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PATHOS DEGREE PRIME GRAPH OF A TREE

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Abstract. Let T be a tree of order n $(n \geq 2)$. A *pathos degree prime graph* of T, written $PDP(T)$, is a graph whose vertices are the vertices and paths of a pathos of T, with two vertices of $PDP(T)$ adjacent whenever the degree of the corresponding vertices of T are unequal and relatively prime; or the corresponding paths $P_i^{'}$ and $P_j^{'}$ $(i \neq j)$ of a pathos of T have a vertex in common; or one corresponds to the path $P^{'}$ and the other to a vertex v and $P^{'}$ begins (or ends) at v such that v is a pendant vertex in T. We look at some properties of this graph operator. For this class of graphs we discuss the planarity; outerplanarity; maximal outerplanarity; minimally nonouterplanarity; Eulerian; and Hamiltonian properties these graphs.

Keywords: Crossing number, inner vertex number, pathos, path number.

AMS Subject Classification: 05C05, 05C45.

1. Introduction

There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graphs, the total graphs, and their generalizations. One such graph operator is called the *degree prime graph.* This was introduced by Sattanathan et al. in [9].

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v, each loop counting as two edges. A *pendant vertex* is a vertex with degree one. We denote by $\Delta(G)$ the maximum degree of the vertex of G. Two integers a and b are said to be *relatively prime* if the only positive integer that divides both of them is one.

Let $G = (V, E)$ be a graph of order n $(n \geq 2)$. The *degree prime graph* of G, denoted by $DP(G)$, is defined as the graph having the same vertex set as G and two vertices are adjacent in $DP(G)$ if and only if their degrees are unequal and relatively prime in G.

An example of a graph and its degree prime graph is given in Figure.1.

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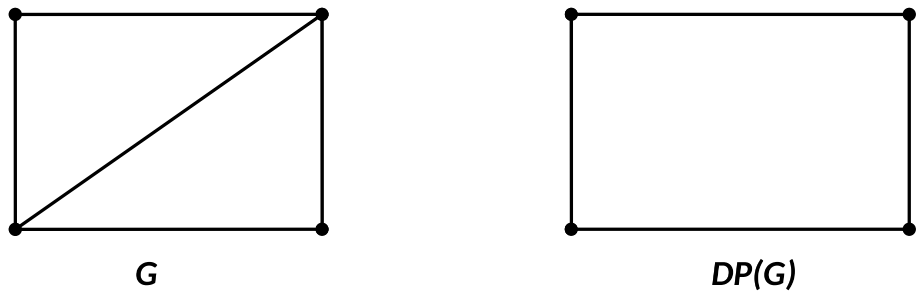


Figure.1

Notations and definitions not introduced here can be found in [4].

The concept of *pathos* of a graph G was introduced by Harary [5] as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The path number of a tree T equals k, where $2k$ is the number of odd degree vertices of T [7].

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common. Muddebihal et al. in [7] extended the concept of pathos of graphs to trees there by introducing a graph operator called a *pathos line graph* of a tree.

A *pathos line graph* of a tree T, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T have a vertex in common or the edge lies on the corresponding path of the pathos.

An example of a tree along with pathos (indicated by dotted lines) and its pathos line graph is shown in Figure.2.

