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# DETERMINATION OF A TIME-DEPENDENT COEFFICIENT IN THE TIME-FRACTIONAL WAVE EQUATION WITH A NON-CLASSICAL BOUNDARY CONDITION

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Abstract. In this paper, an initial-boundary value problem for the time-fractional wave equation is considered. Given an additional condition, a time-dependent coefficient is determined and the existence and uniqueness theorem for small time is proved. An efficient finite difference scheme for solving the inverse problem is also proposed.

Keywords: Fractional wave equation, inverse coefficient problem, existence and uniqueness, non-classical boundary condition, finite difference method.

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## 1. Introduction

Consider the following partial differential equation (PDE) with a fractional derivative at time $t$

$$\partial_t^\alpha u(x,t) = u_{xx}(x,t) + a(t)u(x,t) + f(x,t),\ (x,t) \in \overline{D}_T, \tag{1}$$

where $D_T = \{(x,t) : 0 < x < 1,\ 0 < t < T\}$, $\partial_t^\alpha$ is the left sided Caputo fractional derivative of order $1 < \alpha < 2$ which is defined on the interval $(0,t)$ by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial_{ss} u(x,s)}{(t-s)^{\alpha-1}} ds$$

provided that $\Gamma(\cdot)$ is the Gamma function. Note that the equation (1) is a classical diffusion when $\alpha = 1$, and it is a wave equation when $\alpha = 2$. We consider equation (1) with the following initial and boundary conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ 0 \le x \le 1, \tag{2}$$

$$u(0,t) = 0,\ u_x(1,t) + du_{xx}(1,t) = 0,\ 0 \le t \le T,\ d > 0. \tag{3}$$

On the contrary of the common boundary conditions, the boundary condition (3) contains the term of maximal order $u_{xx}(1,t)$ which is called the non-classical boundary condition.

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Many of the universal electromagnetic, acoustic, and mechanical responses can be modeled accurately using the fractional wave (or diffusion-wave) equations, see [12, 13].

For a given function $a(t)$, $0 \leq t \leq T$, the problem (1)-(3) for the unknown function $u(x,t)$ is called the direct (forward) problem. The direct problem for the fractional diffusion-wave equation in a bounded domain has been investigated in [1, 10, 11]. If $a(t)$, with $0 \leq t \leq T$, is unknown, finding the pair of solution $\{a(t), u(x,t)\}$ from the problem (1)-(3) with the additional condition

$$u(x_0,t) = h(t),\ 0 \leq t \leq T, \tag{4}$$

is called the coefficient inverse problem. Here $0 < x_0 < 1$ is the location of the measurement, and equation (4) indicates that the measurement at a given point $x_0$ is available for any $0 \leq t \leq T$.

In the literature, there is not much work on the inverse problem of determining the time-dependent coefficient from the fractional wave equation. The inverse source Cauchy problem for the time-fractional wave equation was investigated in [8]. Determination of the time-dependent source function for the fractional wave equation with classical boundary conditions and non-classical boundary conditions were studied in [2, 9, 15] and [16, 17], respectively. In [5], the authors identified a space-dependent source term in a multidimensional time-fractional diffusion-wave equation from a part of noisy boundary data. Finding the initial displacement or initial velocity function from the initial-boundary value problem for the fractional wave equation with classical boundary conditions in a bounded domain was considered in [20]. In [21], the fractional order, initial flux speed and the Neumann boundary data were simultaneously determined from the partial observation of the Cauchy boundary data.

Some numerical aspects of the coefficient inverse problems were investigated in [3, 7, 19, 22, 23]. In particular, the authors in [3] converted the inverse problem into a nonlinear minimization problem, and then used the discretize-then-optimize approach to find the diffusion coefficient. In [23], a conjugate gradient method were used to solve a one-dimensional slab problem with two sensor locations. The authors in [7] proposed a high-order direct numerical method to solve the inverse problem involving the heat equation. In [19], central differences approximations have been used to solve the coefficient inverse problem for the Klein-Gordon equation. In [22], a pseudo-spectral method was proposed to solve an inverse problem for the linear Boussinesq-type equation.

In this paper, we consider an initial-boundary value problem for the fractional wave equation with a non-classical boundary condition. Given an additional condition, we determine the time-dependent coefficient and prove the existence and uniqueness theorem for small $T$ by means of the contraction principle. In addition, we propose an efficient direct numerical method for solving the coefficient inverse problem.

The article is organized as following: In Section 2, we present the preliminaries and the auxiliary spectral problem of the problem (1)-(3) and its properties. In Section 3, the series expansion method in terms of eigenfunctions converts the inverse problem to a fixed point problem in a suitable Banach space. Under some consistency and regularity conditions on the initial and boundary data, the existence and uniqueness of the inverse problem is shown by the way that the fixed point problem has unique solution for small $T$. In Section 4, the detailed description of our numerical method is presented. Two numerical experiments involving smooth and non-smooth exact solutions are shown to demonstrate the efficiency of our methods.

