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LOCAL DISTANCE IRREGULAR LABELING OF GRAPHS

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ABSTRACT. We introduce the notion of distance irregular labeling, called the local distance irregular labeling. We define $\lambda : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that the weight calculated at the vertices induces a vertex coloring if $w(u) \neq w(v)$ for any edge uv. The weight of a vertex $u \in V(G)$ is defined as the sum of the labels of all vertices adjacent to u (distance 1 from u), that is $w(u) = \Sigma_{y \in N(u)} \lambda(y)$. The minimum cardinality of the largest label over all such irregular assignment is called the local distance irregularity strength, denoted by $dis_l(G)$. In this paper, we found the lower bound of the local distance irregularity strength of graphs G and also exact values of some classes of graphs namely path, cycle, star graph, complete graph, (n, m)-tadpole graph, unicycle with two pendant, binary tree graph, complete bipartite graphs, sun graph.

Keywords: Distance irregularity labeling, local distance irregularity strength, some families graph.

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1. INTRODUCTION

We consider that all graphs in this paper are finite, simple and connected graph, for detail definition of graph see [6, 2]. There are many types of graph labeling in Gallian [10]. One of graph labeling is magic labeling that was introduced by Sedlacek [11]. Suppose G is magic if the edges of G can be labeled by $1, 2, \ldots, |E(G)|$ so that the sum of the labels of all the edges incident with any vertex is the same. Hartsfield and Ringel [7] introduced the concept of antimagic labeling of a graph. There are a lot of results regarding to antimagic labeling, some of them can be found in Dafik *et. al* [5, 3, 4]. They determined super edge-antimagic total labelings of $mK_{n,n}$ and super edge-antimagicness for a class of disconnected graphs, respectively. Furthermore, Bača *et. al* in [1, 9] found the antimagicness of disjoint union of isomorphism graphs.

The distance $d(u, v)$ is the minimum of the length of the $u-v$ path of G. For a connected graph G of diameter k, a distance-d graph G_d for $d = 1, 2, \ldots, k$ is a graph with the same vertex set $V(G)$ and the edge set consists of the pairs of vertices that lie at distance d apart. Slamin [8] was introduced the distance irregular labeling of graphs. In this labeling, the weight of a vertices in G, is the sum of the labels of all vertices adjacent to u (distance 1 from u), that is $w(u) = \Sigma_{y \in N(u)} \lambda(y)$. The distance irregular labeling of G, denoted by $dis(G)$, is the minimum cardinality of the largest label k over all such irregular assignment.

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The coloring problem started the four-color problem. We define a coloring of G to be an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. On the other hand, the chromatic number has been related with several parameters of graphs, and as a consequence, there exists different types of vertex coloring such as list coloring, total coloring, acyclic coloring, distance-k coloring, etc.

Observation 1.1. [8] *Let u and w be any two adjacent vertices in a connected graph G. If $N(u) - \{w\} = N(w) - \{u\}$, then the labels of u and w must be distinct, that is $\lambda(u) \neq \lambda(w)$.*

2. Results

In this paper, we combine the concept of distance irregular labeling and coloring problem. We introduce the notion of distance irregular labeling, called the local distance irregular labeling.

Definition 2.1. *Suppose $\lambda : V(G) \longrightarrow \{1, 2, \dots, k\}$ such that the weight calculated at the vertices induces a vertex coloring if $w(u) \neq w(v)$ for any edge uv. The weight of a vertex $u \in V(G)$ is defined as the sum of the labels of all vertices adjacent to u (distance 1 from u), that is $w(u) = \Sigma_{y \in N(u)} \lambda(y)$.*

Definition 2.2. *The minimum cardinality of the largest label over all such irregular assignment is called the local distance irregularity strength, denoted by $dis_l(G)$.*

In this paper, we find the new concept of distance irregular labeling namely local distance irregular labeling. We have found the lower bound of local distance irregular labeling and also determine the exact values of local distance irregular labeling of some classes graphs path, cycle, star graph, complete graph, (n, m)-tadpole graph, unicycle with two pendant, binary tree graph, complete bipartite graphs, sun graph in the following theorems.

