + y) &= \psi(\psi^n(x + y)) \\ &\geq \psi(\psi^n(x) + \psi^n(y)) \\ &\geq \psi(\psi^n(x)) + \psi(\psi^n(y)) = \psi^{n+1}(x) + \psi^{n+1}(y). \end{aligned}$$

So inequality (5) is proved by induction.

(7) Similarly, this part is obtain from (4) and (5) obviously.

(8) If $x > y$ instead x by $x-y$ in (4) we have $\psi^{-1}(x) - \psi^{-1}(y) \leq \psi^{-1}(x-y)$, also if $x \leq y$ instead y by $y-x$ in (4) we have $\psi^{-1}(y) - \psi^{-1}(x) \leq \psi^{-1}(y-x)$. Hence in generally we have

$$|\psi^{-1}(x) - \psi^{-1}(y)| \leq \psi^{-1}(|x-y|).$$

□

Now, we introduced the concept of extended b-metric spaces as follows.

Definition 1.1. *Let X be a (nonempty) set. A function $\rho : X \times X \to [0, +\infty)$ is a expanded b-metric if there exists a $\psi \in \Psi$ such that for all $x, y, z \in X$, the following conditions hold:*

(ρ_1) $\rho(x, y) = 0$ *if and only if* $x = y$,
(ρ_2) $\rho(x, y) = \rho(y, x)$,
(ρ_3) $\rho(x, z) \leq \psi(\rho(x, y)) + \psi(\rho(y, z))$.

In this case, the triple (X, ρ, ψ) is called a expanded b-metric space.

A b-metric [7] is a expanded b-metric, with $\psi(t) = st$, for some fixed $s \geq 1$, also every metric is a expanded b-metric, for every $\psi \in \Psi$. For a proper choice of ψ, the notion of strong b-metric, defined by Kirk and Shahzad [15] can be derived.

Example 1.1. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\rho(x, y) = \sinh(d(x, y))$. We show that ρ is a expanded b-metric with $\psi(t) = \sinh(2st)$ for all $t \geq 0$ (and $\psi^{-1}(u) = \frac{1}{2s}\sinh^{-1}(2su)$ for $u \geq 0$). Obviously, conditions (ρ_1) and (ρ_2) of Definition 1.1 are satisfied. Since $\sinh(x)$ is an increasing function, hence for every $x, y \geq 0$ we have*

$$\sinh(x + y) \leq \sinh(2\max\{x, y\}) \leq \sinh(2x) + \sinh(2y).$$

Therefore, for each $x, y, z \in X$, we have

$$\begin{aligned}
\rho(x,z) &= \sinh(d(x,z)) \\
&\leq \sinh(sd(x,y) + sd(y,z)) \leq \sinh(s\sinh(d(x,y)) + s\sinh(d(y,z))) \\
&= \sinh(s\rho(x,y) + s\rho(y,z)) \\
&\leq \sinh(2s\rho(x,y)) + \sinh(2s\rho(y,z)) \\
&= \psi(\rho(x,y)) + \psi(\rho(y,z)).
\end{aligned}$$

So, condition (ρ_3) of Definition 1.1 is also satisfied and ρ is an expanded b-metric. Note that $\sinh|x-y|$ is not a metric on $\mathbb{R}$, as, e.g.,

$$\sinh 5 \approx 74.203 \nleq 3.627 + 10.0179 \approx \sinh 2 + \sinh 3.$$

Similarly, although $d(x, y) = (x-y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, there is no $s \neq 1$ such that $\rho(x, y) = \sinh(x-y)^2$ is a b-metric with parameter s. Indeed, putting $z = 0$ and $y = 1$ we should have $\sinh x^2 \leq s(\sinh(x-1)^2 + \sinh 1)$ which cannot hold for any fixed s and x sufficiently large.

Definition 1.2. *Let (X, ρ, ψ) be a expanded b-metric space. A sequence $\{x_n\}$ in X is said to be:*

(1) convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$, $\rho(x, x_n) < \varepsilon$.

(2) Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $\rho(x_n, x_m) < \varepsilon$.

(3) An expanded b-metric space X is called complete, if every Cauchy sequence is convergent in X.