## 2. Preliminaries and auxiliary spectral problem

Throughout this paper, we use the following definition and lemma:

**Definition 2.1** ([6, 14])**.** *The generalized Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ z\in\mathbb{C},$$

*where $\alpha>0,\ \beta\in\mathbb{R}$.*

**Lemma 2.1** ([14])**.** *Let $0<\alpha<2$, and $\beta\in\mathbb{R}$ be arbitrary. We suppose that $\mu$ is such that $\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$. Then there exists a constant $C_{\alpha,\beta}$ such that*

$$|E_{\alpha,\beta}(z)|\le\frac{C_{\alpha,\beta}}{1+|z|},\ \mu\le|\arg(z)|\le\pi.$$

Consider the spectral problem which is corresponding to the problem (1)-(3)

$$\begin{cases} X''(x)-\lambda X(x)=0,\ 0\le x\le 1,\\ X(0)=0,\ X'(1)-dX(1)=0,\ d>0.\end{cases}\tag{5}$$

This spectral problem has the eigenfunctions $X_n(x)=\sqrt{2}\sin(\sqrt{\lambda_n}x)$, $n=0,1,...$ with positive eigenvalues $\lambda_n$ which are determined from the equation $\cot(\sqrt{\lambda})=d\sqrt{\lambda}$. We assign the zero index to an arbitrary eigenfunction and the remaining eigenfunctions in increasing order of eigenvalues. Let $\lambda_0$ be an arbitrary root of the equation $\cot(\sqrt{\lambda})=d\sqrt{\lambda}$. Consider the spectral problem

$$\begin{cases} X''(x)-\lambda X(x)=0,\ 0\le x\le 1,\\ X(0)=0,\\ X(1)+\frac{1}{d\sin(\sqrt{\lambda_0})}\int_0^1 X(x)\sin(\sqrt{\lambda_0}x)dx=0\end{cases}\tag{6}$$

This spectral problem is equivalent to the spectral problem (5) without the eigenfunction corresponding to the eigenvalue $\lambda_0$ and has the eigenfunctions $X_n(x)=\sqrt{2}\sin(\sqrt{\lambda_n}x)$, $n=1,2,...$ with positive increasing eigenvalues $\lambda_n$ determined from $\cot(\sqrt{\lambda})=d\sqrt{\lambda}$ (see [4]).

The system $X_n(x)=\sqrt{2}\sin(\sqrt{\lambda_n}x)$, $n=1,2,...$ is bi-orthogonal to the system

$$Y_n(x)=\frac{\sqrt{2}}{1+d\sin^2(\sqrt{\lambda_n})}\left[\sin(\sqrt{\lambda_n}x)-\frac{\sin(\sqrt{\lambda_n})}{\sin(\sqrt{\lambda_0})}\sin(\sqrt{\lambda_0}x)\right],\ n=1,2,...$$

and the system $X_n(x)$, $n=1,2,...$ forms a Riesz basis in $L_2[0,1]$. Also, the system $Y_n(x)$, $n=1,2,...$ is a Riesz basis in $L_2[0,1]$ and is complete.

## 3. Solution of the inverse problem

In this section, we will examine the existence and uniqueness of the solution of the inverse initial-boundary value problem for the equation (1) with time-dependent coefficient.

**Definition 3.1.** *A solution of the inverse problem (1)-(4), which we called the classical solution, is a pair of functions $\{a(t),u(x,t)\}$ satisfying $a(t)\in C[0,T]$, $u(x,t)\in C^2([0,1],\mathbb{R})$, and $\partial_t^{\alpha}u(x,t)\in C([0,T],\mathbb{R})$.*

From this definition, the consistency conditions

$$\mathbf{A}_0\text{:}\ \begin{cases} u_0(0) = u_0'(1) + du_0''(1) = 0, \\ u_1(0) = u_1'(1) + du_1''(1) = 0, \\ h(0) = u_0(x_0),\ h'(0) = u_1(x_0), \end{cases}$$

holds for the data $u_0(x), u_1(x) \in C^2[0,1]$ and $h(t) \in C[0,T]$ with $h(t) \neq 0,\ \forall t \in [0,T]$. Since the coefficient $a$ only depends on time and the boundary conditions are homogeneous, the Fourier method is suitable for the problem (1)-(3). That is, we can express $u(x,t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x), \tag{7}$$

where $u_n(t) = \int_0^1 u(x,t) Y_n(x) dx$, $n = 1, 2, ....$

Applying the Fourier method, we can obtain (from the equation (1) and initial conditions (2))

$$\begin{cases} \partial_t^\alpha u_n(t) + \lambda_n u_n(t) = F_n(t; u, a),\ 0 \le t \le T, \\ u_n(0) = u_{0,n},\ u_n'(0) = u_{1,n}, n = 1, 2, ..., \end{cases} \tag{8}$$

where $F_n(t; u, a) = a(t) u_n(t) + f_n(t)$, $f_n(t) = \int_0^1 f(x,t) Y_n(x) dx$, $u_{0,n} = \int_0^1 u_0(x) Y_n(x) dx$, and $u_{1,n} = \int_0^1 u_1(x) Y_n(x) dx$, $n = 1, 2, ....$

The Laplace transform of both sides of (8) yields

$$U_n(s) = \frac{\mathfrak{F}_n(s; u, a)}{s^\alpha + \lambda_n} + \frac{s^{\alpha-1} u_{0,n}}{s^\alpha + \lambda_n} + \frac{s^{\alpha-2} u_{1,n}}{s^\alpha + \lambda_n}.$$

By using the inverse Laplace transform, we obtain the solutions of the Cauchy problems (8) given by

$$\begin{aligned} u_n(t) &= u_{0,n} E_{\alpha,1}(-\lambda_n t^\alpha) + u_{1,n} t E_{\alpha,2}(-\lambda_n t^\alpha) \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) F_n(s; u, a) ds. \end{aligned} \tag{9}$$

Considering (9) into (7) we obtain that

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \big[ & u_{0,n} E_{\alpha,1}(-\lambda_n t^\alpha) + u_{1,n} t E_{\alpha,2}(-\lambda_n t^\alpha) \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) F_n(s; u, a) ds \big] X_n(x). \end{aligned} \tag{10}$$