Lemma 2.1. *Let G be a connected graph on $n \geq 4$ vertices with the chromatic number χ, the minimum degree δ and the maximum degree Δ and there is no vertex having identical neighbours, then we have the lower bound of the local distance irregular labeling of G is*

$$dis_l(G) \geq \lceil \frac{\chi + \delta - 1}{\Delta} \rceil$$

Proof. The smallest weight possible of vertices in G is δ. Since, the weight of vertices in G induces vertex coloring and there are n vertices, then the largest weight of vertices in G is at least $\chi + \delta - 1$. This weight obtained from the sum of at most the maximum degree Δ. Thus, the largest label is at least $\lceil \frac{\chi+\delta-1}{\Delta} \rceil$. □

Theorem 2.1. *Let P_n be a path graph with order $n \geq 3$, then we have the local distance irregular labeling of P_n is*

$$dis_l(P_n) = \begin{cases} 1, & \text{if } n = 3 \\ 2, & \text{if } n \geq 4 \end{cases}$$

Proof. The path P_n is a tree graph with n vertices. The vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1}; 1 \leq i \leq n-1\}$. We have $\Delta(P_n) = 2$, $\delta(P_n) = 1$ and $\chi_(P_n) = 2$. The cardinality of vertex set and edge set, respectively are $|V(P_n)| = n$ and $|E(P_n)| = n - 1$. This proof divided into two cases are as follows.
Case 1: For $n = 3$, we give the label of the vertices $\lambda(v_1) = \lambda(v_2) = \lambda(v_3) = 1$. Then, we have the weight of vertices in P_3 as follows

$$w(v_1) = w(v_3) = 1 \qquad\qquad w(v_2) = 2$$

It is clear that the labeling provide different weight of two adjacent vertices such that we obtain $dis_l(P_3) \leq 1$. It concludes that $dis_l(P_3) = 1$.

Case 2: For $n \geq 4$, Based on Lemma 2.1 that the lower bound of local distance irregular labeling of path graph P_n is $dis_l(P_n) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{2+1-1}{2} \rceil = 1$. However, we can not attain the sharpest lower bound. If all vertices assigned label 1, then there is at least two adjacent vertices which have the same weight namely $w(v_3) = \lambda(v_2) + \lambda(v_4) = 1 + 1 = 2$ and $w(v_4) = \lambda(v_3) + \lambda(v_5) = 1 + 1 = 2$

so that $w(v_3) = w(v_4)$ for edge $v_3v_4 \in E(P_n)$. Hence, the lower bound of the distance irregular coloring of P_n is $dis_l(P_n) \geq 2$. Furthermore, we prove that the upper bound of the local distance irregular labeling of P_n is $dis_l(P_n) \leq 2$. Suppose the label vertices for n is even using the formula

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i,\ i \equiv 1,2,3(\text{mod}4) \\ 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = v_1 \text{ and } v = v_n \\ 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}2) \\ 3, & \text{if } v = v_i,\ i \equiv 1(\text{mod}2) \end{cases}$$

The label vertices for $n \equiv 3(\text{mod}4)$ using formula

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_{n-2} \text{ and } v = v_i,\ i \equiv 1,2,3(\text{mod}4),\ 1 \leq i \leq n-3 \\ 2, & \text{if } v = v_{n-1},\ v = v_n \text{ and } v = v_i,\ i \equiv 0(\text{mod}4),\ 1 \leq i \leq n-3 \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = v_1 \\ 2, & \text{if } v = v_n \text{ and } v = v_i,\ i \equiv 0(\text{mod}2),\ 2 \leq i \leq n-3 \\ 3, & \text{if } v = v_{n-1} \text{ and } v = v_i,\ i \equiv 1(\text{mod}2),\ 2 \leq i \leq n-3 \\ 4, & \text{if } v = v_{n-2} \end{cases}$$

The label vertices for $n \equiv 1(\text{mod}4)$ and $n \neq 5$ using formula

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i,\ i \equiv 1,2,3(\text{mod}4),\ i \neq n-3 \\ 2, & \text{if } v = v_{n-3},\ v = v_n \text{ and } v = v_i,\ i \equiv 0(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = v_1 \\ 2, & \text{if } v = v_n \text{ and } v = v_i,\ i \equiv 0(\text{mod}2),\ 2 \leq i \leq n-3 \\ 3, & \text{if } v = v_{n-1} \text{ and } v = v_i,\ i \equiv 1(\text{mod}2),\ 2 \leq i \leq n-5 \\ 4, & \text{if } v = v_{n-2} \text{ and } v = v_{n-4} \end{cases}$$

It is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of P_n is $dis_l(P_n) \leq 2$. It concludes that $dis_l(P_n) = 2$. □

