TWMS J. App. and Eng. Math. V.13, N.1, 2023, pp. 11-20

K-PRODUCT CORDIAL LABELING OF FAN GRAPHS

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Abstract. Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0, 1, ..., k-1)$. In this paper we prove that fan F_n and double fan DF_n when k=4 and 5 admit k-product cordial labeling.

Keywords: cordial labeling, product cordial labeling, k-product cordial labeling, 4-product cordial graph, 5-product cordial graph.

AMS Subject Classification: 05C78.

1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [4]. While studying graph theory, one that has gained a lot of popularity during the last 60 years is the concept of labelings of graphs due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [13] published a pioneering paper on graph labeling problems. Thereafter many types of graph labeling techniques have been studied by several authors. Gallian [2] in his survey beautifully classified the labelings into graceful labeling and harmonious labelings, variations of graceful labelings, variations of harmonious labelings, magic type labelings, anti-magic type labelings and miscellaneous labelings. Cordial labeling is a weaker version of graceful and harmonious labeling was introduced by Cahit in [1]. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$.

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§ Manuscript received: August 25, 2020; accepted: January 22, 2021.

f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Motivated by the concept of cordial labeling, Sundaram et al. introduced the concept of product cordial labeling in [14]. Let f be a function from $V(G)$ to $\{0,1\}$. For each edge uv, assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i = 0,1)$. Ponraj et al. extended the concept of product cordial labeling and introduced k-product cordial labeling in [12]. Let f be a map from $V(G)$ to $\{0,1,...,k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i,j \in \{0,1,...,k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0,1,...,k-1)$. They proved that k-product cordial labeling of stars, bistars and also 4-product cordial labeling behavior of paths, complete graphs and combs. Inspired by the results in [12], we further studied on k-product cordial labeling and showed that the following graphs admit k-product cordial labeling: union of graphs [6]; cone and double cone graphs [7]; powers of paths [8]; Napier bridge graphs [9]; the maximum number of edges in a 4-product cordial graph of order p is $4\lceil \frac{p-1}{4} \rceil \lfloor \frac{p-1}{4} \rfloor + 3$ [10] and product of graphs [11]. In this work we exhibit that fan F_n and double fan DF_n when k=4 and 5 admit k-product cordial labeling. A fan graph F_n [3], is obtained by joining all the vertices of P_n to a new vertex which is known as the center. The graph $P_n + 2K_1$ is called a double fan [5] denoted by DF_n.

2. Main Results

Theorem 2.1. *The fan F_n is a 4-product cordial graph if and only if $n = 1$ or 4 or 5 or 6 or 8 or 9 or 10 or 13 or 17.*

Proof. Let the vertex set and the edge set of F_n be $V(F_n) = \{v, v_i; 1 \leq i \leq n\}$ and $E(F_n) = \{(v, v_i); 1 \leq i \leq n\} \bigcup \{(v_i, v_{i+1}); 1 \leq i \leq n-1\}$ respectively. 4-product cordial labeling of F_1, F_4, F_5, F_6, F_8, F_9, F_{10}, F_{13} and F_{17} are shown in Table 1.

Table 1

n	v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}
1	3	0																
4	3	0	3	1	2													
5	3	0	3	2	1	1												
6	3	0	2	2	1	1	3											
8	3	0	0	2	3	1	1	3	2									
9	3	0	0	2	3	3	1	1	1	2								
10	3	0	0	2	3	3	1	1	1	2	2							
13	3	0	0	0	2	3	3	1	3	2	2	1	1	1				
17	3	0	0	0	0	2	2	3	3	1	1	3	1	1	1	2	3	2

From the above labeling pattern we have $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$ for all $i,j = 0,1,2,3$.

Conversely, we assume that F_n is a 4-product cordial graph.
Let f be a 4-product cordial labeling of F_n.