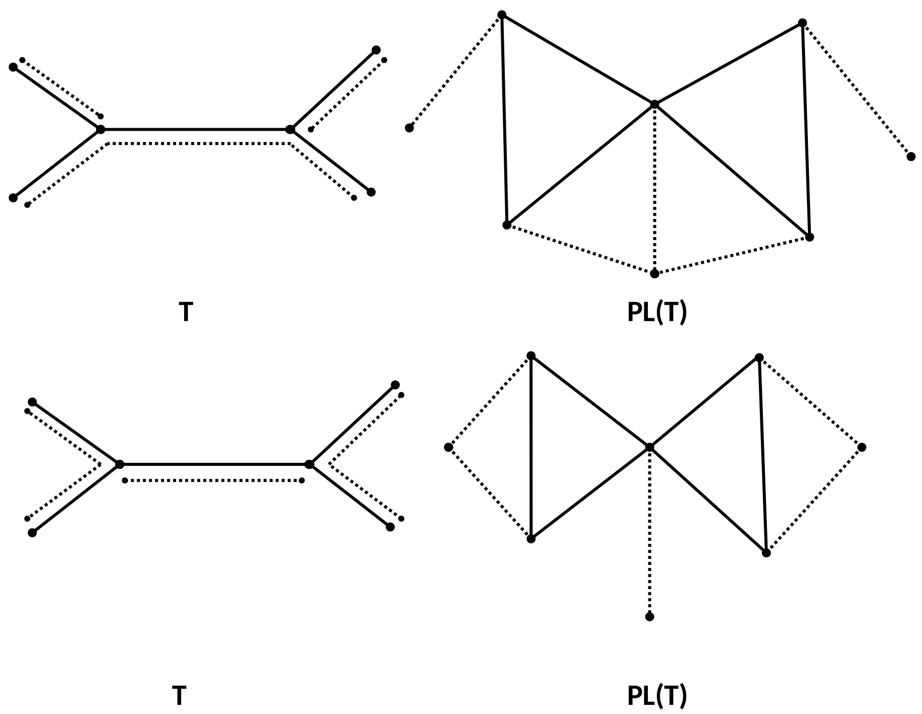


Figure.2

A *pathos vertex* of $PL(T)$ is a vertex corresponding to the path of a pathos of T. For example, for every tree (on the left) of Figure.2, there are three paths of pathos of T, say $P_1^{'}, P_2^{'}$, and $P_3^{'}$. Thus $P_1^{'}, P_2^{'}$, and $P_3^{'}$ are the *pathos vertices* of the corresponding $PL(T)$.

Motivated by the studies above, we introduce a natural generalization of the degree prime graph called a *pathos degree prime graph* of a tree.

2. Definition of $PDP(T)$

A *pathos degree prime graph* of T, written $PDP(T)$, is a graph whose vertices are the vertices and paths of a pathos of T, with two vertices of $PDP(T)$ adjacent whenever the degree of the corresponding vertices of T are unequal and relatively prime; or the corresponding paths $P_i^{'}$ and $P_j^{'}$ $(i \neq j)$ of a pathos of T have a vertex in common; or one corresponds to the path $P^{'}$ and the other to a vertex v and $P^{'}$ begins (or ends) at v such that v is a pendant vertex in T.

See Figure.3 for an example of a tree along with pathos (indicated by dotted lines) and its pathos degree prime graph.

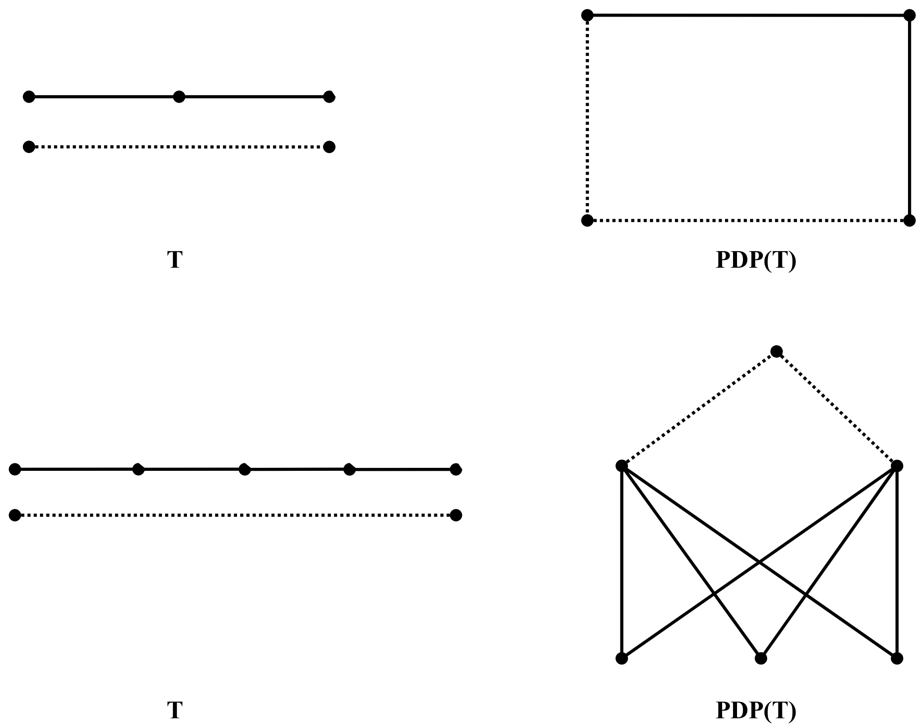


Figure.3

Note that there is freedom in marking the paths of a pathos of a tree T in different ways, provided that the path number k of T is fixed. For example, consider the marking of the paths of pathos of the first and second tree (on the left) of Figure.2, where $k = 3$. Therefore, we conclude that since the order of marking of the paths of a pathos of a tree is not unique, the corresponding pathos degree prime graph is also not unique. This obviously raises the question of the existence of "unique" pathos degree prime graph.

One can easily check that if the path number of a tree is exactly one, i.e., k=1, then the corresponding pathos degree prime graph is unique. For example, the path number of a path P_n on $n \geq 2$ vertices is one. Thus only for the path P_n, we can speak of "the" pathos degree prime graph. Furthermore, one can also observe easily that for different ways of marking of the paths of a pathos of a star graph $K_{1,n}$ on $n \geq 3$ vertices, the corresponding pathos degree prime graphs are isomorphic.

