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ON SOLUTION OF $BG-$VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. In this study, we center upon obtaining the solution of linear bigeometric Volterra integral equations of the second kind in the sense of bigeometric calculus. The method of successive substitutions and resolvent kernel method are applied for solving the linear bigeometric Volterra integral equations of the second kind by using the concept of bigeometric integral. The necessary conditions for the bigeometric continuity and the uniqueness of the solution of linear bigeometric Volterra integral equations of the second kind are given in these methods. Finally, some numerical examples are presented to explain the procedure of the method of successive substitutions and resolvent kernel method.

Keywords: Bigeometric calculus, bigeometric Volterra integral equations, method of successive substitutions, $BG-$resvolvent kernel.

AMS Subject Classification: 45A05, 45D99

1. INTRODUCTION

Grossman and Katz [13] introduced the non-Newtonian calculus comprising of the branches of geometric, harmonic, quadratic, bigeometric, biharmonic and biquadratic calculus, as an alternative to classical calculus. Non-Newtonian calculus provides a wide application areas in science, engineering and mathematics. Bigeometric calculus which is one of the most popular non-Newtonian calculus is worked by many researchers. Rybaczuk and Stopel [19] investigated the fractal growth in material science by using bigeometric calculus. Aniszewska and Rybaczuk [1] used bigeometric calculus on a multiplicative Lorenz System. Córdova-Lepe [5] studied on measure of elasticity in economics by aid of bigeometric calculus. Boruah and Hazarika [2, 3] named bigeometric calculus as G-calculus and investigated basic properties of derivative and integral in the sense of bigeometric calculus and also applications in numerical analysis. Boruah et al. [4] researched solvability of bigeometric differential equations by using numerical methods. Güngör [12] defined Volterra integral equations in the sense of bigeometric calculus and investigated its relationship with bigeometric differential equations. Further details can be found in [6, 7, 8, 9, 11, 14, 20, 21].

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Integral equations have a significant role in engineering, pure and applied mathematics and mathematical physics. Many researchers reformulated and applied different types of methods and techniques for getting the solutions of integral equations. One can find relevant terminology related to integral equations in [15, 16, 17, 18].

A complete ordered field is a system consisting of a set A, four binary operations $\dot{+}, \dot{-}, \dot{\times}, \dot{/}$ for A and an ordering relation $\dot{<}$ for A, all of which behave with respect to A exactly as $+, -, \times, /, <$ behave with respect to the set $\mathbb{R}$ of real numbers. We call A the realm of complete ordered field. A complete ordered field is called arithmetic if its realm is a subset of $\mathbb{R}$. A generator is one-to-one function whose domain is $\mathbb{R}$, the set of all real numbers and whose range is a subset of $\mathbb{R}$. The range of generator α which is called non-Newtonian real line denoted by $\mathbb{R}_\alpha = \{\alpha(x) : x \in \mathbb{R}\}$. $\alpha-$ arithmetic operations are described as below:

$$\begin{array}{ll}
\alpha - \text{addition} & x\dot{+}y = \alpha\left\{\alpha^{-1}(x) + \alpha^{-1}(y)\right\} \\
\alpha - \text{subtraction} & x\dot{-}y = \alpha\left\{\alpha^{-1}(x) - \alpha^{-1}(y)\right\} \\
\alpha - \text{multiplication} & x\dot{\times}y = \alpha\left\{\alpha^{-1}(x) \cdot \alpha^{-1}(y)\right\} \\
\alpha - \text{division} & x\dot{/}y = \alpha\left\{\alpha^{-1}(x) \div \alpha^{-1}(y)\right\} \\
\alpha - \text{order} & x\dot{<}y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)
\end{array}$$

for $x, y \in \mathbb{R}_\alpha$. $\left(\mathbb{R}_\alpha, \dot{+}, \dot{\times}\right)$ is a complete field. We say that α generates α-arithmetic. In particular, the identity function I generates classical arithmetic and the exponential function generates geometric arithmetic [13].

Grosmann and Katz described the $*$-calculus with the help of two arbitrarily selected generators. Let α and β be arbitrarily selected generators and $*$ is the ordered pair of arithmetic (α-arithmetic, β-arithmetic). The following notions are used:

	$\alpha -$ arithmetic	$\beta -$ arithmetic
Realm	A	B
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$(or $-\alpha$)	$\ddot{/}$(or $-\beta$)
Order	$\dot{<}$	$\ddot{<}$.

If the generators α and β are chosen as one of I and exp, the following special calculi are obtained:

Calculus	α	β
Classical	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp.

The ι (iota) which is an isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic uniquely satisfying the following three properties:

(1) ι is one to one,
(2) ι is on A and onto B,
(3) For any numbers x and y in A,

$$\iota\left(x\dot{+}y\right) = \iota(x)\ddot{+}\iota(y)$$
$$\iota\left(x\dot{-}y\right) = \iota(x)\ddot{-}\iota(y)$$
$$\iota\left(x\dot{\times}y\right) = \iota(x)\ddot{\times}\iota(y)$$

$\iota\left(x\dot{/}y\right)=\iota(x)\ddot{/}\iota(y)$
$x\dot{<}y\Leftrightarrow\iota(x)\ddot{<}\iota(y)$.

It turns out that $\iota(x)=\beta\left\{\alpha^{-1}(x)\right\}$ for every x in A and $\iota(\dot{n})=\ddot{n}$ for every integer n [13].

Throughout this study, we will deal with bigeometric calculus which is the $*$-calculus for which $\alpha=\beta=\exp$ as specified above. In other words, one uses geometric arithmetic on function arguments and values in the bigeometric calculus. Thereby, we will start by giving the geometric aritmetic and its necessary properties.