Case (i): If $n \equiv 0(mod\ 4)$ for $n > 8$. Let $n = 4t$, then $|V(F_n)| = 4t+1$ and $|E(F_n)| = 8t-1$. Thus, $v_f(i) = t$ *or* $t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 2t$ *or* $2t-1$ $(i = 0,1,2,3)$. Clearly, $f(v) \neq 0$. Obviously $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the ending of the path. Otherwise

$e_f(0) > 2t$ is not possible. Thus, $e_f(0) = 2t$ *or* $2t+1$. But $e_f(0)$ can not be $2t+1$. Therefore, $e_f(0) = 2t$. Now $v_f(2) = t$ *or* $t+1$. Suppose $v_f(2) = t$, $f(v) \neq 2$ and 2 must be assigned inconsecutively. Otherwise $e_f(0) > 2t$ is not possible. Then, $3t - 2 \leq e_f(2) \leq 3t$ for $t \geq 3$. We get a contradiction to f is a 4-product cordial labeling. The similar argument shows that $v_f(2)$ can neither be $t+1$. Hence, F_n is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$ for $n > 8$.

Case (ii): If $n \equiv 1(mod\ 4)$ for $n > 17$. Let $n = 4t+1$, then $|V(F_n)| = 4t+2$ and $|E(F_n)| = 8t+1$. Thus, $v_f(i) = t$ *or* $t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 2t$ *or* $2t+1$ $(i = 0,1,2,3)$. Clearly, $f(v) \neq 0$. Obviously $v_f(0) = t$. Otherwise $e_f(0) > 2t+1$ is not possible. We assign 0 to the vertices of the path in such a way that $e_f(0) = 2t$ *or* $2t+1$. If $e_f(0) = 2t$. Now $v_f(2) = t$ *or* $t+1$. Suppose $v_f(2) = t$, $f(v) \neq 2$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 2t+1$ is not possible. Then, $3t - 4 \leq e_f(2) \leq 3t$ for $t \geq 5$. We get a contradiction to f is a 4-product cordial labeling. The similar argument shows that $v_f(2)$ can neither be $t+1$. $e_f(0) = 2t+1$ can be dealt with on similar lines. Hence, F_n is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n > 17$.

Case (iii): If $n \equiv 2(mod\ 4)$ for $n = 2$ and $n > 10$. Let $n = 4t+2$, then $|V(F_n)| = 4t+3$ and $|E(F_n)| = 8t+3$. Thus, $v_f(i) = t$ *or* $t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 2t$ *or* $2t+1$ $(i = 0,1,2,3)$. For $n = 2$, $v_f(0) = 0$. Otherwise $e_f(0) > 1$ is not possible. Now $v_f(2) = 1$. Then, we have $e_f(2) > 1$. we get a contradiction to f is a 4-product cordial labeling. Hence, F_n is not a 4-product cordial graph if $n = 2$. For $n > 10$, $f(v) \neq 0$. Obviously $v_f(0) = t$. Otherwise $e_f(0) > 2t+1$ is not possible. We assign 0 to the vertices of the path in such a way that $e_f(0) = 2t$ *or* $2t+1$. If $e_f(0) = 2t$. Then $v_f(2) = t+1$. Clearly, $f(v) \neq 2$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 2t+1$ is not possible. Then, $3t - 1 \leq e_f(2) \leq 3t+3$ for $t \geq 3$. We get a contradiction to f is a 4-product cordial labeling. The similar argument shows that $e_f(0)$ can neither be $2t+1$. Hence, F_n is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n = 2$ and $n > 10$.

Case (iv): If $n \equiv 3(mod\ 4)$ for $n \geq 3$. Let $n = 4t+3$, then $|V(F_n)| = 4t+4$ and $|E(F_n)| = 8t+5$. Thus, $v_f(i) = t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 2t+1$ *or* $2t+2$ $(i = 0,1,2,3)$. Clearly, $f(v) \neq 0$. Obviously $v_f(0) = t+1$ and 0 must be assigned consecutively at the beginning or at the ending of the path. Otherwise $e_f(0) > 2t+2$ is not possible. Thus, $e_f(0) = 2t+2$. Now $v_f(2) = t+1$. Clearly, $f(v) \neq 2$ and 2 must be assigned inconsecutively. Otherwise $e_f(0) > 2t+2$ is not possible. Then, $3t+1 \leq e_f(2) \leq 3t+3$ for $t \geq 0$. We get a contradiction to f is a 4-product cordial labeling. Hence, F_n is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 3$. □

An example of 4-product cordial labeling of F_9 is shown in Figure 1.