In this paper we look at some properties of $PDP(T)$ and study some of the graph labeling techniques satisfied by $PDP(T)$. For this class of graphs we also discuss the planarity; outerplanarity; maximal outerplanarity; minimally nonouterplanarity; Eulerian; and Hamiltonian properties of these graphs.

3. Properties of $PDP(T)$

In this section we study certain properties of pathos degree prime graph.

Observation 3.1. *For any tree T, $DP(T) \subseteq PDP(T)$, where $\subseteq$ is the subgraph notation.*

We shall use P_n, C_n, and K_n to denote a path, a cycle, and a complete graph on n vertices, respectively; and $P_1', P_2', \ldots$ to denote the paths of a pathos of T. Furthermore, we denote a complete bipartite graph by $K_{m,n}$.

The *Dutch Windmill graph* $D_3^{(m)}$, also called a *friendship graph*, is the graph obtained by taking m copies of the cycle graph C_3 with a vertex in common and therefore corresponds to the usual Windmill graph $W_n^{(m)}$. It is therefore natural to extend the definition to $D_n^{(m)}$, consisting of m copies of C_n.

Proposition 3.1. *A pathos degree prime graph $PDP(T)$ of a tree T is a block if and only if $\Delta(T) \geq 2$, for every vertex $v \in T$.*

Proof. Suppose $PDP(T)$ is a block. Assume that $\Delta(T) < 2$, for every vertex $v \in T$. The only tree that has no vertex of degree two is P_2 (or K_2). If $T = P_2$, then $PDP(T) = P_3$, which is not a block, a contradiction.

Conversely, suppose $\Delta(T) \geq 2$, for every vertex $v \in T$. Assume that $\Delta(T) = 2$. Then $T = P_n$ $(n \geq 3)$. Clearly, the path number of T is one, say P'. We consider the following three cases.

Case 1. For $n = 3$, $PDP(T)$ is a cycle C_4, which is a block.

Case 2. For $n = 4$, $PDP(T)$ is a complete bipartite graph $K_{2,3}$, which is also a block.

Case 3. For $n \geq 5$, let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertices of the path P_n. Then $DP(T)$ is the complete bipartite graph $K_{2,n-2}$, which is a block. Since the path number of P_n is one, i.e., P', and P' is adjacent to both v_1 and v_n of $DP(T)$, $PDP(T)$ is also a block.

Assume now that $\Delta(T) \geq 3$, for every vertex $v \in T$. If there exists a vertex of degree three in T, i.e., $T = K_{1,3}$. Let C be the cut-vertex of $K_{1,3}$, and let $P(T) = \{P_1', P_2'\}$ be a pathos set of T. Then $D_4^{(2)} - v$ is the spanning subgraph of $PDP(T)$, where v is a vertex at distance one from C. Clearly, $D_4^{(2)} - v$ is not a block. Furthermore, since the pathos vertices P_1' and P_2' of $PDP(T)$ are adjacent, the number of cut-vertices of $PDP(T)$ becomes zero, and thus $PDP(T)$ is a block. Hence by all the cases above, $PDP(T)$ is a block. This completes the proof. □

While defining any class of graphs, it is desirable to know the order and size of each. Our next result gives a useful property to determine the size of $PDP(T)$. The proof is straightforward, so we omit it.

Property 3.1. *Let T be a tree of order n $(n \geq 3)$. Then the number of edges whose end-vertices are the pathos vertices in $PDP(T)$ is at most $\frac{k(k-1)}{2} = \beta$ (say), where k is the path number of T. In particular, if T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the* *number of edges whose end-vertices are the pathos vertices in $PDP(T)$ is exactly β, i.e., in a pathos degree prime graph of a star graph, the pathos vertices are pairwise adjacent.*

The following result gives the number of pendant vertices in a tree T, which is also needed while determining the size of $PDP(T)$.

Remark 3.1. *Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. Then the number of pendant vertices of T equals $2 + \sum_{d_T(v)\geq 3} (d_T(v) - 2)$.*

Proof. Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. Let M be the number of pendant vertices in T. By the handshaking lemma, we have $\sum_{v\in T} d_T(v) = 2(n-1) = 2n-2$.

$$\Rightarrow -2 = \sum_{v\in T} d_T(v) - 2n$$
$$\Rightarrow -2 = \sum_{v\in T} d_T(v) - \sum_{v\in T} 2$$
$$\Rightarrow -2 = \sum_{v\in T}(d_T(v) - 2).$$

On taking the sum over the vertices of degree one and two, we get

$$-2 = \sum_{d_T(v)=1} (-1) + \sum_{d_T(v)=2} (0) + \sum_{d_T(v)\geq 3} (d_T(v) - 2)$$
$$\Rightarrow -2 = -M + \sum_{d_T(v)\geq 3} (d_T(v) - 2)$$
$$\Rightarrow M = 2 + \sum_{d_T(v)\geq 3} (d_T(v) - 2).$$
□

The maximum number of edges in the degree prime graph $DP(G)$ of a graph G is determined by Sattanathan et al. in [9] as stated in the following result.