If the function exp from $\mathbb{R}$ to $\mathbb{R}^{+}$ which gives $\alpha^{-1}(x)=\ln x$ is selected as a generator, that is to say that α-arithmetic turns into geometric arithmetic. The range of generator exp is denoted by $\mathbb{R}_{\exp}=\{e^{x}:x\in\mathbb{R}\}$. The following notions are used:

geometric addition	$x\oplus y=x.y$
geometric subtraction	$x\ominus y=x/y,\ y\neq 0$
geometric multiplication	$x\odot y=x^{\ln y}$
geometric division	$x\oslash y=x^{\frac{1}{\ln y}},\ y\neq 1$
geometric order	$x\overset{\exp}{<}y\Leftrightarrow\ln x<\ln y$.

$(\mathbb{R}_{\exp},\oplus,\odot)$ is a complete field with geometric zero 1 and geometric identity e. The geometric positive real numbers and geometric negative real numbers are defined by $\mathbb{R}_{\exp}^{+}=\left\{x\in\mathbb{R}_{\exp}:x\overset{\exp}{>}1\right\}$ and $\mathbb{R}_{\exp}^{-}=\left\{x\in\mathbb{R}_{\exp}:x\overset{\exp}{<}1\right\}$, respectively [2, 3, 14].

The exp-absolute value of $x\in\mathbb{R}_{\exp}$ determined by

$$|x|_{\exp}=\begin{cases}x, & x\overset{\exp}{>}1\\ 1, & x=1\\ 1/x, & x\overset{\exp}{<}1\end{cases}$$

and thus $|x|_{\exp}\overset{\exp}{\geq}1$. For $x,y\in\mathbb{R}_{\exp}$, the following relations hold:

$x^{2_{\exp}}=x\odot x=x^{\ln x}$	$x^{-1_{\exp}}=e^{\frac{1}{\ln x}}$	$x^{p_{\exp}}=x^{\ln^{p-1}x}$
$\sqrt{x}^{\exp}=e^{(\ln x)^{\frac{1}{2}}}$	$\sqrt{x^{2_{\exp}}}^{\exp}=\lvert x\rvert_{\exp}$	$e^{n}\odot x=x^{n}$
$x\oplus 1=x$	$x\odot e=x$	$1\ominus e\odot(x\ominus y)=y\ominus x$
$\lvert e^{x}\rvert_{\exp}=e^{\lvert x\rvert}$	$\lvert x\oplus y\rvert_{\exp}\overset{\exp}{\leq}\lvert x\rvert_{\exp}\oplus\lvert y\rvert_{\exp}$	$\lvert x\ominus y\rvert_{\exp}\overset{\exp}{\geq}\lvert x\rvert_{\exp}\ominus\lvert y\rvert_{\exp}$
$\lvert x\odot y\rvert_{\exp}=\lvert x\rvert_{\exp}\odot\lvert y\rvert_{\exp}$	$\lvert x\oslash y\rvert_{\exp}=\lvert x\rvert_{\exp}\oslash\lvert y\rvert_{\exp}$	

The geometric fractional notation $!_{\exp}$ denoted by $n!_{\exp}=e^{n}\odot e^{n-1}\odot\cdots\odot e=e^{n!}$ [2, 3, 14].

Definition 1.1. *Let (x_n) be sequence and x be a point of metric space $\left(\mathbb{R}_{\exp},|.|_{\exp}\right)$. If for every $\varepsilon\overset{\exp}{>}1$, there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that $|x_n\ominus x|_{\exp}\overset{\exp}{<}\varepsilon$ for all $n>n_0$, then it is said that the sequence (x_n) is* exp*-convergent and denoted by ${}^{\exp}\lim\limits_{n\to\infty}x_n=x$ [21].*

Definition 1.2. *Let $f:A\subset\mathbb{R}_{\exp}\to\mathbb{R}_{\exp}$ be a function and $a\in A^{'_{\exp}}$, $b\in\mathbb{R}_{\exp}$. If for every $\varepsilon\overset{\exp}{>}1$ there exists a number $\delta=\delta(\varepsilon)\overset{\exp}{>}1$ such that $|f(x)\ominus b|_{\exp}\overset{\exp}{<}\varepsilon$ for all $x\in A$ whenever $|x\ominus a|_{\exp}\overset{\exp}{<}\delta$, then it is said that the $BG-$limit of the function f at the point a is b and is denoted by ${}^{BG}\lim\limits_{x\to a}f(x)=b$ [2, 20].*

Definition 1.3. *Let $a \in A$ and $f : A \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ be a function. If for every $\varepsilon \overset{\exp}{>} 1$ there exists a number $\delta = \delta(\varepsilon) \overset{\exp}{>} 1$ such that $|f(x) \ominus f(a)|_{\exp} \overset{\exp}{<} \varepsilon$ for all $x \in A$ whenever $|x \ominus a|_{\exp} \overset{\exp}{<} \delta$, then it is said that f is $BG-$continuous at the point $a \in A$* [2].

Definition 1.4. *Let $f : (r,s) \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ and $a \in (r,s)$. If ${}^{BG}\lim\limits_{x\to a} \frac{f(x)\ominus f(a)}{x\ominus a} \exp = \lim\limits_{x\to a} \left[\frac{f(x)}{f(a)}\right]^{\frac{1}{\ln x - \ln a}}$ exits, it is denoted by $f^{BG}(a)$ and called the $BG-$derivative of f at a and said that f is $BG-$differentiable at a* [2, 13, 14].

Definition 1.5. *The $BG-$average of a $BG-$continuous positive function f on $[r,s] \subset \mathbb{R}_{\exp}$ is denoted by $\overset{BG}{M_r^s} f$ and defined as $\exp-$limit of the $\exp-$convergent sequence whose nth term is geometric average of $f(a_1), f(a_2), ..., f(a_n)$ where $a_1, a_2, ..., a_n$ is the $n-$fold $\exp-$partition of $[r,s]$. The $BG-$integral of a $BG-$continuous function f on $[r,s]$ is indicated by ${}^{BG}\int_r^s f(x)\,dx^{BG}$ that is the number $\left(\overset{BG}{M_r^s} f\right)^{(\ln s - \ln r)}$* [3, 13, 14].