Theorem 2.2. *Let C_n be a cycle graph with order $n \geq 4$, then we have the local distance irregular labeling of C_n is*

$$dis_l(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$$

Proof. The cycle C_n is a connected graph with n vertices. The vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(C_n) = \{v_1v_n, v_iv_{i+1}; 1 \leq i \leq n-1\}$. We have $\Delta(C_n) = 2$, $\delta(C_n) = 2$ and $\chi_(C_n) = 2, n$ even; $\chi_(C_n) = 3, n$ odd.. The cardinality of vertex set and edge set, respectively are $|V(C_n)| = n$ and $|E(C_n)| = n$. This proof divided into two cases are as follows.

Case 1: For n is even

Based on Lemma 2.1 that the lower bound of local distance irregular labeling of cycle graph C_n is $dis_l(C_n) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{2+2-1}{2} \rceil = 2$. However, we attain the sharpest lower bound. Furthermore, we prove that the upper bound of the local distance irregular labeling of C_n is $dis_l(C_n) \leq 2$. Suppose the label vertices for $n \equiv 0(\text{mod}4)$ is even using the formula

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i,\ i \equiv 1,2,3(\text{mod}4) \\ 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}2) \\ 3, & \text{if } v = v_i,\ i \equiv 1(\text{mod}2) \end{cases}$$

The label vertices for $n \equiv 2(\text{mod}4)$ using formula

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i,\ i \equiv 1,2,3(\text{mod}4),\ i \neq n \\ 2, & \text{if } v = v_n \text{ and } v = v_i,\ i \equiv 0(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}2) \\ 3, & \text{if } v = v_i,\ i \equiv 1(\text{mod}2),\ i \neq n-1 \\ 4, & \text{if } v = v_{n-1} \end{cases}$$

It is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of C_n is $dis_l(C_n) \leq 2$. It concludes that $dis_l(C_n) = 2$.

Case 2:For n is odd.
Based on Lemma 2.1 that the lower bound of local distance irregular labeling of cycle graph C_n is $dis_l(C_n) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{3+2-1}{2} \rceil = 2$. However, we can not attain the sharpest lower bound. Suppose to contrary that C_n has 2 label vertices then there exists some possible value as follows

- For $n \equiv 1(\text{mod}4)$, We may be construct the label vertices with a periodical irregular assignment $1,1,1,2,1,1,1,2,\ldots,1,1,1,2$ of path P_{n-1} except the last vertices assigned with 1 or 2. If the last vertices assigned with 1, then there is at least two adjacent vertices which have same weight namely $w(v_1) = w(v_2) = 2$. If the last vertices assigned with 2, then there is at least two adjacent vertices which have same weight namely $w(v_n) = w(v_{n-1}) = 3$. it is a contradict.
- For $n \equiv 3(\text{mod}4)$, We may be construct the label vertices with a periodical irregular assignment $1,1,1,2,1,1,1,2,\ldots,1,1,1,2,1,1$ of path P_{n-1} except the last vertices assigned with 1 or 2. If the last vertices assigned with 1, then there is at least two adjacent vertices which have same weight namely $w(v_n) = w(v_{n-1}) = 2$. If the last vertices assigned with 2, then there is at least two adjacent vertices which have same weight namely $w(v_{n-2}) = w(v_{n-1}) = 3$. it is a contradict.

Hence, the lower bound of the local distance irregular labeling of C_n is $dis_l(C_n) \geq 3$. Furthermore, we prove that the upper bound of the local distance irregular labeling of C_n is $dis_l(C_n) \leq 3$. Suppose the label vertices for $n \equiv 1(\text{mod}4)$ is even using the formula

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i,\ i \equiv 1,2,3(\text{mod}4),\ i \neq n \\ 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}4) \\ 3, & \text{if } v = v_n \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}2),\ 2 \leq i \leq n-2 \\ 3, & \text{if } v = v_i,\ i \equiv 1(\text{mod}2),\ 2 \leq i \leq n \\ 4, & \text{if } v = v_1 \text{ and } v = v_{n-1} \end{cases}$$

The label vertices for $n \equiv 3(\text{mod}4)$ using formula

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i,\ i \equiv 1,2,3(\text{mod}4),\ i \neq n \\ 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}4) \\ 3, & \text{if } v = v_n \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 2, & \text{if } v = v_n \text{ and } v = v_i,\ i \equiv 0(\text{mod}2),\ 2 \leq i \leq n-2 \\ 3, & \text{if } v = v_i,\ i \equiv 1(\text{mod}2),\ 2 \leq i \leq n-1 \\ 4, & \text{if } v = v_1 \text{ and } v = v_{n-1} \end{cases}$$