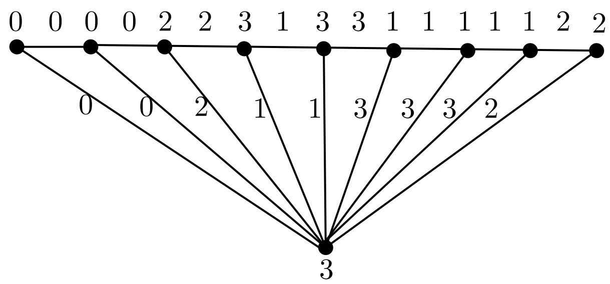

Figure 1: 4-product cordial labeling of F_9

In the next result we prove that the double fan DF_n is a 4-product cordial graph if and only if $n = 1$ *or* 4 *or* 8.

Theorem 2.2. *The double fan DF_n is a 4-product cordial graph if and only if $n = 1$ or 4 or 8.*

Proof. Let the vertex set and the edge set of DF_n be $V(DF_n) = \{u, v, v_i; 1 \leq i \leq n\}$ and $E(DF_n) = \{(u, v_i), (v, v_i); 1 \leq i \leq n\} \bigcup \{(v_i, v_{i+1}); 1 \leq i \leq n-1\}$ respectively. 4-product cordial labeling of DF_1, DF_4 and DF_8 are shown in Table 2.

Table 2

n	v	u	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1	0	1	2							
4	3	1	0	2	3	1				
8	3	1	0	0	2	3	1	1	3	2

From the above labeling pattern we have $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2, 3$.

Conversely, we assume that DF_n is a 4-product cordial graph.
Let f be a 4-product cordial labeling of DF_n.

Case (i): If $n \equiv 0(mod\ 4)$ for $n > 8$. Let $n = 4t$, then $|V(DF_n)| = 4t + 2$ and $|E(DF_n)| = 12t - 1$. Thus, $v_f(i) = t\ or\ t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t\ or\ 3t - 1$ $(i = 0, 1, 2, 3)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. Obviously $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the ending of the path. Otherwise $e_f(0) > 3t$ is not possible. Thus, $e_f(0) = 3t\ or\ 3t + 1$. But $e_f(0)$ can not be $3t + 1$. Therefore, $e_f(0) = 3t$. Now $v_f(2) = t\ or\ t + 1$. Suppose $v_f(2) = t$, $f(v) \neq 2$, $f(u) \neq 2$ and 2 must be assigned inconsecutively. Otherwise $e_f(0) > 3t$ is not possible. Then, $4t - 2 \leq e_f(2) \leq 4t$ for $t \geq 3$. We get a contradiction to f is a 4-product cordial labeling. The similar argument shows that $v_f(2)$ can neither be $t + 1$. Hence, DF_n is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$ for $n > 8$.

Case (ii): If $n \equiv 1(mod\ 4)$ for $n \geq 5$. Let $n = 4t+1$, then $|V(DF_n)| = 4t+3$ and $|E(DF_n)| = 12t+2$. Thus, $v_f(i) = t\ or\ t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t\ or\ 3t + 1$ $(i = 0, 1, 2, 3)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. Obviously $v_f(0) = t$. Otherwise $e_f(0) > 3t + 1$ is not possible. We assign 0 to the vertices of the path in such a way that $e_f(0) = 3t\ or\ 3t + 1$. If $e_f(0) = 3t$, then $v_f(2) = t + 1$. Clearly, $f(v) \neq 2$, $f(u) \neq 2$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 3t + 1$ is not possible. Then, $4t \leq e_f(2) \leq 4t + 4$ for $t \geq 1$. We get a contradiction to f is a 4-product cordial labeling. The similar argument shows that $e_f(0)$ can neither be $3t + 1$. Hence, DF_n is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n \geq 5$.