Theorem 3.1. ([9]) : *Let G be a graph of order n $(n \geq 2)$. Then the maximum number of edges of $DP(G)$ equals $\frac{(n-s)(n+s-1)}{2}$, where s is the number of vertices of even degree in G.*

The following result gives the order and size of $PDP(T)$.

Proposition 3.2. *Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. Then $E(PDP(T)) \leq \frac{(n-s)(n+s-1)}{2} + 2 + \sum_{d_T(v)\geq 3} (deg(v) - 2) + \frac{k(k-1)}{2}$, where s is the number of vertices of even degree in T.*

Proof. Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. By definition, the order of $PDP(T)$ equals the sum of vertices and the path number of T. Thus $V(PDP(T)) = n+k$. The size of $PDP(T)$ equals the sum of size of $DP(T)$; number of pendant vertices of T; and the number of edges whose end-vertices are the pathos vertices. By Property 3.1, Remark 3.1, and Theorem 3.1, $E(PDP(T)) \leq \frac{(n-s)(n+s-1)}{2} + 2 + \sum_{d_T(v)\geq 3} (deg(v) - 2) + \frac{k(k-1)}{2}$. □

We believe that this bound is true but not sharp. We now characterize the trees whose $PDP(T)$ admits certain types of graph labeling such as square sum labeling; strongly square sum labeling; E-cordial labeling; and vertex and edge magic labeling.

A *graph labeling* is the assignment of labels, traditionally represented by integers, to the edges or vertices, or both, of a graph. Arumugam et al. [1] introduced the concept of *square sum labeling* and *strongly square sum labeling* of a graph.

Let $G = (V, E)$ be a (p, q) graph. G is said to be a *square sum* graph if there exists a bijection $f : V(G) \rightarrow \{0, 1, \ldots, p-1\}$ such that the induced function $f^* = E(G) \rightarrow N$ given by $f^*(u, v) = [f(u)]^2 + [f(v)]^2$ for every $(u, v) \in E(G)$ are all distinct. The square sum labeling f is called a *prime sum labeling* if $f^*(u, v)$ is 1 or a prime number $\forall (u, v) \in E(G)$.

Proposition 3.3. *A pathos degree prime graph $PDP(T)$ of a tree T admits square sum labeling if T is either P_2 or P_3.*

Proof. Suppose that $T = P_2$, and let $V(P_2) = \{v_1, v_2\}$. Then the path number of P_2 is one, say P'. By definition, $PDP(T) = P_3$. Let $V(P_3) = \{v_1, v_2, P'\}$ and $E(P_3) = \{(v_1, P'), (P', v_2)\}$. Define $f : V(P_3) \rightarrow \{0, 1, 2\}$ and $f(v_1) = 0; f(P') = 1;$ and $f(v_2) = 2$. Then f induces a function f^* such that $f^*(v_1, P') = [f(v_1)]^2 + [f(P')]^2 = 1$; and $f^*(P', v_2) = [f(P')]^2 + [f(v_2)]^2 = 5$. Clearly, $f^*(v_1, P') \neq f^*(P', v_2)$. Hence f^* is injective and f is a square sum lebeling of $PDP(T)$.

On the other hand, suppose that $T = P_3$, and let $V(P_3) = \{v_1, v_2, v_3\}$. By definition, $PDP(T) = C_4$. Let $V(C_4) = \{v_1, v_2, v_3, P'\}$ and $E(C_4) = \{(v_1, v_2), (v_2, v_3), (v_3, P'), (P', v_1)\}$. Define $f : V(C_4) \rightarrow \{0, 1, 2, 3\}$ and $f(v_1) = 0; f(v_2) = 1; f(v_2) = 2;$ and $f(P') = 3$. Then f induces a function f^* such that $f^*(v_1, v_2) = [f(v_1)]^2 + [f(v_2)]^2 = 1$; $f^*(v_2, v_3) = [f(v_2)]^2 + [f(v_3)]^2 = 5$; $f^*(v_3, P') = [f(v_3)]^2 + [f(P')]^2 = 13$; and $f^*(P', v_1) = [f(P')]^2 + [f(v_1)]^2 = 9$. Clearly, $f^*(u, v) \neq [f(u)]^2 + [f(v)]^2$ for any edge $(u, v) \in E(PDP(T))$. Hence f^* is injective and f is a square sum lebeling of $PDP(T)$. This completes the proof. $\square$

Let $G = (V, E)$ be a (p, q) graph. G is said to be a *strongly square sum* graph if there exists a bijection $f : V(G) \rightarrow \{0, 1, \ldots, p-1\}$ such that $f^*(u, v) = [f(u)]^2 + [f(v)]^2$ for every $(u, v) \in E(G)$ are all distinct and $f^*(E(G))$ consists the first q consecutive numbers of the form $a^2 + b^2, a \leq p-1, a \neq b$, then f is said to be a *strongly square sum labeling of G*.