Remark 1.1. *If f is $BG-$continuous on $[r,s] \subset \mathbb{R}_{\exp}$, then ${}^{BG}\int_r^s f(x)\,dx^{BG} = e^{\int_r^s \frac{\ln f(x)}{x} dx}$* [3, 13, 14].

Theorem 1.1. *Let f and g be $BG-$continuous positive functions on $[r,s] \subset \mathbb{R}_{\exp}$. Then, the following statements hold:*

(1) ${}^{BG}\int_r^s [\lambda \odot f(x) \oplus \mu \odot g(x)]\,dx^{BG} = \lambda \odot {}^{BG}\int_r^s f(x)\,dx^{BG} \oplus \mu \odot {}^{BG}\int_r^s g(x)\,dx^{BG}$ for all $\lambda, \mu \in \mathbb{R}_{\exp}$.

(2) ${}^{BG}\int_r^s f(x)\,dx^{BG} = {}^{BG}\int_r^t f(x)\,dx^{BG} \oplus {}^{BG}\int_t^s f(x)\,dx^{BG}$ where $r \overset{\exp}{<} t \overset{\exp}{<} s$.

(3) If $f(x) \overset{\exp}{\leq} g(x)$ for all $x \in [r,s] \subset \mathbb{R}_{\exp}$, then ${}^{BG}\int_r^s f(x)\,dx^{BG} \overset{\exp}{\leq} {}^{BG}\int_r^s g(x)\,dx^{BG}$

(4) The function f is $\exp-$bounded on $[r,s] \subset \mathbb{R}_{\exp}$.

(5) $\left| {}^{BG}\int_r^s f(x)\,dx^{BG} \right|_{\exp} \overset{\exp}{\leq} {}^{BG}\int_r^s |f(x)|_{\exp}\,dx^{BG}$ [3, 10, 13].

Definition 1.6. *Let $n \in \mathbb{N}$ and A be a nonempty subset of $\mathbb{R}_{\exp}$. The sequence $(f_n) = (f_1, f_2, \ldots, f_n, \ldots)$ is called $BG-$function sequence for functions $f_n : A \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$. Here all functions defined on same set. The sequence $(f_n(x_0))$ is* exp*-sequence in $\mathbb{R}_{\exp}$ for each $x_0 \in A$* [20].

Definition 1.7. *Let us take the $BG-$function sequence (f_n) where $f_n : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$. If the sequence $(f_n(x_0))$ is* exp*-convergent for each $x_0 \in A$, then the $BG-$function sequence (f_n) is called $BG-$convergent. The $BG-$function sequence (f_n), $BG-$pointwise converges or $BG-$converges to the function f, if for any given $\varepsilon \overset{\exp}{>} 1$, there exists a naturel number $n_0 = n_0(x, \varepsilon)$ such that $|f_n(x) \ominus f(x)|_{\exp} \overset{\exp}{<} \varepsilon$ for all $n > n_0$ and for each $x \in A$. We denote $BG-$convergence by ${}^{BG}\lim\limits_{n\to\infty} f_n = f$ ($BG-$pointwise) or $f_n \overset{BG}{\to} f$ ($BG-$pointwise)* [20].

Definition 1.8. *Let us take the $BG-$function sequence (f_n) where $f_n : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$. The $BG-$function sequence (f_n), $BG-$uniform converges to the function f on the set A, if* *for any given* $\varepsilon \overset{\exp}{>} 1$*, there exists a naturel number* n_0 *depends on number* ε *but not depend on variable* x *such that* $|f_n(x) \ominus f(x)|_{\exp} \overset{\exp}{<} \varepsilon$ *for all* $n > n_0$ *and each* $x \in A$*. We denote* $BG-$*uniform convergence by* ${}^{BG}\lim\limits_{n\to\infty} f_n = f$ *(*$BG-$*uniform) or* $f_n \overset{BG}{\to} f$ *(*$BG-$*uniform)* [20].

Definition 1.9. *Let us take* $BG-$*function sequence* (f_n) *with* $f_n : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$*. The infinite* exp-*sum* ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n = f_1 \oplus f_2 \oplus \cdots \oplus f_n \oplus \cdots$ *is called* $BG-$*function series. The* exp-*sum* $S_n = {}_{\exp}\sum\limits_{k=1}^{n} f_k$ *is called* n*th partial* exp-*sum of the series* ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n$ *for* $n \in \mathbb{N}$ [20].

Definition 1.10. *Let the* $BG-$*function series* ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n$ *with* $f_n : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ *and the function* $f : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ *be specified. If the* $\exp -$*partial sums sequence* (S_n)*, where* $S_n = {}_{\exp}\sum\limits_{k=1}^{n} f_k$ *is* $BG-$*uniform convergent to the function* f*, then* ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n$ *is called* $BG-$*uniform convergent to the function* f *on the set* A *and* ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n = f$ *(*$BG-$*uniform) is written* [20].

Theorem 1.2 ($BG-$Weierstrass M-criterion)**.** *If there exist geometric numbers* M_n *such that* $|f_n(x)|_{\exp} \overset{\exp}{<} M_n$ *for all* $x \in A$ *where* $f_n : A \to \mathbb{R}_{\exp}$ *and if the series* ${}_{\exp}\sum\limits_{n=1}^{\infty} M_n$ *is* $\exp -$*convergent, then the series* ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n$ *is* $BG-$*uniform convergent and* exp-*absolutely convergent* [20].

Theorem 1.3. *If the functions* $f_n : [r,s] \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ *are* $BG-$*continuous on* $[r,s] \subset \mathbb{R}_{\exp}$ *for all* $n \in \mathbb{N}$ *and* ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n = f$ *(*$BG-$*uniform), then the function* f *is* $BG-$*continuous on* $[r,s]$ *and* ${}_{\exp}\sum\limits_{n=1}^{\infty} \left({}^{BG}\int\limits_r^s f_n(x)\, dx^{BG} \right) = {}^{BG}\int\limits_r^s f(x)\, dx^{BG}$ [20].