Lemma 1.2. *Let (X,ρ,ψ) be a expanded b-metric space. If ψ is continuous and a sequence $\{x_n\}$ in X converges to x, then x is unique.*

Proof. Let $\{x_n\}$ converges to x and y, then using the rectangle inequality in the expanded b-metric space it is easy to see that

$$\rho(x,y) \leq \psi(\rho(x,x_n)) + \psi(\rho(y,x_n)).$$

Taking the limit as $n \to +\infty$ in the above inequality we obtain $\rho(x,y) = 0$ so $x = y$. □

Lemma 1.3. *Let (X,ρ,ψ) be a expanded b-metric space. If sequence $\{x_n\}$ in X is converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $\lim_{n\to+\infty} x_n = x$ then, by (ρ_3), we obtain

$$\rho(x_n,x_m) \leq \psi(\rho(x_n,x)) + \psi(\rho(x,x_m))$$

Taking the limit as $n,m \to +\infty$ in the above inequality we obtain

$$\lim_{n,m\to+\infty} \rho(x_n,x_m) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

We will need the following simple lemma about the convergent sequences.

Lemma 1.4. *Let (X,ρ,ψ) be a expanded b-metric space with function ψ,*

1. Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then we have

$$\psi^{-2}(\rho(x,y)) \leq \liminf_{n\longrightarrow+\infty} \rho(x_n,y_n) \leq \limsup_{n\longrightarrow+\infty} \rho(x_n,y_n) \leq \psi^2(\rho(x,y)).$$

In particular, if $x = y$, then we have $\lim_{n\to+\infty} \rho(x_n,y_n) = 0$.

2. Suppose that $\{x_n\}$ is convergent to x and $z \in X$ is arbitrary. Then we have

$$\psi^{-1}(\rho(x,z)) \leq \liminf_{n\longrightarrow+\infty} \rho(x_n,z) \leq \limsup_{n\longrightarrow+\infty} \rho(x_n,z) \leq \psi(\rho(x,z)).$$

Proof. 1. Using (ρ_3) in the expanded b-metric space it is easy to see that

$$\begin{aligned}\rho(x,y) &\leq \psi(\rho(x,x_n)) + \psi(\rho(y,x_n))\\ &\leq \psi(\rho(x,x_n)) + \psi\big[\psi(\rho(y,y_n)) + \psi(\rho(x_n,y_n))\big]\end{aligned}$$

and

$$\begin{aligned}\rho(x_n,y_n) &\leq \psi(\rho(x_n,x)) + \psi(\rho(y_n,x))\\ &\leq \psi(\rho(x_n,x)) + \psi\big[\psi(\rho(y_n,y)) + \psi(\rho(x,y))\big].\end{aligned}$$

Taking the lower limit as $n \to +\infty$ in the first inequality and the upper limit as $n \to +\infty$ in the second inequality we obtain the desired result.

2. Using (ρ_3) we see that

$$\rho(x,z) \leq \psi(\rho(x,x_n)) + \psi(\rho(x_n,z)).$$

Taking the lower limit as $n \to +\infty$ in the above inequality we have

$$\rho(x,z) \leq \psi(\liminf_{n\longrightarrow+\infty} \rho(x_n,z)),$$

hence

$$\psi^{-1}(\rho(x,z)) \leq \liminf_{n\longrightarrow+\infty} \rho(x_n,z).$$

Also

$$\rho(x_n,z) \leq \psi(\rho(x_n,x)) + \psi(\rho(z,x)).$$

Taking the the upper limit as $n \to +\infty$ in the above inequality we obtain the desired result. □

Let f and g be two self-mappings on X. A point $x \in X$ is called

(1) a fixed point of f if $f(x) = x$ (fixed point equation);
(2) coincidence point of a pair (f, g) if $fx = gx$ (coincidence point equation).

Notice that solving fixed point equation and coincidence point equations in certain cases is equivalent to solving complementarity and implicit complementarity problems respectively [17].

In this paper, we consider the existence and uniqueness of a fixed point of certain mapping that provides a nonlinear F-contraction in the framework of expanded b-metric space. We shall also express an example to indicate the validity of the presented results.

2. THE MAIN RESULTS

We start with the following useful lemma.

Lemma 2.1. *Let (X, ρ, ψ) be a expanded b-metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to+\infty} \rho(x_n, y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} x_n = t$ for some $t \in X$, then $\lim_{n\to+\infty} y_n = t$.*

Proof. By the triangle inequality in expanded b-metric space, we have

$$\rho(y_n, t) \leq \psi(\rho(y_n, x_n)) + \psi(\rho(x_n, t)).$$

Now, by taking the upper limit when $n \to +\infty$ in above inequality we get

$$\limsup_{n\longrightarrow+\infty} \rho(y_n, t) \leq \psi(\limsup_{n\longrightarrow+\infty} \rho(x_n, y_n)) + \psi(\limsup_{n\longrightarrow+\infty} \rho(x_n, t)) = 0.$$

Hence $\lim_{n\to+\infty} y_n = t$. □

Now, we give the definition of (ϕ, F)-contraction in the setting of expanded b-metric space.