The following result is proved in [1].

Theorem 3.2. *The cycles C_4 and C_5 can be embedded as an induced subgraph of a strongly square sum graph.*

In view of Theorem 3.2, we can state the following result.

Property 3.2. *The pathos degree prime graph of a path P_3 can be embedded as an induced subgraph of a strongly square sum graph.*

The concept of *cordial labeling* was introduced by Cahit [2] as a weaker version of graceful and harmonious labeling. After this, some other labeling techniques were also introduced having the same idea of cordial labeling. Some of them are *cordial labeling, product cordial labeling, and total product labeling.*

Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0, 1\}$ is called a *binary vertex lebeling* of G and $f(v)$ is called the *label* of the vertex v of G under f. For and edge $e = (u, v)$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e = (u, v)) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of G having lables 0 and 1 respecttively, under f and let $e_f(0), e_f(1)$ be the number of edges of G having lables 0 and 1 respectively, under f^*.

A binary lebeling of a graph G is *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is *cordial* if it admits cordial labeling.

Proposition 3.4. *A pathos degree prime graph $PDP(T)$ of a tree T admits E-cordial labeling if T is P_2.*

Proof. Suppose that $T = P_2$, and let $V(P_2) = \{v_1, v_2\}$. By definition, $PDP(T) = P_3$. Let $V(P_3) = \{v_1, v_2, P'\}$ and $E(P_3) = \{(v_1, P'), (P', v_2)\}$. Define $f : E(P_3) \to \{0, 1\}$. For $n = 2$, $f(v_1, P') = 0; f(P', v_2) = 1$. In view of this pattern of labeling, f satisfy the conditions of E-cordial labeling. This completes the proof. □

The authors in [3] introduces the concept of *total labeling* of a graph.

A *total lebeling* of a graph with v vertices and e edges is defined as a one-to-one map taking the vertices and edges onto the integers $1, 2, \ldots, v + e$. Such a labeling is *vertex magic* if the sum of the label on a vertex and the labels on its incident edges is a constant independent of the choice of vertex, and *edge magic* if the sum of an edge label and the label of the end vertices of the edge is constant.

The following result is proved in [8].

Theorem 3.3. ([8]) : *If $n > m+1$, then the complete bipartite graph $K_{m,n}$ has no labeling.*

For a graph G, if there exist a total labeling that is both edge magic and vertex magic, then the graph G is said to be a *totally magic graph*. It is proved in [3] that every tree of order n ($n > 1$) has at least two pendant vertices, and thus K_1 and star graph are the only two magic trees. But in view of Theorem 3.3, $K_{m,n}$ can never be vertex magic for $|m - n| > 1$. Hence no star graph except $K_{1,2}$ is vertex magic. Therefore, we have

Property 3.3. *The pathos degree prime graph of a path P_2 is the only totally magic tree (except K_1).*

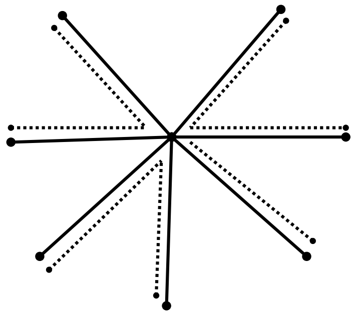
4. Characterization of $PDP(T)$

4.1. **Planar pathos degree prime graphs.** A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such way that its edges intersect only at their end vertices. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called a *plane graph* or *planar embedding of the graph*. We now characterize the graphs whose $PDP(T)$ is planar.

Theorem 4.1. *A pathos degree prime graph $PDP(T)$ of a tree T is planar if and only if T is the star graph $K_{1,n}$ ($2 \leq n \leq 6$).*

Proof. Suppose $PDP(T)$ is planar. Assume that T is $K_{1,n}$ ($n \geq 7$). Suppose $T = K_{1,7}$. Clearly, the each edge in T lie on exactly one cut-vertex, say C. Let $P(T) = \{P_1', P_2', P_3', P_4'\}$ be a pathos set of T. Then $D_4^{(4)} - v$ is the spanning subgraph of $PDP(T)$, where v is a vertex at distance one from the central vertex C. Furthermore, since the pathos vertices P_i' ($1 \leq i \leq 4$) of $PDP(T)$ are pairwise adjacent, the crossing number of $PDP(T)$ becomes one, $\mathrm{cr}(PDP(T)) = 1$ (see Figure.4), a contradiction.