Case (iii): If $n \equiv 2(mod\ 4)$ for $n \geq 2$. Let $n = 4t+2$, then $|V(DF_n)| = 4t+4$ and $|E(DF_n)| = 12t+5$. Thus, $v_f(i) = t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t + 1\ or\ 3t + 2$ $(i = 0, 1, 2, 3)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. Obviously $v_f(0) = t + 1$ then $e_f(0) > 3t + 2$ for $t \geq 0$. We get a contradiction to f is a 4-product cordial labeling. Hence, DF_n is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n \geq 2$.

Case (iv): If $n \equiv 3(mod\ 4)$ for $n \geq 3$. Let $n = 4t+3$, then $|V(DF_n)| = 4t+5$ and $|E(DF_n)| = 12t+8$. Thus, $v_f(i) = t + 1\ or\ t + 2$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t + 2$ $(i = 0, 1, 2, 3)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t + 1,\ or\ t + 2$, then $e_f(0) > 3t + 2$ for $t \geq 0$. We get a contradiction to f is a 4-product cordial labeling. Hence, DF_n is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 3$. □

An example of 4-product cordial labeling of DF_8 is shown in Figure 2.

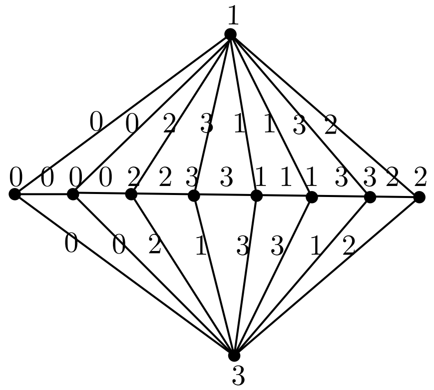

Figure 2: 4-product cordial labeling of DF_8

Theorem 2.3. *The fan F_n is a 5-product cordial graph for all $n \geq 1$ except $n = 3$.*

Proof. Let the vertex set and the edge set of F_n be $V(F_n) = \{v, v_i; 1 \leq i \leq n\}$ and $E(F_n) = \{(v, v_i); 1 \leq i \leq n\} \bigcup \{(v_i, v_{i+1}); 1 \leq i \leq n-1\}$ respectively. We consider the following six cases.

Define $f : V(F_n) \to \{0, 1, 2, 3, 4\}$ as follows:

Case (i): If $n \equiv 0(mod\ 5)$, then

$f(v) = 4,\ f(v_i) = 0\ ;\ 1 \leq i \leq \lfloor \frac{n}{5} \rfloor$.

Subcase (i): If n is odd.

Let $i = \lfloor \frac{n}{5} \rfloor + j\ ;\ 1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor$,

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,\ 6(mod\ 8) \\ 1 & \text{if } j \equiv 2,\ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

Subcase (ii): If n is even.

Let $i = \lfloor \frac{n}{5} \rfloor + j\ ;\ 1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor$,

For $n = 10$, $$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,\ 6(mod\ 8) \\ 4 & \text{if } j \equiv 2,\ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

For $n > 10$, $$f(v_i) = \begin{cases} 2 & \text{if } j \equiv 3(mod\ 4) \\ 3 & \text{if } j \equiv 0(mod\ 4). \end{cases}$$

For $1 \leq j \leq 8$, $$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1(mod\ 4) \\ 4 & \text{if } j \equiv 2(mod\ 4). \end{cases}$$

For $4\lfloor \frac{n}{5} \rfloor - 7 \leq j \leq 4\lfloor \frac{n}{5} \rfloor$, $$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1(mod\ 4) \\ 1 & \text{if } j \equiv 2(mod\ 4). \end{cases}$$

For $9 \leq j \leq 4\lfloor \frac{n}{5} \rfloor - 8$, $$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,\ 6(mod\ 8) \\ 1 & \text{if } j \equiv 2,\ 5(mod\ 8). \end{cases}$$

From the above cases we get

$v_f(0) + 1 = v_f(1) + 1 = v_f(2) + 1 = v_f(3) + 1 = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1$,

$e_f(0) = e_f(1) = e_f(2) = e_f(3) + 1 = e_f(4) = 2\lfloor \frac{n}{5} \rfloor$.