It is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of C_n is $dis_l(C_n) \leq 3$. It concludes that $dis_l(C_n) = 3$. □

Theorem 2.3. *Let K_n be a complete graph with order $n \geq 3$, then we have the local distance irregular labeling of K_n is $dis_l(K_n) = n$.*

Proof. Let K_n be a connected graph with order $n \geq 3$. The vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(K_n) = \{v_i v_{i+k}; 1 \leq i \leq n, 1 \leq k \leq n-i\}$. The cardinality of vertex set and edge set, respectively are $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$. Every vertices of K_n is adjacent to all other vertices. Suppose $u, v \in V(K_n)$ such that $N(u) - \{v\} = N(v) - \{u\}$, based on Observation 1.1 that $\lambda(u) \neq \lambda(v)$ then the labels of all vertices in K_n are distinct, consequently that $dis_l(K_n) \geq n$. We define the label vertices using formula $\lambda(v_i) = i$ for $1 \leq i \leq n$. This labeling provides vertex-weight as follows

$$w(v_i) = \frac{n(n+1)}{2} - i; 1 \leq i \leq n$$

It is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of K_n is $dis_l(K_n) \leq n$. It concludes that $dis_l(K_n) = n$. □

Theorem 2.4. *Let S_n be a star graph with order $n \geq 4$, then we have the local distance irregular labeling of S_n is $dis_l(S_n) = 1$.*

Proof. Let S_n be a connected graph with order $n \geq 4$. The vertex set $V(S_n) = \{v, v_1, v_2, \ldots, v_n\}$ and edge set $E(S_n) = \{vv_i; 1 \leq i \leq n\}$. The cardinality of vertex set and edge set, respectively are $|V(S_n)| = n + 1$ and $|E(S_n)| = n$. We have $\Delta(S_n) = n$, $\delta(S_n) = 1$ and $\chi_(P_n) = 2$. Based on Lemma 2.1 that the lower bound of local distance irregular labeling of star graph S_n is $dis_l(S_n) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{2+1-1}{n} \rceil = 1$. However, we attain the sharpest lower bound. Furthermore, we prove the upper bound of the local distance irregular labeling of star graph is $dis_l(S_n) \leq 1$. We define the label vertices using formula $\lambda(v) = \lambda(v_i) = 1$ for $1 \leq i \leq n$. This labeling provides vertex-weight such as $w(v) = n$ and $w(v_i) = 1$. Each leaf adjacent to central vertex and $w(v) \neq w(v_i)$. Therefore, it is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of S_n is $dis_l(S_n) \leq 1$. It concludes that $dis_l(S_n) = 1$. □

Theorem 2.5. *Let $T_{n,m}$ be a tadpole graph with order $n, m \geq 3$, then we have the local distance irregular labeling of $T_{n,m}$ is $dis_l(T_{n,m}) = 2$.*

Proof. The (m, n)-tadpole graph, also called a dragon graph, is the graph obtained by joining a cycle graph C_n to a path graph P_m with a bridge. The vertex set $V(T_{n,m}) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$ and edge set $E(T_{n,m}) = \{v_1 v_n, v_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{v_1 u_1, u_j u_{j+1}; 1 \leq j \leq m-1\}$. The cardinality of vertex set and edge set, respectively are $|V(T_{n,m})| = n+m$ and $|E(T_{n,m})| = n+m$. We have $\Delta(T_{n,m}) = n$, $\delta(T_{n,m}) = 1$ and $\chi_(T_{n,m}) = 3$. Based on Lemma 2.1 that the lower bound of local distance irregular labeling of (m, n)-tadpole graph $T_{n,m}$ is $dis_l(T_{n,m}) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{3+1-1}{n} \rceil = 2$. However, we attain the sharpest lower bound. Furthermore, we prove the upper bound of the local distance irregular labeling of $(n, m)-$tadpole graph $T_{n,m}$ is $dis_l(T_{n,m}) \leq 2$. We define the label vertices for $n \equiv 0(\text{mod}4)$ using formula.