Conversely, suppose that $T = K_{1,n}$ ($2 \leq n \leq 6$). We consider the following three cases.
Case 1. If $T = K_{1,2} = P_3$, then $PDP(T) = C_4$, which is planar.
Case 2. For $n = 3$ and 4, the path number of T is two. Then $D_4^{(2)} - v$ and $D_4^{(2)}$, respectively, is the spanning subgraph of $PDP(T)$. Since the pathos vertices of $PDP(T)$ are pairwise adjacent, the crossing number of $PDP(T)$ becomes zero, $\mathrm{cr}(PDP(T)) = 0$.
Case 3. For $n = 5$ and 6, the path number of T is three. Then $D_4^{(3)} - v$ and $D_4^{(3)}$, respectively, is the spanning subgraph of $PDP(T)$. Since the pathos vertices of $PDP(T)$ are pairwise adjacent, the crossing number of $PDP(T)$ becomes zero, $\mathrm{cr}(PDP(T)) = 0$. Therefore, by all the cases above, $PDP(T)$ is planar. This completes the proof. □



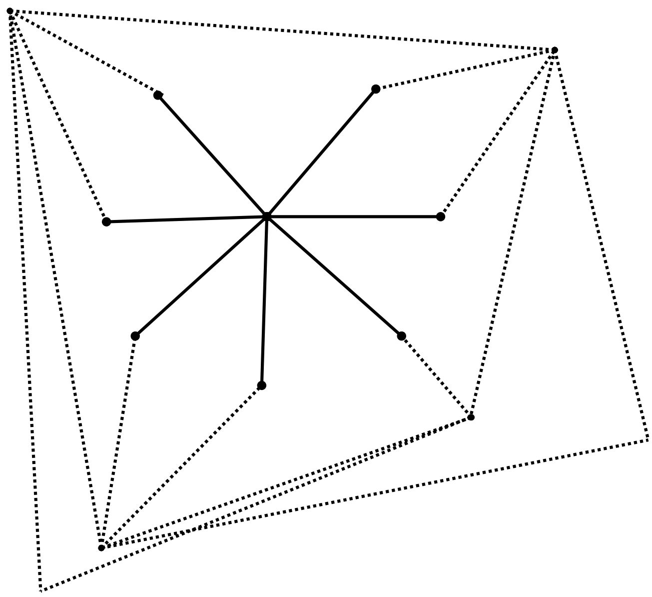


Figure.4

Note that the path number of the star graph $K_{1,8}$ is four and the corresponding pathos vertices are pairwise adjacent in $PDP(T)$. This shows that the crossing number of $PDP(T)$ is one. Therefore, the necessity of Theorem 4.1 can also be proved by assuming $T = K_{1,8}$.

We now establish a characterization of graphs whose $PDP(T)$ are outerplanar; maximal outerplanar; minimally nonouterplanar; and crossing number one.

For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be *outerplanar*, i.e., $i(G) = 0$.

Theorem 4.2. *A pathos degree prime graph $PDP(T)$ of a tree T is outerplanar if and only if $\Delta(T) \leq 2$, for every vertex $v \in T$, and T contains exactly one vertex of degree two.*

Proof. Suppose $PDP(T)$ is outerplanar. Assume that $\Delta(T) \leq 2$ and T contains two vertices of degree two. Then $T \simeq P_4$. By Case 2 of sufficiency of Proposition 3.1, $PDP(T)$ is a complete bipartite graph $K_{2,3}$ (see Figure.5). Clearly, $i(PDP(T)) = 1$, and hence $PDP(T)$ is nonouterplanar, a contradiction. On the other hand, if there exists a vertex of degree three in T. Then $T \simeq K_{1,3}$. Let C be the cut-vertex of $K_{1,3}$, and let $P(T) =$

$\{P_1^{'}, P_2^{'}\}$ be a pathos set of T. Then $D_4^{(2)} - v$ is the spanning subgraph of $PDP(T)$, where v is a vertex at distance one from C. Since the pathos vertices $P_1^{'}$ and $P_2^{'}$ are adjacent in $PDP(T)$, the inner vertex number of $PDP(T)$ becomes exactly one, i.e., $i(PDP(T)) = 1$ (see Figure.6), again a contradiction.

Conversely, suppose that $\Delta(T) \leq 2$, for every vertex $v \in T$, and T contains exactly one vertex of degree two. Then $T \simeq P_3$. By definition, $PDP(T) = C_4$ (see Figure.3), which is outerplanar. This completes the proof. □