Case (ii): If $n \equiv 1(mod\ 5)$.
For $n = 1$, $f(v) = 1$ and $f(v_1) = 4$.
For $n > 1$,
Subcase (i): If n is even.
Assign labels to the vertices v and v_i $(1 \leq i \leq n-1)$ as in Case (i) Subcase (i), then assign 1 to v_n.
Subcase (ii): If n is odd.
Assign labels to the vertices v and v_i $(1 \leq i \leq n-1)$ as in Case (i) Subcase (ii), then assign 1 to v_n.
From this label we get
$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) + 1 = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1$,
$e_f(0) + 1 = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 = e_f(4) = 2\lfloor \frac{n}{5} \rfloor + 1$.
Case (iii): If $n \equiv 2(mod\ 5)$.
For $n = 2$, $f(v) = 1$, $f(v_1) = 4$ and $f(v_2) = 2$.
For $n > 2$,
Subcase (i): If n is odd.
Assign labels to the vertices v and v_i $(1 \leq i \leq n-2)$ as in Case (i) Subcase (i), then assign 1 and 2 to v_{n-1} and v_n respectively.
Subcase (ii): If n is even.
Assign labels to the vertices v and v_i $(1 \leq i \leq n-2)$ as in Case (i) Subcase (ii), then assign 1 and 2 to v_{n-1} and v_n respectively.
From this label we get
$v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) + 1 = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1$,
$e_f(0) + 1 = e_f(1) + 1 = e_f(2) = e_f(3) = e_f(4) = 2\lfloor \frac{n}{5} \rfloor + 1$.
Case (iv): If $n \equiv 3(mod\ 5)$ where $n > 3$, then
$f(v) = 4$, $f(v_{n-2}) = 1$, $f(v_{n-1}) = 2$, $f(v_n) = 3$, $f(v_i) = 0$; $1 \leq i \leq \lfloor \frac{n}{5} \rfloor - 1$
$f(v_{\lfloor \frac{n}{5} \rfloor + 2}) = 2$, $f(v_{\lfloor \frac{n}{5} \rfloor + 3}) = 3$, $f(v_{\lfloor \frac{n}{5} \rfloor + 4}) = 0$.
Let $i = \lfloor \frac{n}{5} \rfloor + 4 + j$; $1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor - 4$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,\ 6(mod\ 8) \\ 4 & \text{if } j \equiv 2,\ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

If n is even, $f(v_{\lfloor \frac{n}{5} \rfloor}) = 1$, $f(v_{\lfloor \frac{n}{5} \rfloor + 1}) = 4$.
If n is odd, $f(v_{\lfloor \frac{n}{5} \rfloor}) = 4$, $f(v_{\lfloor \frac{n}{5} \rfloor + 1}) = 1$.
Then, we have
$v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1$,
$e_f(0) = e_f(1) = e_f(2) = e_f(3) = e_f(4) = 2\lfloor \frac{n}{5} \rfloor + 1$.
Case (v): If $n \equiv 4(mod\ 5)$, then
$f(v) = 4$, $f(v_{n-2}) = 1$, $f(v_{n-1}) = 2$, $f(v_n) = 3$, $f(v_i) = 0$; $1 \leq i \leq \lceil \frac{n}{5} \rceil$.
Subcase (i): If n is odd.
Let $i = \lceil \frac{n}{5} \rceil + j$; $1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor$.