2. Solving $BG-$Volterra Integral Equations

An equation of the unknown function $u(x)$ with $\mathbb{R}_{\exp}-$valued, is generated the following form

$$u(x) = f(x) \oplus \lambda \odot {}^{BG}\int\limits_a^x K(x,s) \odot u(s)\, ds^{BG} \tag{1}$$

is called linear $BG-$Volterra integral equation of the second kind where λ is a $\mathbb{R}_{\exp}-$parameter. If the unkown function $u(x)$ is only under the $BG-$integral sign in form as

$$f(x) = \lambda \odot {}^{BG}\int\limits_a^x K(x,s) \odot u(s)\, ds^{BG},$$

the equation is called linear $BG-$Volterra integral equation of the the first kind. If $f(x) = 1$, the equation is called homogeneous. The functions $f(x)$ and $K(x,s)$ are specified $\mathbb{R}_{\exp}-$valued functions. The function $K(x,s)$ is called the kernel of the $BG-$Volterra integral equation [12].

2.1. **Solving by Using the Method of Successive Substitutions.** This method introduces the solution of the integral equation in a series form through evaluating single integral and multiple integrals as well. In this method, we substitute successively for $u(x)$ its value as given by equation (1). We obtain that

$$\begin{aligned} u(x) &= f(x) \oplus \lambda \odot {}^{BG}\int_a^x K(x,s) \odot \left\{ f(s) \oplus \lambda \odot {}^{BG}\int_a^s K(s,s_1) \odot u(s_1)\, ds_1^{BG} \right\} ds^{BG} \\ &= f(x) \oplus \lambda \odot {}^{BG}\int_a^x K(x,s) \odot f(s)\, ds^{BG} \\ &\oplus \lambda^{2_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \odot u(s_1)\, ds_1^{BG} ds^{BG}. \end{aligned} \tag{2}$$

Again we substitute for $u(s_1)$ its value as given by (1) into the right side of (2), we get

$$\begin{aligned} u(x) &= f(x) \oplus \lambda \odot {}^{BG}\int_a^x K(x,s) \odot f(s)\, ds^{BG} \oplus \lambda^{2_{\exp}} \odot \Bigg[{}^{BG}\int_a^x K(x,s) \\ &\odot {}^{BG}\int_a^s K(s,s_1) \odot \left\{ f(s_1) \oplus \lambda \odot {}^{BG}\int_a^{s_1} K(s_1,s_2) \odot u(s_2)\, ds_2^{BG} \right\} ds_1^{BG} ds^{BG} \Bigg] \\ &= f(x) \oplus \lambda \odot {}^{BG}\int_a^x K(x,s) \odot f(s)\, ds^{BG} \\ &\oplus \lambda^{2_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \odot f(s_1)\, ds_1^{BG} ds^{BG} \\ &\oplus \lambda^{3_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \odot {}^{BG}\int_a^{s_1} K(s_1,s_2) \odot u(s_2)\, ds_2^{BG} ds_1^{BG} ds^{BG}. \end{aligned}$$

Proceeding in the same manner, we obtain

$$\begin{aligned} u(x) &= f(x) \oplus \lambda \odot {}^{BG}\int_a^x K(x,s) \odot f(s)\, ds^{BG} \oplus \lambda^{2_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \\ &\odot {}^{BG}\int_a^s K(s,s_1) \odot f(s_1)\, ds_1^{BG} ds^{BG} \oplus \cdots \oplus \lambda^{n_{\exp}} \odot \Bigg[{}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \\ &\odot {}^{BG}\int_a^{s_1} K(s_1,s_2) \odot \cdots \odot {}^{BG}\int_a^{s_{n-2}} K(s_{n-2},s_{n-1}) \odot f(s_{n-1})\, ds_{n-1}^{BG} \cdots ds_1^{BG} ds^{BG} \Bigg] \oplus R_{n+1}(x) \end{aligned}$$

where

$$\begin{aligned} R_{n+1} &= \lambda^{(n+1)_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \odot {}^{BG}\int_a^{s_1} K(s_1,s_2) \\ &\odot \cdots \odot {}^{BG}\int_a^{s_n-1} K(s_{n-1},s_n) \odot u(s_n)\, ds_n^{BG} \cdots ds_1^{BG} ds^{BG} \end{aligned} \tag{3}$$

is the remainder after n terms. This leads us to the consideration of the following infinite $\exp-$series:

$$\begin{aligned} &f(x) \oplus \lambda \odot {}^{BG}\int_a^x K(x,s) \odot f(s)\, ds^{BG} \oplus \lambda^{2_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \odot f(s_1)\, ds_1^{BG} ds^{BG} \\ &\oplus \lambda^{3_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \odot {}^{BG}\int_a^{s_1} K(s_1,s_2) \odot f(s_2)\, ds_2^{BG} ds_1^{BG} ds^{BG} \oplus \cdots. \end{aligned} \tag{4}$$

Theorem 2.1. *Consider the $BG-$ Volterra integral equation (1). If the following conditions hold*

- **i)** *$K(x,s)$ is a $BG-$continuous function in the rectangle for which $a \overset{\exp}{\leq} x \overset{\exp}{\leq} b$, $a \overset{\exp}{\leq} s \overset{\exp}{\leq} b$ and $K(x,s) \neq 1$,*
- **ii)** *$f(x)$ is a $BG-$continuous function on $a \overset{\exp}{\leq} x \overset{\exp}{\leq} b$ and $f(x) \neq 1$,*
- **iii)** *λ is a constant in $\mathbb{R}_{\exp}$,*

then the equation (1) has one and only one $BG-$continuous solution in $[a,b] \subset \mathbb{R}_{\exp}$ and this solution is given by the $\exp-$*absolutely and $BG-$uniformly convergent series (4).*