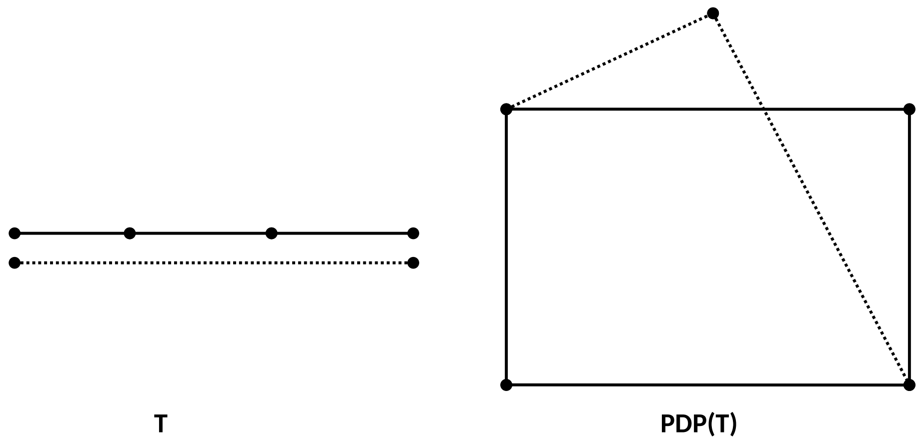


Figure.5

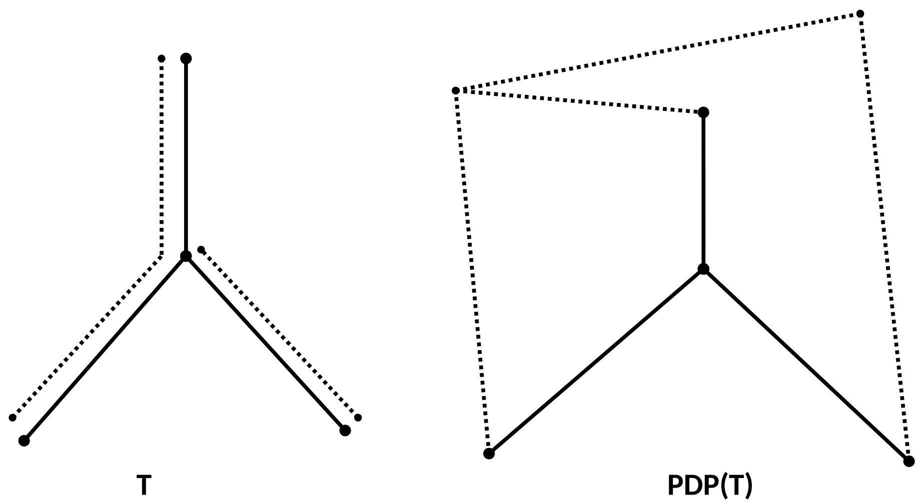


Figure.6

An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity.

Theorem 4.3. *For any tree T, a pathos degree prime graph $PDP(T)$ is not maximal outerplanar.*

Proof. We use contradiction. Suppose that $PDP(T)$ is maximal outerplanar. We consider the following four cases.

Case 1. Suppose that $T = K_{1,n}$ $(n \geq 7)$. By Theorem 4.1, $PDP(T)$ is nonplanar, a contradiction.
Case 2. Suppose that $T = K_{1,n}$ $(3 \leq n \leq 6)$. For $n = 4$ and 6, $D_4^{(2)}$ and $D_4^{(4)}$, respectively, is the spanning subgraph of $PDP(T)$. Next, for $n = 3$ and 5, $D_4^{(2)} - v$ and $D_4^{(4)} - v$, respectively, is the spanning subgraph of $PDP(T)$. Clearly, the inner vertex number of these spanning subgraphs is zero. Since all the pathos vertices of these spanning subgraphs are pairwise adjacent, the inner vertex number of $PDP(T)$ will be at least one. Thus $PDP(T)$ is nonouterplanar, a contradiction.
Case 3. Suppose that T is P_4. By necessity of Theorem 4.2, $PDP(T)$ is nonouterplanar, a contradiction.
Case 4. Suppose that T is P_3. Then $PDP(T) = C_4$, which is not maximal outerplanar, again a contradiction. Hence by all the cases above, $PDP(T)$ is not maximal outerplanar, which contradicts the assumption that $PDP(T)$ is maximal outerplanar. This completes the proof. □

The following characterization of minimally nonouterplanar graphs in [6] is well known.

Theorem 4.4. ([6]) : *A graph G is minimally nonouterplanar if and only if the inner vertex number of G is one, i.e., $i(G) = 1$.*

Theorem 4.5. *A pathos degree prime graph $PDP(T)$ of a tree T is minimally nonouterplanar if and only if T is either P_4 or $K_{1,3}$.*

Proof. Suppose $PDP(T)$ is minimally nonouterplanar. Assume that $T = K_{1,n}$ $(n \geq 4)$. If $T = K_{1,4}$. By Case 2 of sufficiency of Theorem 4.1, $\mathrm{cr}(PDP(T)) = 0$, but $i(PDP(T)) = 2$, a contradiction. On the other hand, assume that $T = P_n$ $(n \geq 5)$. By Case 3 of sufficiency of Proposition 3.1, $PDP(T) = K_{2,n-2}$. Clearly, $i(PDP(T)) \geq 2$, again a contradiction.