For $n = 9$,
$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1(mod\ 4) \\ 4 & \text{if } j \equiv 2(mod\ 4) \\ 2 & \text{if } j \equiv 3(mod\ 4) \\ 3 & \text{if } j \equiv 0(mod\ 4). \end{cases}$$

For $n > 9$, $$f(v_i) = \begin{cases} 2 & \text{if } j \equiv 3(mod\ 4) \\ 3 & \text{if } j \equiv 0(mod\ 4). \end{cases}$$

For $1 \le j \le 8$, $$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1(mod\ 4) \\ 4 & \text{if } j \equiv 2(mod\ 4). \end{cases}$$

For $4\lfloor \frac{n}{5} \rfloor - 3 \le j \le 4\lfloor \frac{n}{5} \rfloor$, $$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1(mod\ 4) \\ 1 & \text{if } j \equiv 2(mod\ 4). \end{cases}$$

For $9 \le j \le 4\lfloor \frac{n}{5} \rfloor - 4$, $$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,\ 6(mod\ 8) \\ 1 & \text{if } j \equiv 2,\ 5(mod\ 8). \end{cases}$$

Subcase (ii): If n is even.
Let $i = \lceil \frac{n}{5} \rceil + j$; $1 \le j \le 4\lfloor \frac{n}{5} \rfloor$,

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,\ 6(mod\ 8) \\ 1 & \text{if } j \equiv 2,\ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

Then, we have
$v_f(0) = v_f(1) = v_f(2) = v_f(3) = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1$,
$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) = e_f(4) + 1 = 2\lfloor \frac{n}{5} \rfloor + 2$.
Case (vi): If $n = 3$, then $|V(F_3)| = 4$ and $|E(F_3)| = 5$. Thus, $v_f(i) = 0$ *or* 1 $(i = 0, 1, 2, 3, 4)$ and $e_f(i) = 1$ $(i = 0, 1, 2, 3, 4)$. Clearly, $v_f(0) = 0$. Otherwise $e_f(0) > 1$ is not possible. Now $v_f(i) = 1$ $(i = 1, 2, 3, 4)$. Then, $e_f(0) = 0$ is not possible. Hence, F_3 is not a 5-product cordial graph.

□

An example of 5-product cordial labeling of F_{13} is shown in Figure 3.

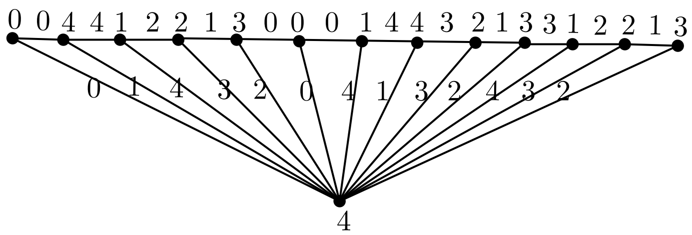

Figure 3: 5-product cordial labeling of F_{13}

Theorem 2.4. *The double fan DF_n is a 5-product cordial graph for all $n \ge 1$ except $n = 2$ and $n \equiv 3(mod\ 5)$.*

Proof. Let the vertex set and the edge set of DF_n be $V(DF_n) = \{u, v, v_i; 1 \le i \le n\}$ and $E(DF_n) = \{(u, v_i), (v, v_i); 1 \le i \le n\} \bigcup \{(v_i, v_{i+1}); 1 \le i \le n-1\}$ respectively. We consider the following five cases.
Define $f : V(DF_n) \to \{0, 1, 2, 3, 4\}$ as follows:
For $n = 2$, we showed that [Theorem 2.3, Case (vi)] F_3 is not a 5-product cordial graph and we have $F_3 \cong DF_2$.
Hence, DF_2 is not a 5-product cordial graph.
Case (i): If $n \equiv 0(mod\ 5)$, then
$f(u) = 3$, $f(v) = 4$,$f(v_i) = 0$; $1 \le i \le \lfloor \frac{n}{5} \rfloor$.