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i \text{ and } v = u_j,\ i \equiv 1, 2, 3(\text{mod}4),\ j \equiv 0, 1, 2(\text{mod}4) \\ 2, & \text{if } v = v_i \text{ and } v = u_j,\ i \equiv 0(\text{mod}4),\ j \equiv 3(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = u_m \text{ for } m \equiv 1, 2, 3(\text{mod}4) \\ 2, & \text{if } v = v_i,\ v = u_j \text{ and } v = u_m,\ i \equiv 0(\text{mod}2),\ j \equiv 1(\text{mod}2) \text{ and } m \equiv 0(\text{mod}4) \\ 3, & \text{if } v = v_i \text{ and } v = u_j,\ i \equiv 1(\text{mod}2),\ j \equiv 0(\text{mod}2) \\ 4, & \text{if } v = v_1 \end{cases}$$

The label vertices for $n \equiv 3(\text{mod}4)$ using formula.

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i \text{ and } v = u_j,\ i, j \equiv 0, 1, 2(\text{mod}4) \\ 2, & \text{if } v = v_i \text{ and } v = u_j,\ i, j \equiv 0(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = u_m \text{ for } m \equiv 1, 2, 3(\text{mod}4) \\ 2, & \text{if } v = v_i,\ v = u_j \text{ and } v = u_m,\ i, j \equiv 1(\text{mod}2),\ i \neq 1 \text{ and } m \equiv 0(\text{mod}4) \\ 3, & \text{if } v = v_i \text{ and } v = u_j,\ i, j \equiv 0(\text{mod}2) \\ 4, & \text{if } v = v_1 \end{cases}$$

The label vertices for $n \equiv 1, 2(\text{mod}4)$ using formula.

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i \text{ and } v = u_j,\ i \equiv 1,2,3(\text{mod}4),\ j \equiv 0,2,3(\text{mod}4) \\ 2, & \text{if } v = v_i \text{ and } v = u_j,\ i \equiv 0(\text{mod}4),\ j \equiv 1(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = u_m \text{ for } m \equiv 1,3,0(\text{mod}4) \\ 2, & \text{if } v = v_i,\ v = u_j \text{ and } v = u_m,\ i \equiv 0(\text{mod}2),\ j \equiv 1(\text{mod}2) \text{ and } m \equiv 2(\text{mod}4) \\ 3, & \text{if } v = v_i \text{ and } v = u_j,\ i \equiv 1(\text{mod}2),\ j \equiv 0(\text{mod}2) \\ 5, & \text{if } v = v_1 \end{cases}$$

Therefore, it is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of $(n, m)-$tadpole graph $T_{n,m}$ is $dis_l(T_{n,m}) \leq 2$. It concludes that $dis_l(T_{n,m}) = 2$. □

Theorem 2.6. *Let C_n^2 be a unicycle with two pendants of order $n \geq 3$, then we have the local distance irregular labeling of C_n^2 is $dis_l(C_n^2) = 2$.*

Proof. Let C_n^2 be a unicycle with two pendants of order $n \geq 3$, the vertex set $V(C_n^2) = \{v_1, v_2, \ldots, v_n, u_1, u_2\}$ and edge set $E(C_n^2) = \{v_1v_n, v_iv_{i+1}, v_1u_1, v_2u_2; 1 \leq i \leq n-1\}$. The cardinality of vertex set and edge set, respectively are $|V(C_n^2)| = n + 2$ and $|E(C_n^2)| = n + 2$. We have $\Delta(C_n^2) = 3$, $\delta(C_n^2) = 1$ and $\chi_(C_n^2) = 3$. Based on Lemma 2.1 that the lower bound of local distance irregular labeling of C_n^2 is $dis_l(C_n^2) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{3+1-1}{3} \rceil = 1$. However, we can not attain the sharpest lower bound. If all vertices assigned label 1, then there is at least two adjacent vertices which have the same weight namely $w(v_3) = \lambda(v_2) + \lambda(v_4) = 1 + 1 = 2$ and $w(v_4) = \lambda(v_3) + \lambda(v_5) = 1 + 1 = 2$ so that $w(v_3) = w(v_4)$ for edge $v_3v_4 \in E(C_n^2)$. Hence, the lower bound of the local distance irregular labeling of C_n^2 is $dis_l(C_n^2) \geq 2$. Furthermore, we prove the upper bound of the local distance irregular labeling of C_n^2 is $dis_l(C_n^2) \leq 2$. We define the label vertices for $n \equiv 0(\text{mod}4)$ using formula.