Proof. Since $K(x,s)$ is $BG-$continuous on $a \overset{\exp}{\leq} x, s \overset{\exp}{\leq} b$, there is $M \overset{\exp}{\geq} 1$ such that

$$|K(x,s)|_{\exp} \overset{\exp}{\leq} M \tag{5}$$

on the interval $a \overset{\exp}{\leq} x, s \overset{\exp}{\leq} b$. Because $f(x)$ is a $BG-$continuous function on $a \overset{\exp}{\leq} x \overset{\exp}{\leq} b$, there is $m \overset{\exp}{\geq} 1$ such that

$$|f(x)|_{\exp} \overset{\exp}{\leq} m. \tag{6}$$

Let us take the general term $A_n(x)$ of the series (4), we can write

$$\begin{aligned} A_n(x) &= \lambda^{n_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \odot {}^{BG}\int_a^{s_1} K(s_1,s_2) \odot \\ &\cdots \odot {}^{BG}\int_a^{s_{n-2}} K(s_{n-2},s_{n-1}) \odot f(s_{n-1})\, ds_{n-1}^{BG} \cdots ds_1^{BG} ds^{BG}. \end{aligned}$$

From (5) and (6), we have

$$\left|A_n(x)\right|_{\exp} = \left|\lambda^{n_{\exp}} \odot {}^{BG}\int_a^x K(x,s) \odot {}^{BG}\int_a^s K(s,s_1) \odot {}^{BG}\int_a^{s_1} K(s_1,s_2) \odot \right.$$
$$\left. \cdots \odot {}^{BG}\int_a^{s_{n-2}} K(s_{n-2},s_{n-1}) \odot f(s_{n-1})\, ds_{n-1}^{BG} \cdots ds_1^{BG} ds^{BG}\right|_{\exp}$$
$$\overset{\exp}{\leq} |\lambda|_{\exp}^{n_{\exp}} \odot m \odot M^{n_{\exp}} \odot \frac{(x\ominus a)^{n_{\exp}}}{n!_{\exp}} \exp, \quad \left(a \overset{\exp}{\leq} x \overset{\exp}{\leq} b\right)$$
$$\overset{\exp}{\leq} |\lambda|_{\exp}^{n_{\exp}} \odot m \odot M^{n_{\exp}} \odot \frac{(b\ominus a)^{n_{\exp}}}{n!_{\exp}} \exp .$$

Since

$$\begin{aligned} & {}^{\exp}\lim_{n\to\infty} \left| \frac{|\lambda|_{\exp}^{(n+1)_{\exp}} \odot m \odot M^{(n+1)_{\exp}} \odot \frac{(b\ominus a)^{(n+1)_{\exp}}}{(n+1)!_{\exp}} \exp}{|\lambda|_{\exp}^{n_{\exp}} \odot m \odot M^{n_{\exp}} \odot \frac{(b\ominus a)^{n_{\exp}}}{n!_{\exp}} \exp} \exp \right|_{\exp} \\ = \ & {}^{\exp}\lim_{n\to\infty} \left| \frac{|\lambda|_{\exp} \odot M \odot (b\ominus a)}{e^{n+1}} \exp \right|_{\exp} \\ = \ & \lim_{n\to\infty} e^{\left| \frac{\ln\left(\frac{b}{a}\right)^{\ln M . |\ln \lambda|}}{n+1} \right|} = 1 \overset{\exp}{<} e \end{aligned}$$

the series ${}_{\exp}\sum_{n=1}^{\infty} |\lambda|_{\exp}^{n_{\exp}} \odot m \odot M^{n_{\exp}} \odot \frac{(b\ominus a)^{n_{\exp}}}{n!_{\exp}} \exp$ is $\exp-$convergent for all values $m, \lambda, M, (b\ominus a)$ from $\exp-$ratio test. Therefore the series (4) is $BG-$uniformly and $\exp-$absolutely convergent by $BG-$Weierstrass $M-$criterion. If (1) has continuous solution, it must be expressed by (4). If $u(x)$ is continuous on $a \leq_{\exp} x \leq_{\exp} b$, there exits a constant N in $\mathbb{R}_{\exp}$ such that ${}_{\exp}\max |u(x)|_{\exp} = N$. Consequently, from the equation (3) we find

$$\left|R_{n+1}(x)\right|_{\exp} \overset{\exp}{\leq} |\lambda|_{\exp}^{(n+1)_{\exp}} \odot N \odot M^{(n+1)_{\exp}} \odot \frac{(x\ominus a)^{(n+1)_{\exp}}}{(n+1)!_{\exp}} \exp, \quad \left(a \overset{\exp}{\leq} x \overset{\exp}{\leq} b\right)$$
$$\overset{\exp}{\leq} |\lambda|_{\exp}^{(n+1)_{\exp}} \odot N \odot M^{(n+1)_{\exp}} \odot \frac{(b\ominus a)^{(n+1)_{\exp}}}{(n+1)!_{\exp}} \exp .$$

Because of

$${}^{\exp}\lim_{n\to\infty} |\lambda|_{\exp}^{(n+1)_{\exp}} \odot N \odot M^{(n+1)_{\exp}} \odot \frac{(b\ominus a)^{(n+1)_{\exp}}}{(n+1)!_{\exp}} \exp = 1$$

we have ${}^{\exp}\lim_{n\to\infty} R_{n+1}(x) = 1$. Hence we see that $u(x)$ satisfying (1), is the continuous function given by the series (4). □

2.2. **Solving by Using BG-Resolvent Kernel.** Let the function $f(x)$ be $BG-$continuous on $1 \overset{\exp}{\leq} x \overset{\exp}{\leq} a$ and the function $K(x,s)$ be $BG-$continuous on $1 \overset{\exp}{\leq} s \overset{\exp}{\leq} x$ and