For the determination of $a(t)$, one can derive that

$$a(t) = \frac{1}{h(t)} \left[ \partial_t^\alpha h(t) - f(x_0, t) - u_{xx}(x_0, t) \right]$$

from equation (1) with the additional data (4). Considering the equation (10) in the equation of $a(t)$, we get

$$a(t) = \frac{1}{h(t)} \left[ \partial_t^\alpha h(t) - f(x_0, t) - \sum_{n=1}^{\infty} \lambda_n u_n(t) X_n(x_0) \right]. \tag{11}$$

Alternatively, we can rewrite the equation above as

$$a(t) = \tfrac{1}{h(t)} \left[\partial_t^\alpha h(t) - f(x_0, t) \right.$$

$$- \sum_{n=1}^{\infty} \lambda_n \left( u_{0,n} E_{\alpha,1}(-\lambda_n t^\alpha) + u_{1,n} t E_{\alpha,2}(-\lambda_n t^\alpha) \right. \tag{12}$$

$$\left. \left. + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) F_n(s; u, a) ds \right) X_n(x_0) \right].$$

Thus we have reduced the problem (1)-(4) to the system (10)-(11) with respect to the unknown functions $a(t)$ and $u(x,t)$.

Now let us denote $z = [a(t), u(x,t)]^T$ and rewrite the system of equations (10)-(11) in the operator form

$$z = \Phi(z), \tag{13}$$

where $\Phi = [\phi_0, \phi_1]^T$ and $\phi_1$ and $\phi_0$ are equal to the right hand side of (11) and (10), respectively as

$$\phi_0(z) = \frac{1}{h(t)} \left[ \partial_t^\alpha h(t) - f(x_0, t) - \sum_{n=1}^{\infty} \lambda_n u_n(t) X_n(x_0) \right], \tag{14}$$

$$\phi_1(z) = \sum_{n=1}^{\infty} u_n(t) X_n(x), \tag{15}$$

where

$$u_n(t) = u_{0,n} E_{\alpha,1}(-\lambda_n t^\alpha) + u_{1,n} t E_{\alpha,2}(-\lambda_n t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) F_n(s; u, a) ds.$$

Let us introduce the functional space

$$B_{2,T}^{3/2} = \left\{ u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x) : \; u_n(t) \in C[0,T], \right.$$
$$\left. J_T(u) = \left[ \sum_{n=1}^{\infty} \left( \lambda_n^{3/2} \|u_n(t)\|_{C[0,T]} \right)^2 \right]^{1/2} < +\infty \right\}$$

with the norm $\|u(x,t)\|_{B_{2,T}^{3/2}} \equiv J_T(u)$ which relates the Fourier coefficients of the function $u(x,t)$ by the eigenfunctions $X_n(x)$, $n = 1, 2, \ldots$. It is shown in Appendix that $B_{2,T}^{3/2}$ is Banach space. Obviously, $E_T^{3/2} = B_{2,T}^{3/2} \times C[0,T]$ with the norm $\|z\|_{E_T^{3/2}} = \|u(x,t)\|_{B_{2,T}^{3/2}} + \|a(t)\|_{C[0,T]}$ is also a Banach space for $z = [a(t), u(x,t)]^T$.

Let us show that $\Phi$ maps $E_T^{3/2}$ onto itself continuously. In other words, we need to show that $\phi_0(z) \in C[0,T]$ and $\phi_1(z) \in B_{2,T}^{3/2}$ for arbitrary $z = [a(t), u(x,t)]^T$ with $a(t) \in C[0,T]$ and $u(x,t) \in B_{2,T}^{3/2}$.

We will use the following assumptions on the data of problem (1)-(4):

$$(A_1): \begin{cases} u_0(x) \in C^2[0,1], \\ u_0(1) + \frac{1}{d \sin(\sqrt{\lambda_0})} \int_0^1 u_0(x) \sin\left(\sqrt{\lambda_0} x\right) dx = 0, \end{cases}$$

$$(A_2): \begin{cases} u_1(x) \in C^2[0,1], \\ u_1(1) + \frac{1}{d\sin\left(\sqrt{\lambda_0}\right)} \int_0^1 u_1(x)\sin\left(\sqrt{\lambda_0}x\right)dx = 0, \end{cases}$$

$$(A_3): \begin{cases} f(x,t) \in C(\overline{D}_T), \\ f_x, f_{xx}, f_{xxx} \in C[0,1], \forall t \in [0,T], \\ f(0,t) = f_{xx}(0,t) = f_x(1,t) + df_{xx}(1,t) = 0, \\ f(1,t) + \frac{1}{d\sin\left(\sqrt{\lambda_0}\right)} \int_0^1 f(x,t)\sin\left(\sqrt{\lambda_0}x\right)dx = 0. \end{cases}$$

By using integration by parts under the assumptions $(A_0) - (A_3)$, it is easy to see that

$$u_{0,n} = \frac{1}{\sqrt{\lambda_n}}\eta_n, \; u_{1,n} = \frac{1}{\sqrt{\lambda_n}}\xi_n, \; f_n(t) = \frac{1}{\lambda_n^{3/2}}\gamma_n(t),$$

where $\eta_n = \frac{\sqrt{2}}{1+d\sin^2(\sqrt{\lambda_n})} \int_0^1 u_0(x)\cos\left(\sqrt{\lambda_n}x\right)dx$, $\xi_n = \frac{\sqrt{2}}{1+d\sin^2(\sqrt{\lambda_n})} \int_0^1 u_1(x)\cos\left(\sqrt{\lambda_n}x\right)dx$ and $\gamma_n(t) = \frac{-\sqrt{2}}{1+d\sin^2(\sqrt{\lambda_n})} \int_0^1 f_{xxx}(x,t)\cos\left(\sqrt{\lambda_n}x\right)dx$.

