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CONTRACTION AND DOMINATION IN FUZZY GRAPHS

S. RAMYA[1], S. LAVANYA[2*], §

Abstract. Fuzzy sets and logics is a true crowning achievement of the century. Among the variety of exemplary changes in science and technology, the concept of uncertainty played a significant role, which led to the development of fuzzy sets, which in turn helped in the transition from graph theory to fuzzy graph theory. This paper familiarizes an improved concept in fuzzy graphs, called contraction. Two types of contraction namely edge contraction and neighbourhood contraction are introduced. We developed these two concepts in fuzzy graphs and analyse its effect on domination number and edge domination number. Any research is meaningful only by its contribution to the society. The modern world and the field of networks are inseparable. We have applied our concept to a wired network problem.

Keywords: Fuzzy graph, Domination, Edge Domination, Contraction, Edge contraction, Neighbourhood Contraction.

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1. Introduction

A graph is the diagrammatic representation of the information. In mathematics the field of graph theory has witnessed a tremendous growth. But when there is vagueness in describing the information Graph theory cannot be appropriate to describe the information diagrammatically. The seed for fuzzy theory was sown by Zadeh [20] to deal with uncertainty or vagueness in description of the problem. Rosenfeld introduced Fuzzy graph theory [13] by applying fuzzy theory to Graph theory, which was further cherished by Moderson [16].

In the course of study of graph theory, domination is one of the most interesting topics. The same is the case in fuzzy graph theory also. Haynes *et. al.,* [3] Fundamentals of dominations in graphs gave a remarkable beginning to the big idea of domination in graphs. The definition of Edge domination was stated by Arumugam and Velammal [1].The critical concept with respect to domination parameters was studied by Thakkar [16]. Along with domination, contraction also plays a vital role in Graph theory. Edge contraction was

[1] R. M. K. Engineering College, Tamilnadu, India.
e-mail: sra.sh@rmkec.ac.in; ORCID: https://orcid.org/0000-0002-3018-6569.
[2] Bharathi Women's College, Tamilnadu, India.
e-mail: lavanyaprasad1@gmail.com; ORCID: https://orcid.org/0000-0002-6108-3298.
[*] Corresponding author.
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studied by Thomas Wolle and Hans [17]. Domination and total domination contraction was analyzed by Jia Huang *et. al.,* [4] and Siska Dwi Oktavia *et. al.,* [11]. Kamath and Prameela Kolake [5] introduced neighborhood contraction in graph theory.

There is legion of articles published in domination and its variants in fuzzy graph theory. In fuzzy graphs, Somasundaram laid the foundation for the big idea of domination [14, 15], which was further developed by Nagoore Gani *et. al.,* [7, 9]. Velammal and Thiagarajan [18] defined edge domination in fuzzy graphs. Ramya and Lavanya [12] modified the statement of edge domination and introduced the critical concept in connection to the same parameter.

In a fuzzy graph G_f edge contraction is the instance of joining the two end vertices a & b of an edge $u = ab$ by removing the edge u. Since a long time the concept of edge addition and deletion have dragged much attention of various researchers. A huge number of results dealing in change of domination parameter due to addition or deletion of edges have been published so far. In parallel to edge addition and deletion, another similar operation performed is edge subdivision and contraction. In this paper we have made an attempt to define edge contraction, and as an extension of that we defined neighborhood contraction. We discuss edge contraction and neighborhood contraction in some special classes of fuzzy graphs and effect of contraction on domination number. All the basic definitions are considered as defined in [12] and [15].

2. Preliminaries

Definition 2.1. [12] *Let $G = (\sigma, \mu)$ be a fuzzy graph on (V, E). A subset D of V is called a dominating set of G if every vertex not in D is adjacent to at least one vertex in D.*

Definition 2.2. [15] *The fuzzy domination number $\gamma(G)$ is the cardinality of the minimal fuzzy dominating set of G.*

Definition 2.3. [12] *Let $G = (\sigma, \mu)$ be a fuzzy graph on (V, E). A subset S of VXV is called an edge dominating set of G if every edge not in S is incident to some edges in S.*

Definition 2.4. [12] *The cardinality of a minimal edge dominating set is called as the edge domination number of G and is denoted by $\gamma'(G)$.*

3. Edge Contraction

In this section, we define a new operation called edge contraction in fuzzy graphs.