$1 \overset{\exp}{\leq} x \overset{\exp}{\leq} a$. In this method, we will research the solution of the equation

$$u(x) = f(x) \oplus \lambda \odot {}^{BG}\int_1^x K(x,s) \odot u(s)\, ds^{BG} \tag{7}$$

in the form of an infinite $BG-$power series with respect to λ as

$$u(x) = u_0(x) \oplus \lambda \odot u_1(x) \oplus \lambda^{2_{\exp}} \odot u_2(x) \oplus \cdots \oplus \lambda^{n_{\exp}} \odot u_n(x) \oplus \cdots \tag{8}$$

If the series is written instead of $u(x)$ in (7), we obtain

$$\begin{aligned} & u_0(x) \oplus \lambda \odot u_1(x) \oplus \lambda^{2_{\exp}} \odot u_2(x) \oplus \cdots \oplus \lambda^{n_{\exp}} \odot u_n(x) \oplus \cdots \\ = \; & f(x) \oplus \lambda \odot {}^{BG}\int_1^x K(x,s) \odot \left[u_0(s) \oplus \lambda \odot u_1(s) \oplus \cdots \oplus \lambda^{n_{\exp}} \odot u_n(s) \oplus \cdots\right] ds^{BG}. \end{aligned}$$

Comparing the coefficients of like powers of λ, we get

$$\begin{aligned} u_0(x) &= f(x) \\ u_1(x) &= {}^{BG}\int_1^x K(x,s) \odot u_0(s)\, ds^{BG} \\ u_2(x) &= {}^{BG}\int_1^x K(x,s) \odot u_1(s)\, ds^{BG} \\ &\vdots \\ u_n(x) &= {}^{BG}\int_1^x K(x,s) \odot u_{n-1}(s)\, ds^{BG}. \end{aligned}$$

If substituting the first equality into second equality, we get

$$u_1(x) = {}^{BG}\int_1^x K(x,s) \odot f(s)\, ds^{BG}.$$

Since $f(x)$ and $K(x,s)$ are $BG-$continuous functions, we find $u_2(x)$ as in the following form

$$\begin{aligned}
u_2(x) &= {}^{BG}\int_1^x K(x,s) \odot u_1(s)\, ds^{BG} \\
&= {}^{BG}\int_1^x K(x,s) \odot \left[{}^{BG}\int_1^s K(s,s_1) \odot f(s_1)\, ds_1^{BG} \right] ds^{BG} \\
&= {}^{BG}\int_1^x K(x,s) \odot e^{\int_1^s \frac{\ln K(s,s_1)^{\ln f(s_1)}}{s_1} ds_1} ds^{BG} \\
&= e^{\int_1^x \frac{\ln K(x,s)}{s} \int_1^s \frac{\ln f(s_1) \ln K(s,s_1)}{s_1} ds_1 ds} \\
&= e^{\int_1^x \frac{\ln f(s_1)}{s_1} \int_{s_1}^x \frac{\ln K(x,s) \ln K(s,s_1)}{s} ds ds_1} \\
&= e^{\int_1^x \frac{\ln f(s_1)}{s_1} \ln e^{\int_{s_1}^x \frac{\ln K(x,s) \ln K(s,s_1)}{s} ds} ds_1} \\
&= {}^{BG}\int_1^x f(s_1) \odot \left[{}^{BG}\int_{s_1}^x K(x,s) \odot K(s,s_1)\, ds^{BG} \right] ds_1^{BG} \\
&= {}^{BG}\int_1^x K_{(2)}(x,s_1) \odot f(s_1)\, ds_1^{BG}
\end{aligned}$$

where $K_{(2)}(x,s_1) = {}^{BG}\int_{s_1}^x K(x,s) \odot K(s,s_1)\, ds^{BG}$. By proceeding similar method, we get

$$u_n(x) = {}^{BG}\int_1^x K_{(n)}(x,s) \odot f(s)\, ds^{BG},\ n \geq 1 \tag{9}$$

where

$$\begin{aligned}
K_{(1)}(x,s) &= K(x,s) \\
K_{(n+1)}(x,s) &= {}^{BG}\int_s^x K(x,z) \odot K_{(n)}(z,s)\, dz^{BG},\ n \geq 1
\end{aligned} \tag{10}$$

The $K_{(n)}$ are called the $BG-$iterated kernel. Benefiting from the expressions (9), the equality (8) can be written as follow

$$u(x) = f(x) \oplus {}_{\exp}\sum_{n=1}^{\infty} \lambda^{n_{\exp}} \odot {}^{BG}\int_1^x K_{(n)}(x,s) \odot f(s)\, ds^{BG}. \tag{11}$$

The function $R(x,s;\lambda)$ defined by

$$R(x,s;\lambda) = {}_{\exp}\sum_{n=0}^{\infty} \lambda^{n_{\exp}} \odot K_{(n+1)}(x,s) \tag{12}$$

is called $BG-$resolvent kernel for the $BG-$Volterra integral equation (7). Since $K(x,s)$ is $BG-$continuous on $1 \overset{\exp}{\leq} s \overset{\exp}{\leq} x \overset{\exp}{\leq} a$, there exists $M \overset{\exp}{\geq} 1$ such that $|K(x,s)|_{\exp} \overset{\exp}{\leq} M$ on $1 \overset{\exp}{\leq} s \overset{\exp}{\leq} x \overset{\exp}{\leq} a$. Hence, we can write the following statements

$$\left|K_{(1)}(x,s)\right|_{\exp} = |K(x,s)|_{\exp} \overset{\exp}{\leq} M$$

$$\begin{aligned}
\left|K_{(2)}(x,s)\right|_{\exp} &= \left|{}^{BG}\!\int_s^x K(x,z) \odot K_{(1)}(z,s)\, dz^{BG}\right|_{\exp} \\
&\overset{\exp}{\leq} {}^{BG}\!\int_s^x |K(x,z)|_{\exp} \odot |K(z,s)|_{\exp}\, dz^{BG} \\
&\overset{\exp}{\leq} {}^{BG}\!\int_s^x M^{2_{\exp}} dz^{BG} \\
&= e^{\int_s^x \frac{\ln M^{\ln M}}{z} dz} \\
&= e^{(\ln M)^2 \left(\ln \frac{x}{s}\right)} \\
&= M^{2_{\exp}} \odot (x \ominus s)
\end{aligned}$$