Subcase (i): If n is odd.
Let $i = \lfloor \frac{n}{5} \rfloor + j \ ; \ 1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, \ 6(mod\ 8) \\ 4 & \text{if } j \equiv 2, \ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3, \ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4, \ 0(mod\ 8). \end{cases}$$

Subcase (ii): If n is even.
Let $i = \lfloor \frac{n}{5} \rfloor + j \ ; \ 1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor$,

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1, \ 6(mod\ 8) \\ 1 & \text{if } j \equiv 2, \ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3, \ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4, \ 0(mod\ 8). \end{cases}$$

Then, we have
$v_f(0) + 1 = v_f(1) + 1 = v_f(2) + 1 = v_f(3) = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1$,
$e_f(0) = e_f(1) = e_f(2) + 1 = e_f(3) = e_f(4) = 3\lfloor \frac{n}{5} \rfloor$.
Case (ii): If $n \equiv 1(mod\ 5)$.
For $n = 1$, $f(u) = 3, \ f(v) = 4, \ f(v_1) = 1$.
For $n > 1$,
Subcase (i): If n is odd.
Assign labels to the vertices u, v and v_i $(1 \leq i \leq n-1)$ as in Case (i) Subcase (i), then assign 1 to v_n.
Subcase (ii): If n is even, then
$f(u) = 3, \ f(v) = 4, \ f(v_i) = 0 \ ; \ 1 \leq i \leq \lfloor \frac{n}{5} \rfloor$,
$f(v_{n-4}) = 4, \ f(v_{n-3}) = 1, \ f(v_{n-2}) = 2, \ f(v_{n-1}) = 3, \ f(v_n) = 1$.
Let $i = \lfloor \frac{n}{5} \rfloor + j \ ; \ 1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor - 4$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, \ 6(mod\ 8) \\ 4 & \text{if } j \equiv 2, \ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3, \ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4, \ 0(mod\ 8). \end{cases}$$

From this label we get
$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1$,
$e_f(0) + 1 = e_f(1) + 1 = e_f(2) = e_f(3) + 1 = e_f(4) = 3\lfloor \frac{n}{5} \rfloor + 1$.
Case (iii): If $n \equiv 2(mod\ 5)$ where $n > 2$, then
$f(u) = 3, \ f(v) = 4, \ f(v_{n-1}) = 1, \ f(v_n) = 2, \ f(v_i) = 0 \ ; \ 1 \leq i \leq \lfloor \frac{n}{5} \rfloor - 1$,
$f(v_{\lfloor \frac{n}{5} \rfloor}) = 4, \ f(v_{\lfloor \frac{n}{5} \rfloor + 1}) = 1, \ f(v_{\lfloor \frac{n}{5} \rfloor + 2}) = 2, \ f(v_{\lfloor \frac{n}{5} \rfloor + 3}) = 3, \ f(v_{\lfloor \frac{n}{5} \rfloor + 4}) = 0$.
Subcase (i): If n is odd.
Let $i = \lfloor \frac{n}{5} \rfloor + 4 + j \ ; \ 1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor - 4$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, \ 6(mod\ 8) \\ 4 & \text{if } j \equiv 2, \ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3, \ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4, \ 0(mod\ 8). \end{cases}$$

Subcase (ii): If n is even.
Let $i = \lfloor \frac{n}{5} \rfloor + 4 + j \ ; \ 1 \leq j \leq 4\lfloor \frac{n}{5} \rfloor - 4$,

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,\ 6(mod\ 8) \\ 1 & \text{if } j \equiv 2,\ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

Then, we have

$v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1,$

$e_f(0) = e_f(1) = e_f(2) = e_f(3) = e_f(4) = 3\lfloor \frac{n}{5} \rfloor + 1.$

Case (iv): If $n \equiv 3(mod\ 5)$. Let $n = 5t + 3$, then $|V(DF_n)| = 5t + 5$ and $|E(DF_n)| = 15t + 8$. Thus, $v_f(i) = t+1$ $(i = 0, 1, 2, 3, 4)$ and $e_f(i) = 3t+1$ *or* $3t+2$ $(i = 0, 1, 2, 3, 4)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t + 1$, then $e_f(0) > 3t + 2$ for $t \geq 0$. Therefore, $|e_f(0) - e_f(j)| > 1$ for all $i, j = 1, 2, 3, 4$.