Two mappings $A, B : X \longrightarrow X$ is said to be a (ϕ, F)-contraction (or nonlinear F-contraction) if there exist the functions $F : (0, +\infty) \longrightarrow \mathbb{R}$ and $\phi : (0, +\infty) \longrightarrow (0, +\infty)$ satisfying

(H1) for all $t_1, t_2 > 0, t_1 > t_2$ implies $F(t_1) > F(t_2)$;
(H2) for any sequence $\{t_n\} \subset (0, +\infty), t_n \longrightarrow 0$ if and only if $F(t_n) \longrightarrow -+\infty$;
(H3) $\liminf_{s\longrightarrow t^+} \phi(s) > 0$ for all $t \geq 0$;
(H4) $\phi(\rho(Bx, By)) + F(\psi(\rho(Ax, Ay))) \leq F(\rho(Bx, By))$ for all $x, y \in X$ such that $Ax \neq Ay$.

Now, by taking as $\psi(t) = t$ and $B = I$(Identity map) we give the definition of (ϕ, F)-contraction in the setting of metric space see e.g. Dariusz and Wardowski [11] and also [1, 3, 12, 13, 14].

First we shall announce a coincidence point result concerning nonlinear F-contractions.

Theorem 2.1. *Let (X, ρ, ψ) be a complete expanded b-metric space and let $A, B : X \longrightarrow X$ be (ϕ, F)-contraction such that $A(X) \subseteq B(X)$.*

(1) *If B is surjective then A, B have a coincidence point.*
(2) *If B is bijection then A, B have a unique coincidence point.*

Proof. Take any $x_0, x_1 \in X$ such that $y_0 = Ax_0 = Bx_1$ and define the sequence $y_n = Ax_n = Bx_{n+1}$, $n = 0, 1, 2, \cdots$. Denote the sequence $\gamma_n = \rho(y_{n-1}, y_n), n \in \mathbb{N}$. If there exist $n_0 \in \mathbb{N}$ such that $\gamma_{n_0} = 0$ then $y_{n_0} = y_{n_0+1}$, that is $Bx_{n_0+1} = Ax_{n_0+1}$ hence A, B have a coincidence point. Therefore without loss of generality we can assume that $\gamma_n > 0$ for all $n \in \mathbb{N}$. From (H4) we have

$$\begin{aligned} & F(\psi(\rho(y_n, y_{n+1}))) + \phi(\rho(y_{n-1}, y_n)) \\ = \ & F(\psi(\rho(Ax_n, Ax_{n+1}))) + \phi(\rho(Bx_n, Bx_{n+1})) \\ \leq \ & F(\rho(Bx_n, Bx_{n+1})) = F(\rho(y_{n-1}, y_n)) \ for \ all \ n \in \mathbb{N}. \end{aligned}$$

That is

$$F(\gamma_{n+1}) \leq F(\psi(\gamma_{n+1})) \leq F(\gamma_n) - \phi(\gamma_n),$$

hence

$$F(\gamma_n) \leq F(\gamma_{n-1}). \tag{1}$$

From the inequality (2.1) and from (H1) we get that γ_n is decreasing, and hence, $\gamma_n \longrightarrow t$, for $t \geq 0$. From (H3) there exists $c > 0$ and $n_0 \in \mathbb{N}$ such that $\phi(\gamma_n) > c$ for all $n \geq n_0$. In consequence, we have

$$\begin{aligned} F(\gamma_{n+1}) &\leq F(\psi(\gamma_{n+1})) = F(\psi(\rho(y_n, y_{n+1}))) \\ &\leq F(\rho(y_{n-1}, y_n)) - \phi(\rho(y_{n-1}, y_n)) \\ &\leq F(\psi(\rho(y_{n-1}, y_n))) - \phi(\rho(y_{n-1}, y_n)) \\ &\leq F(\rho(y_{n-2}, y_{n-1})) - \phi(\rho(y_{n-2}, y_{n-1})) - \phi(\rho(y_{n-1}, y_n)) \\ &\vdots \\ &\leq F(\rho(y_0, y_1)) - \sum_{i=1}^{n-1} \phi(\rho(y_{i-1}, y_i)) \\ &= F(\gamma_1) - \sum_{i=1}^{n-1} \phi(\gamma_i). \end{aligned}$$