$$\begin{aligned}
\left|K_{(3)}(x,s)\right|_{\exp} &= \left|{}^{BG}\!\int_s^x K(x,z) \odot K_{(2)}(z,s)\, dz^{BG}\right|_{\exp} \\
&\overset{\exp}{\leq} {}^{BG}\!\int_s^x |K(x,z)|_{\exp} \odot \left|K_{(2)}(z,s)\right|_{\exp} dz^{BG} \\
&\overset{\exp}{\leq} {}^{BG}\!\int_s^x M^{3_{\exp}} \odot (z \ominus s)\, dz^{BG} \\
&= e^{\int_s^x \frac{(\ln M)^3 \ln\left(\frac{z}{s}\right)}{z} dz} \\
&= e^{(\ln M)^3 \ln^2 \frac{x}{s}} \\
&= M^{3_{\exp}} \odot \frac{(x \ominus s)^{2_{\exp}}}{2!_{\exp}} \exp
\end{aligned}$$

$$\vdots$$

$$\left|K_{(n+1)}(x,s)\right|_{\exp} \overset{\exp}{\leq} M^{(n+1)_{\exp}} \odot \frac{[x \ominus s]^{n_{\exp}}}{n!_{\exp}} \exp .$$

by the recursion formulas (10). Therefore the series (12) is $\exp-$absolutely convergent and $BG-$uniformly convergent on $1 \overset{\exp}{\leq} s \overset{\exp}{\leq} x \overset{\exp}{\leq} a$ from $BG-$Weierstrass M-criterion.

Theorem 2.2. *If the function $f(x)$ is $BG-$continuous on $1 \overset{\exp}{\leq} x \overset{\exp}{\leq} a$ and $K(x,s)$ is $BG-$continuous on $1 \overset{\exp}{\leq} s \overset{\exp}{\leq} x \overset{\exp}{\leq} a$, then*

$$f(x) \oplus \lambda \odot \ {}^{BG}\int_1^x R(x,s;\lambda) \odot f(s)\, ds^{BG}$$

is unique continuous solution of (7).

Proof. Since the order of $BG-$integration and $\exp-$summation is interchanged by Theorem 1.3, we get

$$\begin{aligned} & f(x) \oplus \lambda \odot \ {}^{BG}\int_1^x R(x,s;\lambda) \odot f(s)\, ds^{BG} \\ = \ & f(x) \oplus \lambda \odot \ {}_{\exp}\sum_{n=0}^{\infty} \left(\lambda^{n_{\exp}} \odot \ {}^{BG}\int_1^x K_{(n+1)}(x,s) \odot f(s)\, ds^{BG} \right) \\ = \ & f(x) \oplus \ {}_{\exp}\sum_{n=0}^{\infty} \left(\lambda^{(n+1)_{\exp}} \odot \ {}^{BG}\int_1^x K_{(n+1)}(x,s) \odot f(s)\, ds^{BG} \right) \\ = \ & f(x) \oplus \ {}_{\exp}\sum_{n=0}^{\infty} \lambda^{(n+1)_{\exp}} \odot u_{n+1}(x) \\ = \ & u(x) \end{aligned}$$

by expression (12). □

Theorem 2.3. *Under the hypothesis of Theorem 2.2, the $BG-$resolvent kernel $R(x,s;\lambda)$ satisfies the following equation*

$$R(x,s;\lambda) = K(x,s) \oplus \lambda \odot \ {}^{BG}\int_s^x K(x,z) \odot R(z,s;\lambda)\, dz^{BG}.$$

Proof. We know that the $BG-$resolvent kernel is

$$R(x,s;\lambda) = \ {}_{\exp}\sum_{n=0}^{\infty} \lambda^{n_{\exp}} \odot K_{(n+1)}(x,s)$$

where iterated kernels are given by (10). Hence, we find

$$\begin{aligned} R(x,s;\lambda) &= K_{(1)}(x,s) \oplus \ {}_{\exp}\sum_{n=1}^{\infty} \lambda^{n_{\exp}} \odot K_{(n+1)}(x,s) \\ &= K(x,s) \oplus \ {}_{\exp}\sum_{n=1}^{\infty} \lambda^{n_{\exp}} \odot \left({}^{BG}\int_s^x K(x,z) \odot K_{(n)}(z,s)\, dz^{BG} \right) \\ &= K(x,s) \oplus \lambda \odot \ {}_{\exp}\sum_{n=1}^{\infty} \lambda^{(n-1)_{\exp}} \odot \ {}^{BG}\int_s^x K(x,z) \odot K_{(n)}(z,s)\, dz^{BG} \\ &= K(x,s) \oplus \lambda \odot \ {}_{\exp}\sum_{n=0}^{\infty} \lambda^{n_{\exp}} \odot \ {}^{BG}\int_s^x K(x,z) \odot K_{(n+1)}(z,s)\, dz^{BG}. \end{aligned}$$

Changing the order of exp −summation and $BG-$integration, we have

$$\begin{aligned} R(x,s;\lambda) &= K(x,s) \oplus \lambda \odot {}^{BG}\int_s^x K(x,z) \odot \left({}_{\exp}\sum_{n=0}^{\infty} \lambda^{n_{\exp}} \odot K_{(n+1)}(z,s) \right) dz^{BG} \\ &= K(x,s) \oplus \lambda \odot^{BG} \int_s^x K(x,z) \odot R(z,s;\lambda)\, dz^{BG}. \end{aligned}$$