First, let us show that $\phi_0(z) \in C[0,T]$. Under the assumptions $(A_0)$-$(A_3)$ and using Cauchy-Schwartz inequality and Bessel inequality, we obtain from (14) that

$$\max_{0\le t\le T}|\phi_0(t)| \le \frac{1}{\min\limits_{0\le t\le T}|h(t)|}\left[\max_{0\le t\le T}|\partial_t^\alpha h(t)| - \max_{0\le t\le T}|f(x_0 t)| + d_1\left\{\widetilde{C}_{\alpha,1}\left(\sum_{n=1}^\infty|\eta_n|^2\right)^{1/2}\right.\right.$$
$$\left. + T\widetilde{C}_{\alpha,2}\left(\sum_{n=1}^\infty|\xi_n|^2\right)^{1/2} + \frac{TC_{\alpha,\alpha}}{\alpha}\left(\sum_{n=1}^\infty\left(\max_{0\le t\le T}|\gamma_n(t)|\right)^2\right)^{1/2}\right\}$$
$$\left. + \frac{TC_{\alpha,\alpha}}{\alpha}\max_{0\le t\le T}|a(t)|\,d_2\left(\sum_{n=1}^\infty\left(\lambda_n^{3/2}\max_{0\le t\le T}|u_n(t)|\right)^2\right)^{1/2}\right], \tag{16}$$

where $d_1 = \left(\sum_{n=1}^\infty \frac{1}{\lambda_n}\right)^{1/2}$, $d_2 = \left(\sum_{n=1}^\infty \frac{1}{\lambda_n^3}\right)^{1/2}$ and $\widetilde{C}_{\alpha,i} = \frac{C_{\alpha,i}}{k^\alpha}$ with $i = 1,2$ for $0 < k \le t \le T$. Therefore, the right hand side of (16) is bounded for $\phi_0(z) \in C[0,T]$.

Next, let us show that $\phi_1(z) \in B_{2,T}^{3/2}$. That is, we only need to show that

$$J_T(\phi_1) = \left[\sum_{n=1}^\infty\left(\lambda_n^{3/2}\max_{0\le t\le T}|\phi_{1n}(t)|\right)^2\right]^{1/2} < +\infty,$$

where $\phi_{1n}(t)$ is equal to the right hand side of $u_n(t)$ as in (9). After some manipulations on the last equality under the assumptions $(A_0)$-$(A_3)$, we get

$$\sum_{n=1}^\infty\left(\lambda_n^{3/2}\max_{0\le t\le T}|\phi_{1n}(t)|\right)^2 \le 2\left(\widetilde{C}_{\alpha,1}\right)^2\sum_{n=1}^\infty|\eta_n|^2 + 2T^2\left(\widetilde{C}_{\alpha,2}\right)^2\sum_{n=1}^\infty|\xi_n|^2 \tag{17}$$
$$+\left(\frac{TC_{\alpha,\alpha}}{\alpha}\right)^2\left[\sum_{n=1}^\infty\left(\max_{0\le t\le T}|\gamma_n(t)|\right)^2 + \left(\max_{0\le t\le T}|a(t)|\right)^2\sum_{n=1}^\infty\left(\lambda_n^{3/2}\max_{0\le t\le T}|u_n(t)|\right)^2\right].$$

From the Bessel inequality and $\left[\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\left\|u_n(t)\right\|_{C[0,T]}\right)^2\right]^{1/2} < +\infty$, the series on the right side of the last inequality is convergent. Thus $J_T(\phi_1) < +\infty$ and $\phi_1(z)$ belongs to the space $B_{2,T}^{3/2}$.

We now show that $\Phi$ is a contraction mapping on $E_T^{3/2}$. Let $z_i = \left[a^i(t), u^i(x,t)\right]^T$ with $i = 1, 2$ be any two elements in $E_T^{3/2}$. We know that $\|\Phi(z_1) - \Phi(z_2)\|_{E_T^{3/2}} = \|\phi_0(z_1) - \phi_0(z_2)\|_{C[0,T]} + \|\phi_1(z_1) - \phi_1(z_2)\|_{B_{2,T}^{3/2}}$. Under the assumptions $(A_0) - (A_3)$ and the equations (16)-(17), we can obtain that

$$\|\Phi(z_1) - \Phi(z_2)\|_{E_T^{3/2}} \leq A(T)C(a^1, u^2)\|z_1 - z_2\|_{E_T^{3/2}},$$

where $A(T) = \frac{TC_{\alpha,\alpha}}{\alpha}\left(1 + \frac{d_2}{\min\limits_{0\leq t\leq T}|h(t)|}\right)$ and $C(a^1, u^2)$ is the constant that depends on the norm of $\left\|a^1(t)\right\|_{C[0,T]}$ and $\left\|u^2(x,t)\right\|_{B_{2,T}^{3/2}}$. Since $A(T)$ has limit zero as $T$ tends to zero, it means that for sufficient small $T$, the operator $\Phi$ is a contraction mapping which maps $E_T^{3/2}$ onto itself continuously. Thus, according to the Banach fixed point theorem, there exists a unique solution of (13). To summarize, we have proved the following theorem:

**Theorem 3.1** (Existence and uniqueness)**.** *Suppose the assumptions $(A_0) - (A_3)$ are satisfied. Then the inverse problem (1)-(4) has a unique solution for small $T$.*

## 4. Numerical Solution

In this section, we will present our proposed numerical method for solving the inverse problem (1)-(4), and demonstrate its numerical performance on two cases with smooth and non-smooth $a(t)$, respectively.