Therefore:

$$\begin{aligned} F(\gamma_n) &\leq F(\gamma_{n-1}) - \phi(\gamma_{n-1}) \leq \cdots \leq F(\gamma_1) - \sum_{i=1}^{n-1} \phi(\gamma_i) \\ &= F(\gamma_1) - \sum_{i=1}^{n_0-1} \phi(\gamma_i) - \sum_{i=n_0}^{n-1} \phi(\gamma_i) < F(\gamma_1) - (n - n_0)c, \ n > n_0. \end{aligned}$$

Tending with $n \longrightarrow +\infty$ we get $F(\gamma_n) \longrightarrow - + \infty$ and, by $(H2)$, $\gamma_n \longrightarrow 0$. To show that $\{y_n\}$ is the Cauchy sequence. Suppose on the contrary that $\{y_n\}$ is not Cauchy. From $(H1)$ the set Δ of all discontinuity points of F is at most countable. There exists $\varepsilon > 0$, $\psi(\varepsilon) \notin \Delta$ such that for every $k \geq 0$ we can find two subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that n_k is the smallest index for which

$$k \leq m_k < n_k \ and \ \rho(y_{m_k}, y_{n_k}) > \varepsilon. \tag{2}$$

This means that

$$\rho(y_{m_k}, y_{n_k-1}) \leq \varepsilon. \tag{3}$$

By (H4) we have

$$F(\psi(\rho(Ax_{m_k}, Ax_{n_k}))) \leq F(\rho(Bx_{m_k}, Bx_{n_k})) - \phi(\rho(Bx_{m_k}, Bx_{n_k})).$$

Hence

$$F(\psi(\rho(y_{m_k}, y_{n_k}))) \leq F(\rho(y_{m_k-1}, y_{n_k-1})),$$

therefore,

$$\psi(\rho(y_{m_k}, y_{n_k})) \leq \rho(y_{m_k-1}, y_{n_k-1}).$$

From the rectangular inequality, we get

$$\begin{aligned}
\psi(\varepsilon) &< \psi(\rho(y_{m_k}, y_{n_k})) \leq \rho(y_{m_k-1}, y_{n_k-1}) \\
&\leq \psi(\rho(y_{m_k-1}, y_{m_k})) + \psi(\rho(y_{m_k}, y_{n_k-1})) \\
&\leq \psi(\gamma_{m_k}) + \psi(\varepsilon).
\end{aligned}$$

Taking the limit as $k \to +\infty$, we get

$$\lim_{k\to+\infty} \psi(\rho(y_{m_k}, y_{n_k})) = \lim_{k\to+\infty} \rho(y_{m_k-1}, y_{n_k-1}) = \psi(\varepsilon), \tag{4}$$

since ψ is continuous, we get

$$\lim_{k\to+\infty} \rho(y_{m_k}, y_{n_k}) = \varepsilon. \tag{5}$$

Finally observe that from (H4) we get

$$\begin{aligned}
\phi(\rho(y_{m_k-1}, y_{n_k-1}) &= \phi(\rho(Bx_{m_k}, Bx_{n_k})))) \\
&\leq F(\rho(Bx_{m_k}, Bx_{n_k})) - F(\psi(\rho(Ax_{m_k}, Ax_{n_k}))) \\
&= F(\rho(y_{m_k-1}, y_{n_k-1})) - F(\psi(\rho(y_{m_k}, y_{n_k}))), k \geq 0.
\end{aligned}$$

Now, from the above inequality, using (2.4), (2.5) and the fact that F is continuous at $\psi(\varepsilon)$ one gets

$$\begin{aligned}
\liminf_{s\longrightarrow\psi(\varepsilon)^+} \phi(s) &\leq \liminf_{k\longrightarrow+\infty} \phi(\rho(y_{m_k-1}, y_{n_k-1})) \\
&\leq \lim_{k\longrightarrow+\infty} [F(\rho(y_{m_k-1}, y_{n_k-1})) - F(\psi(\rho(y_{m_k}, y_{n_k})))] \\
&= F(\psi(\varepsilon)) - F(\psi(\varepsilon)) = 0,
\end{aligned}$$

which contradicts (H3). Therefore $\{y_n\}$ is Cauchy. The completeness of X there exist $u \in X$ such that

$$\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Bx_{n+1} = u.$$

Since B is surjective there exist $v \in X$ such that $Bv = u$. Now, we show that $Av = u$. Suppose on the contrary that $Av \neq u$. We can find a subsequence $\{y_{m_k}\}$ of $\{y_n\}$ such that $Av \neq y_{m_k} = Ax_{m_k}$. For $x = v, y = x_{m_k}$, from $(H4)$ we have

$$\begin{aligned}
F(\psi(\rho(Av, Ax_{m_k}))) &\leq F(\rho(Bv, Bx_{m_k})) - \phi(\rho(Bv, Bx_{m_k})) \\
&\leq F(\rho(Bv, Bx_{m_k})).
\end{aligned}$$