□

2.3. **Numerical Examples.**

Example 2.1. *Solve the $BG-$Volterra integral equation* $u(x) = e \oplus {}^{BG}\int_1^x u(s)\, ds^{BG}$ *by using successive substitutions method.*
The solution of the $BG-$Volterra integral equation is obtained as

$$\begin{aligned} u(x) &= e \oplus {}^{BG}\int_1^x e \odot e ds^{BG} \oplus {}^{BG}\int_1^x e \odot {}^{BG}\int_1^s e \odot e ds_1^{BG} ds^{BG} \\ &\oplus {}^{BG}\int_1^x e \odot {}^{BG}\int_1^s e \odot {}^{BG}\int_1^{s_1} e \odot e ds_2^{BG} ds_1^{BG} ds^{BG} \oplus \cdots \\ &= e \oplus {}^{BG}\int_1^x e ds^{BG} \oplus {}^{BG}\int_1^x {}^{BG}\int_a^s e ds_1^{BG} ds^{BG} \oplus {}^{BG}\int_1^x {}^{BG}\int_1^s {}^{BG}\int_1^{s_1} e ds_2^{BG} ds_1^{BG} ds^{BG} \oplus \cdots \\ &= e \oplus e^{\int_1^x \frac{\ln e}{s} ds} \oplus {}^{BG}\int_1^x \left(e^{\int_1^s \frac{\ln e}{s_1} ds_1} \right) ds^{BG} \oplus {}^{BG}\int_1^x {}^{BG}\int_1^s \left(e^{\int_1^{s_1} \frac{\ln e}{s_2} ds_2} \right) ds_1^{BG} ds^{BG} \oplus \cdots \\ &= e \oplus e^{\ln x} \oplus e^{\int_1^x \frac{\ln s}{s} ds} \oplus {}^{BG}\int_1^x \left(e^{\int_1^s \frac{\ln s_1}{s_1} ds_1} \right) \oplus \cdots \\ &= e \oplus e^{\ln x} \oplus e^{\frac{\ln^2 x}{2!}} \oplus e^{\frac{\ln^3 x}{3!}} \oplus \cdots \\ &= e \oplus x \oplus \frac{x^{2_{\exp}}}{2!_{\exp}} \oplus \frac{x^{3_{\exp}}}{3!_{\exp}} \oplus \cdots \\ &= e^x. \end{aligned}$$

Example 2.2. *With the help of the $BG-$resolvent kernel find the solution of $BG-$Volterra integral equation* $u(x) = e^x \oplus {}^{BG}\int_1^x e^{x \ominus s} \odot u(s)\, ds^{BG}$.
By definition of $BG-$iterated kernels, we find

$$K_{(1)}(x,s) = K(x,s) = e^{x \ominus s}$$

$$\begin{aligned}
K_{(2)}(x,s) &= {}^{BG}\int_s^x K(x,z)\odot K_{(1)}(z,s)\,dz^{BG} \\
&= {}^{BG}\int_s^x e^{(x\ominus z)}\odot e^{(z\ominus s)}dz^{BG} \\
&= {}^{BG}\int_s^x e^{(x\ominus s)}dz^{BG} \\
&= e^{x\ominus s}\odot(x\ominus s)
\end{aligned}$$

$$\begin{aligned}
K_{(3)}(x,s) &= {}^{BG}\int_s^x K(x,z)\odot K_{(2)}(z,s)\,dz^{BG} \\
&= {}^{BG}\int_s^x e^{(x\ominus z)}\odot e^{(z\ominus s)}\odot(z\ominus s)\,dz^{BG} \\
&= e^{(x\ominus s)}\odot {}^{BG}\int_s^x (z\ominus s)\,dz^{BG} \\
&= e^{(x\ominus s)}\odot e^{\int_s^x\left(\frac{\ln z-\ln s}{z}\right)dz} = e^{(x\ominus s)}\odot e^{\frac{(\ln x-\ln s)^2}{2}} \\
&= e^{x\ominus s}\odot\frac{(x\ominus s)^{2_{\exp}}}{2!_{\exp}}
\end{aligned}$$

and so on. In general, we have

$$K_{n+1}(x,s)=e^{x\ominus s}\odot\frac{(x\ominus s)^{n_{\exp}}}{n!_{\exp}},\ n\in\mathbb{N}.$$

Hence, the BG−resolvent kernel is obtained by

$$R(x,s;\lambda)={}_{\exp}\sum_{n=0}^{\infty}\lambda^{n_{\exp}}\odot e^{x\ominus s}\odot\frac{(x\ominus s)^{n_{\exp}}}{n!_{\exp}}=e^{x\ominus s}\odot_{\exp}\sum_{n=0}^{\infty}\lambda^{n_{\exp}}\odot\frac{(x\ominus s)^{n_{\exp}}}{n!_{\exp}}.$$

Therefore,

$$R(x,s;\lambda)=e^{x\ominus s}\odot e^{\lambda\odot(x\ominus s)}.$$

The required solution of the given BG−integral equation is

$$\begin{aligned}
u(x) &= e^x\oplus {}^{BG}\int_1^x R(x,s;e)\odot e^s ds^{BG} \\
&= e^x\oplus {}^{BG}\int_1^x e^{x\ominus s}\odot e^{e\odot(x\ominus s)}\odot e^s ds^{BG} \\
&= e^x\oplus e^{\int_1^x\frac{\ln e^{\frac{x^2}{s}}}{s}ds} \\
&= e^x\oplus e^{x^2}\ominus e^x \\
&= e^{x^2}.
\end{aligned}$$

3. Conclusions

The method of successive substitutions is applied to solve bigeometric Volterra integral equations of the second kind by using the concept of bigeometric integral and also, the necessary conditions for $BG-$continuous and uniqueness of this solution is given in this method. The resolvent kernel method is applied to solve bigeometric Volterra integral equations of the second kind by aid of obtaining $BG-$iterated kernel in the sense of bigeometric calculus and the necessary conditions are given for the solution to be $BG-$continuous and uniquene in this method. Finally, some numerical examples are presented to illustrate these methods.

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