We consider the inverse problem defined on $\overline{D_T} = \{(x,t) : 0 \leq x \leq 1,\ 0 \leq t \leq T\}$ for a given final time $T$. We partition the temporal domain into $N$ elements, and let $t_n := n\Delta t$ for $n = 0, 1, \ldots, N$, where $\Delta t = T/N$ is the uniform time step size. Similarly, we divide the spatial domain into $M$ elements using the grid points $x_k := k\Delta x$ for $k = 1, 2, \ldots, M$ with $\Delta x = 1/M$. To avoid confusion with the notation $x_0$ in equation (4), in this section we use $x^\star$ as the location of the measurement, and $x_0$ as one of the spatial grid points.

We then discretize the inverse problem using finite difference approximations. At $(x,t) = (x_k, t_n)$, $\partial_t^\alpha u$ can be approximated by

$$\partial_t^\alpha u|_{(x_k,t_n)} \approx \frac{(\Delta t)^{-\alpha}}{\Gamma(3-\alpha)}\sum_{i=0}^{n-1} d_{n,i}(U_k^{i+2} - 2U_k^{i+1} + U_k^i). \tag{18}$$

Here $d_{n,i} := (n-i)^{2-\alpha} - (n-i-1)^{2-\alpha}$, and $U_k^i$ represents the numerical approximation of $u$ at $(x,t) = (x_k, t_i)$. It was proved that the approximation in (18) is of first-order accuracy [18]. We use the standard second-order central difference approximation for the term $u_{xx}$ in (1). To deal with the non-classical boundary condition in (3), we introduce a ghost-cell with the left endpoint to be $x_M = 1$ and the right endpoint to be $x_{M+1} := 1 + 1/M$, and use second-order approximations for both $u_x$ and $u_{xx}$. That is,

$$0 = u_x(1, t_n) + du_{xx}(1, t_n) \approx \frac{U_{M+1}^n - U_{M-1}^n}{2\Delta x} + d\frac{U_{M+1}^n - 2U_M^n + U_{M-1}^n}{(\Delta x)^2}.$$

The equation above leads to

$$U_{M+1}^n = \frac{2d}{d+\frac{\Delta x}{2}} U_M^n - \frac{d-\frac{\Delta x}{2}}{d+\frac{\Delta x}{2}} U_{M-1}^n. \tag{19}$$

We further substitute the numerical solution $U_{M+1}^n$ given by (19) into the second-order central difference approximation for $u_{xx}(x_M, t_n)$, and obtain

$$u_{xx}(x_M, t_n) \approx \frac{U_{M+1}^n - 2U_M^n + U_{M-1}^n}{(\Delta x)^2} = \frac{U_{M-1}^n - U_M^n}{(d+\frac{\Delta x}{2})\Delta x}. \tag{20}$$

The left boundary condition $u(0,t) = 0$ gives $U_0^n = 0$ for all $n$, and thus we have

$$u_{xx}(x_1, t_n) \approx \frac{U_2^n - 2U_1^n}{(\Delta x)^2}. \tag{21}$$

To impose the condition $u(x^\star, t) = h(t)$, we can use the interpolation of the numerical solutions at the grid points to approximate $u(x^\star, t)$. For example, given the spatial grid points $\{x_i\}_{i=0}^M$, suppose $x^\star \in (x_k, x_{k+1})$, then we have

$$h(t_n) = u(x^\star, t_n) \approx \frac{x_{k+1} - x^\star}{\Delta x} U_k^n + \frac{x^\star - x_k}{\Delta x} U_{k+1}^n.$$

Alternatively, we can define the mesh in such a way that the point $x^\star$ coincides with one of the grid points. Let us suppose $x^\star = x_{k^\star}$ for some $k^\star$, then $h(t_n) \approx U_{k^\star}^n$. Based on the discussion above, (1)-(4) can be approximated by the following equations:

$$\begin{aligned}
&\tfrac{(\Delta t)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{n-1} d_{n,i}(U_k^{i+2} - 2U_k^{i+1} + U_k^i) = \frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{(\Delta x)^2} + a^n U_k^n + f_k^n, \\
&\tfrac{(\Delta t)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{n-1} d_{n,i}(U_1^{i+2} - 2U_1^{i+1} + U_1^i) = \frac{U_2^n - 2U_1^n}{(\Delta x)^2} + a^n U_1^n + f_1^n, \\
&\tfrac{(\Delta t)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{n-1} d_{n,i}(U_M^{i+2} - 2U_M^{i+1} + U_M^i) = \frac{U_{M-1}^n - U_M^n}{(d+\frac{\Delta x}{2})\Delta x} + a^n U_M^n + f_M^n, \\
&U_j^0 = u_0(x_j), \quad U_j^1 = U_j^0 + \Delta t u_1(x_j), \quad U_0^n = 0, \\
&U_{k^\star}^n = h(t_n),
\end{aligned} \tag{22}$$

for $k = 2, 3, \ldots, M-1$, $j = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$. Here $a^n := a(t_n)$ and $f_k^n := f(x_k, t_n)$. Note that we have already applied (20) and (21) to the approximation of the time fractional wave equation (1).