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i,\ i \equiv 1,2,3(\text{mod}4) \\ 2, & \text{if } v = u_1,\ v = u_2 \text{ and } v = v_i,\ i \equiv 0(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = u_1 \text{ and } v = u_2 \\ 2, & \text{if } v = v_i,\ i \equiv 0(\text{mod}2),\ i \neq 2 \\ 3, & \text{if } v = v_i,\ i \equiv 1(\text{mod}2),\ i \neq 1 \\ 4, & \text{if } v = v_2 \\ 5, & \text{if } v = v_1 \end{cases}$$

The label vertices for $n \equiv 3(\text{mod}4)$ using formula.

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i,\ v = v_1 \text{ and } v = u_1,\ i \equiv 0,2,3(\text{mod}4) \\ 2, & \text{if } v = v_i \text{ and } v = u_2,\ i \equiv 1(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = u_1 \text{ and } v = u_2 \\ 2, & \text{if } v = v_i,\ i \equiv 1(\text{mod}2),\ i \neq 1 \\ 3, & \text{if } v = v_i \text{ and } v = v_1,\ i \equiv 0(\text{mod}2),\ i \neq 2 \\ 4, & \text{if } v = v_2 \end{cases}$$

The label vertices for $n \equiv 2(\text{mod}4)$ using formula.

$$\lambda(v) = \begin{cases} 1, & \text{if } v = v_i \text{ and } v = u_1,\ i \equiv 1,2,3(\text{mod}4) \\ 2, & \text{if } v = u_2 \text{ and } v = v_i,\ i \equiv 0(\text{mod}4) \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v)=\begin{cases}1, & \text{if } v=u_1 \text{ and } v=u_2\\ 2, & \text{if } v=v_i,\ i\equiv 0(\text{mod}2),\ i\neq 2\\ 3, & \text{if } v=v_i,\ i\equiv 1(\text{mod}2)\\ 4, & \text{if } v=v_2\end{cases}$$

The label vertices for $n\equiv 1(\text{mod}4)$ using formula.

$$\lambda(v)=\begin{cases}1, & \text{if } v=u_1,\ v=u_2 \text{ and } v=v_i,\ i\equiv 1,2,0(\text{mod}4)\\ 2, & \text{if } v=u_2 \text{ and } v=v_i,\ i\equiv 3(\text{mod}4)\end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v)=\begin{cases}1, & \text{if } v=u_1 \text{ and } v=u_2\\ 2, & \text{if } v=v_i,\ i\equiv 1(\text{mod}2),\ i\neq 1\\ 3, & \text{if } v=v_1 \text{ and } v=v_i,\ i\equiv 0(\text{mod}2)\\ 4, & \text{if } v=v_2\end{cases}$$

Therefore, it is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of C_n^2 is $dis_l(C_n^2)\leq 2$. It concludes that $dis_l(C_n^2)=2$. □

Theorem 2.7. *Let M_n be a sun graph with order $n\geq 3$, then we have the local distance irregular labeling of M_n is $dis_l(M_n)=2$.*

Proof. Let M_n be a unicycle with n pendants of order $n\geq 3$, the vertex set $V(M_n)=\{v_1,v_2,\dots,v_n,u_1,u_2,\dots,u_n\}$ and edge set $E(M_n)=\{v_1v_n,v_iv_{i+1};1\leq i\leq n-1\}\cup\{v_iu_i;1\leq i\leq n\}$. The cardinality of vertex set and edge set, respectively are $|V(M_n)|=2n$ and $|E(M_n)|=2n$. We have $\Delta(M_n)=3,\ \delta(M_n)=1$ and $\chi_(M_n)=3$. Based on Lemma 2.1 that the lower bound of local distance irregular labeling of M_n is $dis_l(M_n)\geq\lceil\frac{\chi+\delta-1}{\Delta}\rceil=\lceil\frac{3+1-1}{3}\rceil=1$. However, we can not attain the sharpest lower bound. If all vertices assigned label 1, then there is at least two adjacent vertices which have the same weight namely $w(v_3)=\lambda(v_2)+\lambda(v_4)=1+1=2$ and $w(v_4)=\lambda(v_3)+\lambda(v_5)=1+1=2$ so that $w(v_3)=w(v_4)$ for edge $v_3v_4\in E(C_n^2)$. Hence, the lower bound of the local distance irregular labeling of M_n is $dis_l(M_n)\geq 2$. Furthermore, we prove the upper bound of the local distance irregular labeling of M_n is $dis_l(M_n)\leq 2$. We define the label vertices for n is even using formula.