Hence, DF_n is not a 5-product cordial graph if $n \equiv 3(mod\ 5)$.

Case (v): If $n \equiv 4(mod\ 5)$.

Subcase (i): If n is odd.

Assign labels to the vertices u, v and v_i $(1 \leq i \leq n - 3)$ as in Case (i) Subcase (ii) then assign 1, 2 and 3 to v_{n-2}, v_{n-1} and v_n respectively.

Subcase (ii): If n is even.

Assign labels to the vertices u, v and v_i $(1 \leq i \leq n - 3)$ as in Case (i) Subcase (i) then assign 1, 2 and 3 to v_{n-2}, v_{n-1} and v_n respectively.

From this we have

$v_f(0) + 1 = v_f(1) + 1 = v_f(2) + 1 = v_f(3) = v_f(4) + 1 = \lceil \frac{n}{5} \rceil + 1,$

$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 = e_f(4) + 1 = 3\lceil \frac{n}{5} \rceil.$

□

An example of 5-product cordial labeling of DF_{12} is shown in Figure 4.

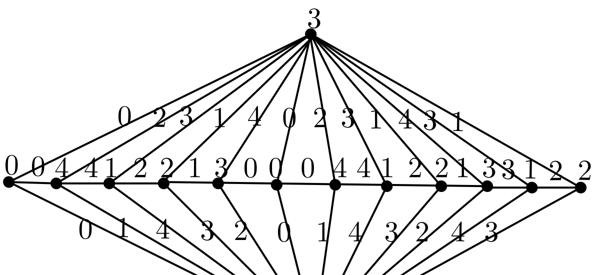

Figure 4: 5-product cordial labeling of DF_{12}

3. Conclusions

In this paper we prove that the graphs fan F_n and double fan DF_n when k=4 and 5 admit k-product cordial labeling.

References

[1] Cahit, I., (1987), Cordial graphs: a weaker version of graceful and harmonious graphs, Ars Combin., 23, pp. 201-207.

[2] Gallian, J. A., (2019), A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, # DS6.

[3] Gross, J. and Yellen, J.,(1999), Graph theory and its application, CRC Press, Boca Raton.

[4] Harary, F., (1972), Graph Theory, Addison-Wesley, Reading, Massachusetts.

[5] Jeyanthi, P. and Vijayalakshmi, M., (2019), Further results on 3-product cordial labeling, Proyecciones Journal of Mathematics, Vol. 38, No 2, pp. 191-202.

[6] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M. Z., (2022), k-Product cordial behaviour of union of graphs, Journal of the Indonesian Mathematical Society, Vol. 28, No. 1, 1-7.

[7] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M. Z., (2022), k-Product cordial labeling of cone graphs, International Journal of Mathematical Combinatorics, Vol. 2, 72-80.

[8] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M. Z., k-Product cordial labeling of powers of paths, Jordan Journal of Mathematics and Statistics, to appear.

[9] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M. Z., k-Product cordial labeling of Napier bridge graphs, Nepal Journal of Mathematical Sciences, to appear.

[10] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M. Z., Further results on k-product cordial labeling, TWMS J. App. and Eng. Math., to appear.

[11] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M. Z., k-Product cordial labeling of product of graphs, Discrete Mathematics, Algorithms and Applications, to appear.

[12] Ponraj, R., Sivakumar, M. and Sundaram, M., (2012), k-product cordial labeling of Graphs, Int. J. Contemp. Math. Sciences, Vol. 7, No. 15, pp. 733 - 742.

[13] Rosa, A., (1967), On certain valuations of the vertices of a graph, Theory of Graphs, Internat. Sympos., ICC Rome, Paris, Dunod, pp. 349-355.

[14] Sundaram, M., Ponraj, R. and Somasundaram, S., (2004), Product cordial labeling of graphs, Bulletin of Pure and Applied Sciences, 23E(1), pp. 155-163.

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