We now describe our proposed numerical scheme for solving the inverse problem (1)-(4) based on equations in (22). We first compute $a^0$ using

$$a^0 = \frac{\partial_t^\alpha h(t)|_{t=0} - u_0''(x^\star) - f(x^\star, 0)}{h(0)}, \tag{23}$$

where $x^\star$ is the location of the measurement, i.e., $u(x^\star, t) = h(t)$. Since we have assumed that $h(t) \neq 0$, $\forall t \in [0, T]$ in the consistency conditions $\mathbf{A}_0$, equation (23) is well-defined. We then initialize $U_k^0$ and $U_k^1$ for $k = 0, 1, \ldots, M$ using

$$U_k^0 = u_0(x_k) \quad \text{and} \quad U_k^1 = U_k^0 + \Delta t u_1(x_k). \tag{24}$$

At general $t = t^n$, we update $a^n$ and $U_k^n$ in alternating order. For $n = 1, 2, \dots, N$, we compute $a^n$ using

$$a^n = \left[ \frac{(\Delta t)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{n-1} d_{n,i} \left( h^{i+2} - 2h^{i+1} + h^i \right) - \frac{U_{k^\star+1}^n - 2U_{k^\star}^n + U_{k^\star-1}^n}{(\Delta x)^2} - f_{k^\star}^n \right] \frac{1}{h^n}, \quad (25)$$

where $k^\star$ is the index such that $k^\star \Delta x = x^\star$ and $h^i := h(i\Delta t)$. Note that $U_k^n$ for $k = 0, 1, \dots, M$ have been computed in the previous steps.

Next, we update $U_k^{n+1}$ using the first three equations in (22). For $k = 1, 2, \dots, M-1$, we update $U_k^{n+1}$ as follows:

$$\begin{aligned} U_k^{n+1} = 2U_k^n - U_k^{n-1} + \tfrac{(\Delta t)^\alpha \cdot \Gamma(3-\alpha)}{d_{n,n-1}} \left[ \tfrac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{(\Delta x)^2} + a^n U_k^n + f_k^n \right] \\ - \tfrac{1}{d_{n,n-1}} \sum_{i=0}^{n-2} d_{n,i} (U_k^{i+2} - 2U_k^{i+1} + U_k^i). \end{aligned} \quad (26)$$

For $k = M$, we update $U_M^{n+1}$ using

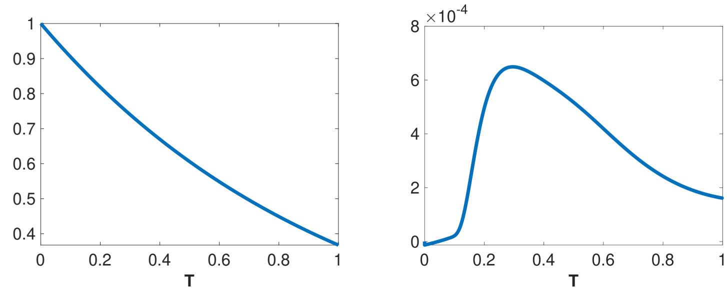
$$\begin{aligned} U_M^{n+1} = 2U_M^n - U_M^{n-1} + \tfrac{(\Delta t)^\alpha \cdot \Gamma(3-\alpha)}{d_{n,n-1}} \left[ \tfrac{U_{M-1}^n - U_M^n}{(d + \frac{\Delta x}{2})\Delta x} + a^n U_M^n + f_M^n \right] \\ - \tfrac{1}{d_{n,n-1}} \sum_{i=0}^{n-2} d_{n,i} (U_M^{i+2} - 2U_M^{i+1} + U_M^i). \end{aligned} \quad (27)$$

Due to the left boundary condition, $U_k^n = 0$ for $k = 0$. To summarize, we initialize the numerical solution $a^0$, $U_k^0$ and $U_k^1$ using (23) and (24), and then update $a^n$ and $U_k^{n+1}$ for $n = 1, 2, \dots, N$ in alternating order using (25)-(27).

**Example 1.** In this example, we consider the inverse problem (1)-(4) with smooth $a(t)$. We choose the following values for the parameters $\alpha$, $d$ and $x^\star$: $\alpha = 1.5$, $d = 1$ and $x^\star = 0.5$. We use the following data:

$$\begin{aligned} f(x,t) &= \tfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)}(1 - e^{-x}) + (1 + t^2)e^{-x} - e^{-t}(1 + t^2)(1 - e^{-x}), \\ u_0(x) &= 1 - e^{-x}, \quad u_1(x) = 0, \quad h(t) = (1 - e^{-1/2})(1 + t^2), \end{aligned} \quad (28)$$