$$\lambda(v)=\begin{cases}1, & \text{if } v=u_i;\ i \text{ is odd and } v=v_i;\ 1\leq i\leq n\\ 2, & \text{if } v=u_i;\ i \text{ is even}\end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v)=\begin{cases}1, & \text{if } v=u_i,\ 1\leq i\leq n\\ 3, & \text{if } v=v_i,\ i \text{ is odd}\\ 4, & \text{if } v=v_i,\ i \text{ is even}\end{cases}$$

The label vertices for n is odd using formula.

$$\lambda(v)=\begin{cases}1, & \text{if } v=u_i;\ i \text{ is odd},\ i\neq 1 \text{ and } v=v_i;\ 1\leq i\leq n-1\\ 2, & \text{if } v=v_n,\ v=u_1 \text{ and } v=u_i;\ i \text{ is even}\end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v)=\begin{cases}1, & \text{if } v=u_i,\ 1\leq i\leq n-1\\ 2, & \text{if } v=u_n\\ 3, & \text{if } v=v_i,\ i \text{ is odd},\ i\neq 1\\ 4, & \text{if } v=v_i,\ i \text{ is even}\\ 5, & \text{if } v=v_1\end{cases}$$

Therefore, it is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of M_n is $dis_l(M_n)\leq 2$. It concludes that $dis_l(M_n)=2$. □

Theorem 2.8. *Let $T_{2,h}$ be a complete binary tree graph of height $h\geq 3$, then we have the local distance irregular labeling of $T_{2,h}$ is $dis_l(T_{2,h})=2$.*

Proof. Let $T_{2,h}$ be a tree graph with height $h \geq 3$, the vertex set $V(T_{2,h}) = \{u, u_k^i; 1 \leq i \leq h, 1 \leq k \leq 2^h\}$ and edge set $E(T_{2,h}) = \{uu_1^1, uu_2^1\} \cup \{u_k^i u_{2k-1}^{i+1}, u_k^i u_{2k-1}^{i+1}; 1 \leq i \leq h-1, 1 \leq k \leq 2^i\}$. The cardinality of vertex set and edge set, respectively are $|V(T_{2,h})| = 2^{h+1} - 1$ and $|E(T_{2,h})| = 2^{h+1} - 2$. We have $\Delta(T_{2,h}) = 3$, $\delta(T_{2,h}) = 1$ and $\chi_(T_{2,h}) = 2$. Based on Lemma 2.1 that the lower bound of local distance irregular labeling of $T_{2,h}$ is $dis_l(T_{2,h}) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{2+1-1}{3} \rceil = 1$. However, we can not attain the sharpest lower bound. If all vertices assigned label 1, then there is at least two adjacent vertices which have the same weight namely $w(v_1^1) = w(v_1^2)$ for edge $v_1^1 v_1^2 \in E(T_{2,h})$. Hence, the lower bound of the local distance irregular labeling of $T_{2,h}$ is $dis_l(T_{2,h}) \geq 2$. Furthermore, we prove the upper bound of the local distance irregular labeling of $T_{2,h}$ is $dis_l(T_{2,h}) \leq 2$. We define the label vertices for h odd using formula.

$$\lambda(v) = \begin{cases} 1, & \text{if } v = u \text{ and } v = u_k^i;\ i \text{ is even, } 1 \leq i \leq h,\ 1 \leq k \leq 2^i \\ 2, & \text{if } v = u_k^i;\ i \text{ is odd, } 1 \leq i \leq h,\ 1 \leq k \leq 2^i \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 1, & \text{if } v = u_k^h,\ 1 \leq k \leq 2^h \\ 3, & \text{if } v = u_k^i,\ i \text{ is odd, } 1 \leq i \leq h-1 \\ 4, & \text{if } v = u \\ 6, & \text{if } v = u_k^i,\ i \text{ is even, } 1 \leq i \leq h \end{cases}$$

The label vertices for h even using formula.

$$\lambda(v) = \begin{cases} 1, & \text{if } v = u \text{ and } v = u_k^i;\ i \text{ is even, } 1 \leq i \leq h,\ 1 \leq k \leq 2^i \\ 2, & \text{if } v = u_k^i;\ i \text{ is odd, } 1 \leq i \leq h,\ 1 \leq k \leq 2^i \end{cases}$$