That is

$$\psi(\rho(Av, Ax_{m_k})) \leq \rho(Bv, Bx_{m_k}).$$

Tending $k \to +\infty$ by Lemma 1.4 we get

$$\begin{aligned}
\rho(Av, u) &= \psi(\psi^{-1}(\rho(Av, u))) \leq \limsup_{k\longrightarrow+\infty} \psi(\rho(Av, Ax_{m_k})) \\
&\leq \limsup_{k\longrightarrow+\infty} \rho(Bv, Bx_{m_k}) \\
&\leq \psi(\rho(Bv, u)) = \psi(0) = 0.
\end{aligned}$$

Hence $\rho(Av, u) = 0$, that is A, B have a coincidence point.

Now, let B is bijection and let there exist $u, v \in X$ such that $Av = Bv$ and $Au = Bu$. To show that $Av = Au$. Suppose on the contrary that $Av \neq Au$. For $x = v, y = u$, from (H4) we get

$$\begin{aligned} F(\psi(\rho(Av, Au))) &\leq F(\rho(Bv, Bu)) - \phi(\rho(Bv, Bu)) \\ &< F(\rho(Bv, Bu)) = F(\rho(Av, Au)). \end{aligned}$$

That is

$$\psi(\rho(Av, Au)) < \rho(Av, Au),$$

which contradiction. Therefore, $Av = Au$ so $Bv = Bu$, since B is one to one it follows that $u = v$. That is A, B have a unique coincidence point. □

For every $b \geq 1$, taking $\psi(t) = bt$ in Theorem 2.1, one obtains the following Corollary.

Corollary 2.1. *Let (X, d) be a complete b-metric space for $b \geq 1$ and let $A, B : X \longrightarrow X$ be (ϕ, F)-contraction such that $A(X) \subseteq B(X)$.*

(1) *If B is surjective then A, B have a coincidence point.*
(2) *If B is bijection then A, B have a unique coincidence point.*

Corollary 2.2. *Let (X, ρ, ψ) be a complete expanded b-metric space and let $A : X \longrightarrow X$ be a (ϕ, F)-contraction, then A has unique fixed point.*

Proof. It is enough set $B = I$ identity map in Theorem 2.1. □

Corollary 2.3. *Let (X, ρ, ψ) be a complete expanded b-metric space and let $T : X \longrightarrow X$. Suppose that there exists $\lambda \in]0, 1[$ such that*

$$\psi(\rho(Tx, Ty)) \leq \lambda\rho(x, y) \ for\ all\ x, y \in X.$$

Then T has unique fixed point.

Proof. The result follows from Corollary 2.2, by taking as functions

$$F(t) = Ln(t) \ and \ \phi(s) = -Ln(\lambda),$$

for all $t, s > 0$. □

Example 2.1. *Let $X = [0, +\infty)$ and $\rho : X \times X \longrightarrow \mathbb{R}$ be defined by $\rho(x, y) = \sinh|x - y|$. If define $\psi(t) = \sinh(2t)$, then (X, ρ, ψ) is a complete expanded b-metric space. Define a mapping $T : X \longrightarrow X$ by $Tx = \frac{1}{4}\sinh^{-1}(x)$. By Lemma 1.1, for all $x, y \in X$ with $\frac{1}{2} \leq q < 1$, we have*

$$\begin{aligned} \psi(\rho(Tx, Ty)) &= \sinh(2\rho(Tx, Ty)) \\ &= \sinh(2(\sinh|\frac{1}{4}(\sinh^{-1}(x) - \sinh^{-1}(y))|)) \\ &\leq \sinh(2(\frac{1}{4}\sinh(|\sinh^{-1}(x) - \sinh^{-1}(y)|))) \\ &\leq \sinh(\frac{1}{2}(\sinh(\sinh^{-1}(|x - y|)))) \\ &\leq \frac{1}{2}\sinh(|x - y|) \\ &\leq q\rho(x, y). \end{aligned}$$

Hence, since all the conditions of Corollary 2.3 are satisfied, then T has a unique fixed point $x = 0$.

The following corollary give Theorem of Dariusz Wardowski [11].

Corollary 2.4. *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a (ϕ, F)-contraction. Then T has unique fixed point.*

Proof. The result follows from Corollary 2.2, by taking as $\psi(t) = t$. □

Acknowledgement. The authors would like to extend their gratitude to their universities.

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