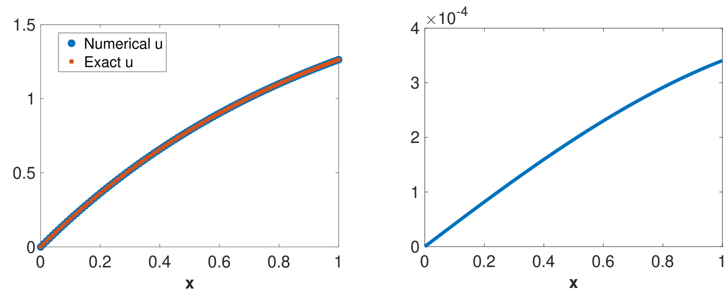
for $0 \le x, t \le 1$. The exact solution to the inverse problem with data (28) is $u(x,t) = (1+t^2)(1-e^{-x})$ and $a(t) = e^{-t}$. In our simulations, we choose $\Delta t = 10^{-4}$ and $\Delta x = 10^{-2}$ so that the error in space and time is at the same order of magnitude. The numerical solution and the error of $a(t)$ are shown in Figure 1, from which we can observe that our numerical solution captures the feature of exponential decay in $a(t)$ quite well. Numerical results also show that the maximum absolute error of $a(t)$ is $6.4914 \times 10^{-4}$ which is at the same order of magnitude as $\Delta t$. The numerical and exact solutions of $u$ at $T = 1$ are shown in Figure 2(A). We can see that the numerical solution coincides with the exact solution. When we further compute its error (see also Figure 2(B)), we find that the absolute value of error increases as $x$ increases, and the maximum absolute error of $u$ at $T = 1$ is $3.406 \times 10^{-4}$. The surface plots for the numerical solution of $u$ and its error are given in Figure 3. For $x, t \in [0, 1]$, the absolute maximum error of $u$ occurs at $x = t = 1$. The results indicate that our numerical solution is in good agreement with the exact solution.




(A) Numerical solution of $a(t)$ for $0 \le t \le 1$

(B) Error of $a(t)$ for $0 \le t \le 1$

FIGURE 1. Numerical solution and error of $a(t)$ for $0 \le t \le 1$ in example 1.




(A) Numerical and exact solutions of $u$ at $T = 1$

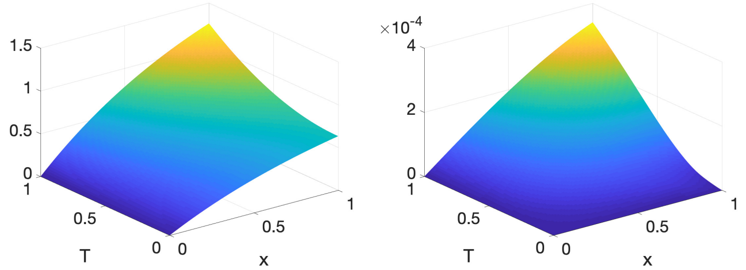
(B) Error of $u$ at $T = 1$

FIGURE 2. Numerical solution, exact solution and the error of $u$ at $T = 1$ in example 1.

**Example 2.** We then consider the inverse problem (1)-(4) with non-smooth $a(t)$. We take $\alpha = 1.2$, $d = 1$ and $x^\star = 0.2$, and use the following data:

$$\begin{aligned} f(x,t) &= -\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}(4x - x^2) + 2 - t^2 + \mathbf{1}_{[0.25,0.75)}(t) \cdot (1 - t^2/2)(4x - x^2), \\ u_0(x) &= 4x - x^2, \quad u_1(x) = 0, \quad h(t) = 0.76(1 - t^2/2), \end{aligned} \tag{29}$$

where $\mathbf{1}_{[0.25,0.75)}(t)$ is an indicator function. The exact solution to this inverse problem is $u(x,t) = (4x - x^2)(1 - t^2/2)$ and $a(t) = -\mathbf{1}_{[0.25,0.75)}(t)$. Note that in this example, the exact solution $a(t)$ is a discontinuous step function. We use the same mesh size and time step size as in Example 1, i.e., $\Delta t = 10^{-4}$ and $\Delta x = 10^{-2}$. As can be seen in Figure 4, the numerical solution of $a$ is an accurate approximation of the exact solution, with the maximum absolute error being $6.8762 \times 10^{-4}$. This value is slightly larger than the maximum absolute error of $a$ in example 1, but they are both at the same order of magnitude. We can also see the jumps of error at $T = 0.25$ and $0.75$, which is due to the discontinuity at these time. Similar to the results in example 1, the numerical solution of $u$ coincides with the exact solution at $T = 1$ and the absolute error of $u$ increases as $x$

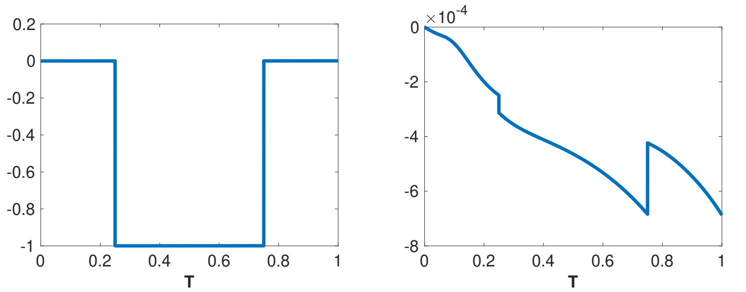



(A) Numerical solution of $u(x,t)$

(B) Error of $u(x,t)$

FIGURE 3. Numerical solution and error of $u(x,t)$ for $0 \leq x,t \leq 1$ in example 1.

increases (see Figure 5). The maximum absolute error of $u$ at $T = 1$ is $7.6514 \times 10^{-4}$, which is also the maximum absolute error of $u$ for all $0 \leq x,t \leq 1$ (see Figure 6). This example demonstrates that our numerical method also leads to accurate numerical solution for the inverse problem with non-smooth exact solution $a(t)$.