This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 2, & \text{if } v = u_k^h,\ 1 \leq k \leq 2^h \\ 3, & \text{if } v = u_k^i,\ i \text{ is odd, } 1 \leq i \leq h \\ 4, & \text{if } v = u \\ 6, & \text{if } v = u_k^i,\ i \text{ is even, } 1 \leq i \leq h-1 \end{cases}$$

Therefore, it is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of $T_{2,h}$ is $dis_l(T_{2,h}) \leq 2$. It concludes that $dis_l(T_{2,h}) = 2$. □

Observation 2.1. *Let $K_{n,m}$ be k-partite graph with $n, m \geq 2$. If the vertices $u, v \in V(K_{n,m})$ in every partite class are not adjacent, then $N(u) = N(v)$ with u, v belong to the same partite class.*

Theorem 2.9. *Let $K_{m,n}$ be a complete bipartite graph of $n, m \geq 3$, then we have the local distance irregular labeling of $K_{m,n}$ is*

$$dis_l(K_{m,n}) = \begin{cases} 1, & \textit{if } n \neq m \\ 2, & \textit{if } n = m \end{cases}$$

Proof. Let $K_{m,n}$ be a complete bipartite graph of $n, m \geq 3$. The vertex set $V(K_{m,n}) = \{v_i, u_j; 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(K_{m,n}) = \{v_i u_j; 1 \leq i \leq n, 1 \leq j \leq m\}$. The cardinality of vertex set and edge set, respectively are $|V(K_{m,n})| = n + m$ and $|E(K_{m,n})| = nm$. We have $\Delta(K_{m,n}) = m$, $\delta(K_{m,n}) = n$ and $\chi_(K_{m,n}) = 2$. The proof divided into two cases as follows.

Case 1: For $n \neq m$ ($n < m$ or $n > m$), based on Lemma 2.1 that the lower bound of local distance irregular labeling of $K_{m,n}$ is $dis_l(K_{m,n}) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{2+n-1}{m} \rceil = 1$. However, we attain the sharpest lower bound. Furthermore, we prove the upper bound of the local distance irregular labeling of $K_{m,n}$ is $dis_l(K_{m,n}) \leq 1$. We define the label vertices using formula includes $\lambda(v_i) = \lambda(u_j) = 1$. This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} m, & \text{if } v = v_i,\ 1 \leq i \leq n \\ n, & \text{if } v = u_j,\ 1 \leq j \leq m \end{cases}$$

Therefore, it is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of $K_{m,n}$ is $dis_l(K_{m,n}) \leq 1$. It concludes that $dis_l(K_{m,n}) = 1$.

Case 2: For $n = m$, based on Lemma 2.1 that the lower bound of local distance irregular labeling of $K_{m,n}$ is $dis_l(K_{m,n}) \geq \lceil \frac{\chi+\delta-1}{\Delta} \rceil = \lceil \frac{2+m-1}{m} \rceil = 2$. However, we attain the sharpest lower bound. Furthermore, we prove the upper bound of the local distance irregular labeling of $K_{m,n}$ is $dis_l(K_{m,n}) \leq 2$. We define the label vertices using formula includes $\lambda(v_i) = 1$ and $\lambda(u_j) = 2$. This labeling provides vertex-weight as follows.

$$w(v) = \begin{cases} 2m, & \text{if } v = v_i,\ 1 \leq i \leq n \\ n, & \text{if } v = u_j,\ 1 \leq j \leq m \end{cases}$$

Therefore, it is easy to see that w is vertex coloring. Hence, we obtain the upper bound of the local distance irregular labeling of $K_{m,n}$ is $dis_l(K_{m,n}) \leq 2$. It concludes that $dis_l(K_{m,n}) = 2$. □

3. Conclusions

In this section, we conclude that local distance irregular labeling is a new invariant. We find the exact value of local distance irregular labeling of some families graph, namely path graph, cycle graph, star graph, complete graph, tadpole graph, unicycle with two pendant graph, binary tree graph, complete bipartite graph, and sun graph. There is some open problems for research in this area. We state a few interesting problems as follows.

Open Problem 3.1. *Determine the local distance irregular labeling for graph operation namely corona product, Cartesian product, joint, comb product and others.*

Open Problem 3.2. *Determine the local distance irregular labeling for some family graph namely unicycle, related wheel, tree graph and cayley graph.*

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