Conversely, suppose that T is either P_4 or $K_{1,3}$. By necessity of Theorem 4.2, $i(PDP(T)) = 1$, and thus Theorem 4.4 implies that $PDP(T)$ is minimally nonouterplanar. This completes the proof. □

The least number of edge crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by $\mathrm{cr}(G)$.

Theorem 4.6. *A pathos degree prime graph $PDP(T)$ of a tree T has crossing number one if and only if T is either $K_{1,7}$ or $K_{1,8}$.*

Proof. Suppose that $PDP(T)$ has crossing number one. Assume that $T = K_{1,n}$ $(n \geq 9)$. If $K_{1,9}$, then $D_4^{(5)} - v$ is the spanning subgraph of $PDP(T)$. Since all the pathos vertices of these spanning subgraphs are pairwise adjacent, the crossing number of $PDP(T)$ is more than one, a contradiction.

Conversely, suppose that T is $K_{1,7}$ or $K_{1,8}$. By necessity of Theorem 4.1, $\mathrm{cr}(PDP(T)) = 1$. This completes the proof. □

4.2. **Eulerian pathos degree prime graphs.** A *tour* of a connected graph G is a closed walk that traverses each edge of G at least once, and an *Euler tour* one that traverses each edge exactly once (in other words, a closed Euler trail). A graph is *Eulerian* if it admits an Euler tour.

We now investigate the Eulerian property of $PDP(T)$. The following result is well known.

Theorem 4.7. (***F. Harary*** [4]) : *A connected graph G is Eulerian if and only if each vertex in G has even degree.*

Theorem 4.8. *A pathos degree prime graph $PDP(T)$ of a tree T is Eulerian if and only if T is $K_{1,4n-2}$ $(n \geq 1)$.*

Proof. Suppose that $PDP(T)$ is Eulerian. We consider the following two cases.
Case 1. Assume that $T = K_{1,2n+1}$ for $n \geq 1$. Clearly, the degree of the central vertex C of T is $2n+1$, which is odd for $n \geq 1$. Since the degree of C remains unchanged in $PDP(T)$, Theorem 4.7 implies that $PDP(T)$ is non-Eulerian, a contradiction.
Case 2. Assume that $T = K_{1,4n}$ for $n \geq 1$. Then $D_4^{(2n)}$ for $n \geq 1$, is the spanning subgraph of $PDP(T)$. Clearly, the degree of each vertex in $D_4^{(2n)}$ is even. Since all the pathos vertices of $D_4^{(2n)}$ are pairwise adjacent, the degree of every pathos vertex of $PDP(T)$ is incremented by $2n-1$ for $n \geq 1$. Thus $d_{PDP(T)}(P^{'}) = 2 + 2n - 1 = 2n+1$. Since the degree of every pathos vertex of $PDP(T)$ is odd, Theorem 4.7 implies that $PDP(T)$ is non-Eulerian, a contradiction.

Conversely, suppose that T is $K_{1,4n-2}$ $(n \geq 1)$. If $T = K_{1,2}$, then $PDP(T) = C_4$, which is Eulerian. If $T = K_{1,4n+2}$ $(n \geq 1)$, then $D_4^{(2n+1)}$ for $n \geq 1$, is the spanning subgraph of $PDP(T)$. Clearly, the degree of each vertex in $D_4^{(2n+1)}$ is even. Since all the pathos vertices of $D_4^{(2n+1)}$ are pairwise adjacent, the degree of every pathos vertex of $PDP(T)$ is incremented by $2n$ for $n \geq 1$. Thus $d_{PDP(T)}(P^{'}) = 2 + 2n = 2(n+1)$. Since the degree of every pathos vertex of $PDP(T)$ is even, Theorem 4.7 implies that $PDP(T)$ is Eulerian. This completes the proof. □

4.3. **Hamiltonian pathos degree prime graphs.** A *Hamiltonian cycle* is a cycle that visits each vertex exactly once (except for the vertex that is both the initial and end, which is visited twice). A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph.*

We characterize the graphs whose $PDP(T)$ is Hamiltonian.

Theorem 4.9. *A pathos degree prime graph $PDP(T)$ of a tree T is Hamiltonian if T is either P_3 or P_4.*

Proof. Suppose that $T = P_3$. Then $PDP(T) = C_4$, which is Hamiltonian. On the other hand, if $T = P_4$, then $PDP(T)$ is isomorphic to the house graph, which is also Hamiltonian. This completes the proof. □

5. Conclusion

In this paper we have defined a graph operator called a pathos degree prime graph of a tree. We do not know of the directed path number of digraphs. Finding the directed path number of a digraph seems to be interesting one and it leads to the study of many digraph operators. What one can say about the properties of these digraph operators? All these facts highlight a wide scope for further studies in this direction.

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