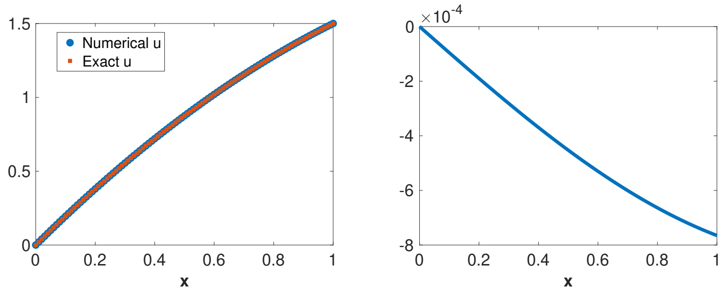

(A) Numerical solution of $a(t)$ for $0 \leq t \leq 1$

(B) Error of $a(t)$ for $0 \leq t \leq 1$

FIGURE 4. Numerical solution and error of $a(t)$ for $0 \leq t \leq 1$ in example 2.

## 5. Conclusion

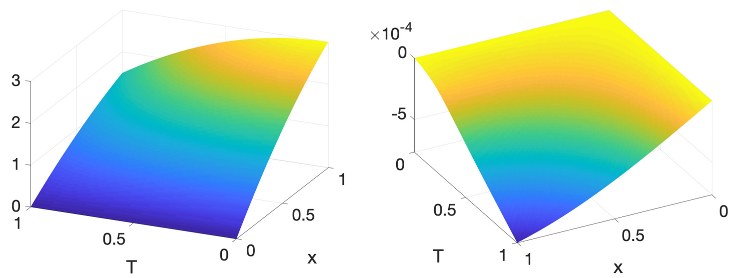
The paper considers the problem of determining the time-dependent coefficient in a time-fractional wave equation with non-classical boundary condition from the additional measurement. The consideration of the non-classical boundary conditions for the inverse problem of the time-fractional wave equation is the novelty of this work. In addition, the existence and uniqueness of the solution on a sufficiently small time interval is proved by means of the contraction principle. The key step of the proof is to establish a fixed-point system using the Fourier method, the Laplace transformation of the fractional derivative and the generalized Mittag-Leffler function. Such a form of the system brings along computations that are technically more simple than in the case of the usual variational approach. We also propose an efficient finite-difference-based direct method to compute




(A) Numerical and exact solution of $u$ at $T = 1$

(B) Error of $u$ at $T = 1$

FIGURE 5. Numerical solution, exact solution and the error of $u$ at $T = 1$ in example 2.




(A) Numerical solution of $u(x, t)$

(B) Error of $u(x, t)$

FIGURE 6. Numerical solution and error of $u(x, t)$ for $0 \leq x, t \leq 1$ in example 2.

the numerical solution of the inverse problem. Some numerical examples with smooth and non-smooth coefficient are performed to demonstrate the accuracy of our proposed numerical method.

## 6. Appendix

In this section, we show that the space $B^{3/2}_{2,T}$ is a Banach space. Since a Banach space is a complete normed space, we need to demonstrate that the normed space $B^{3/2}_{2,T}$ is complete. If every Cauchy sequence in $B^{3/2}_{2,T}$ converges, the space $B^{3/2}_{2,T}$ is said to be complete.

We consider any Cauchy sequence $\{u^m(x, t)\}$ in $B^{3/2}_{2,T}$, writing

$$u^m(x, t) = \sum_{n=1}^{\infty} u_n^{(m)}(t) X_n(x).$$

Since $\{u^m(x,t)\}$ is a Cauchy sequence, for every $\varepsilon > 0$ there is an $N$ such that for all $m, r > N$

$$\|u^m(x,t) - u^r(x,t)\|^2_{B^{3/2}_{2,T}} = \sum_{n=1}^{\infty} \left( \lambda_n^{3/2} \max_{0 \le t \le T} \left| u_n^{(m)}(t) - u_n^{(r)}(t) \right| \right)^2 < \varepsilon^2.$$

It follows that for every $n = 1, 2, ...$ we have

$$\max_{0 \le t \le T} \left| u_n^{(m)}(t) - u_n^{(r)}(t) \right| < \varepsilon.$$

Since $C[0,T]$ is complete, $u_n^{(m)}(t) \to u_n(t)$ as $m \to \infty$. Using these limits, we define $u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x)$ and show that $u(x,t) \in B^{3/2}_{2,T}$ and $u^m(x,t) \to u(x,t)$.

We have for all $m, r > N$

$$\sum_{n=1}^{k} \left( \lambda_n^{3/2} \max_{0 \le t \le T} \left| u_n^{(m)}(t) - u_n^{(r)}(t) \right| \right)^2 < \varepsilon^2, \quad (k = 1, 2, ...).$$

Letting $r \to \infty$, we obtain for all $m > N$

$$\sum_{n=1}^{k} \left( \lambda_n^{3/2} \max_{0 \le t \le T} \left| u_n^{(m)}(t) - u_n(t) \right| \right)^2 < \varepsilon^2, \quad (k = 1, 2, ...).$$

We may let $k \to \infty$, then for all $m > N$

$$\sum_{n=0}^{\infty} \left( \lambda_n^{3/2} \max_{0 \le t \le T} \left| u_n^{(m)}(t) - u_n(t) \right| \right)^2 < \varepsilon^2.$$

This implies that $u^m(x,t) \to u(x,t)$ and $u^m(x,t) - u(x,t) \in B^{3/2}_{2,T}$. Since $u^m(x,t) \in B^{3/2}_{2,T}$, $u(x,t) = u^m(x,t) + (u(x,t) - u^m(x,t)) \in B^{3/2}_{2,T}$. Thus $B^{3/2}_{2,T}$ is complete normed space.

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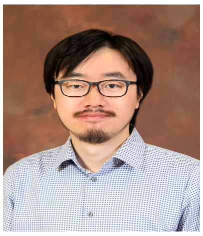

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