Definition 3.1. *Let $G_f = (\sigma, \mu)$ be a fuzzy graph and let uv be the edge of G_f, then the edge contracted fuzzy graph with respect to the edge uv is denoted by $G_f \setminus uv$ and is a graph with vertex set $V' = [V \setminus \{u, v\} \cup \{w\}]$, where $\sigma(G_f \setminus uv) = \sigma(G_f)$ $\forall$ vertices $x \in V$, and*

$$\sigma(w) = \wedge(\sigma(u), \sigma(v))$$

and two vertices x and y are adjacent in $G_f \setminus uv$ if any one of the following conditions hold.

- *(i) If $x, y \in V$ & x, y are adjacent in G_f, then $\mu_{G_{f\setminus uv}}(xy) = \mu_{G_f}(xy)$.*
- *(ii) If $x \in V$, & $y = w$ then x and y are adjacent in $G_f \setminus uv$ if either x is adjacent to u or v then $\mu_{G_{f\setminus uv}}(xy) = \wedge(\sigma(x), \sigma(w))$.*
- *(iii) $x \in V$, & $y = w$ then x and y are adjoining in $G_f \setminus uv$ if x is adjacent to u and v in G_f then $\mu_{G_{f\setminus uv}}(xy) = \wedge[\mu_G(xu), \mu_{G_f}(xv)]$.*

Example 3.1.

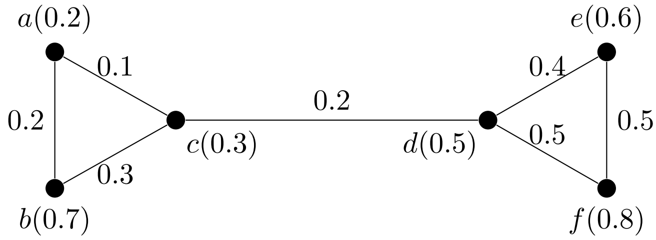

FIGURE 1

The edge contracted fuzzy graph with respect to the edge cd is given below by the graph $G_f \setminus cd$

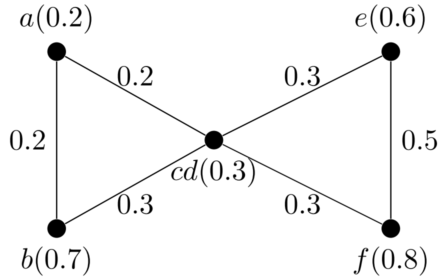

FIGURE 2

3.1. **Results.**

Theorem 3.1. *If* $G_f = (\sigma, \mu)$ *be a fuzzy graph. If* $E' \subset E(G_f)$*, then* $|V(G_f/E')| = |V(G_f)| - \vee_{\forall\ xy \in E'}[\sigma(x), \sigma(y)]$

Proof. Let the set E' consists of only one edge say xy. Then by the definition of edge contraction the vertex set of $G_f \setminus xy$ is given by $[V \setminus \{u, v\} \cup \{w\}]$, where $\sigma(G_f \setminus xy) = \sigma(G_f)$ for all vertices in V and $\sigma(w) = \wedge(\sigma(x), \sigma(y))$. Clearly cardinality of the contracted fuzzy graph $G_f \setminus xy$ will be equal to the cardinality of the vertex set of V minus the $\vee[\sigma(x), \sigma(y)]$ if $E' \subset E(G_f)$, then

$|V(G_f/E')| = |V(G_f)| - \vee_{\forall\ xy \in E'}[\sigma(x), \sigma(y)]$. □

Theorem 3.2. *Let* $G_f = (\sigma, \mu)$ *be a fuzzy graph. If* $E' \subset E(G)$*, then* $|E(G_f/E'| \leq |E(G_f)| - |E'|$

Proof. We have from subdivision (i),(ii) & (iii) of definition 2.1
$\mu_{G_{f \setminus uv}}(xy) = \mu_{G_f}(xy)$, x, y are adjacent in G_f &

$$\mu_{G_{f \setminus uv}}(xy) = \begin{cases} \wedge(\sigma(x), \sigma(w)) \text{ if either x is adjacent to u or v in } \mathrm{G_f} \\ \wedge[\mu_{G_f}(xu), \mu_{G_f}(xv)] \text{ if x is adjacent to both u and v in } \mathrm{G_f} \end{cases}$$

Hence we have $|E(G_f/E')| \leq |E(G)| - |E'|$. □

Theorem 3.3. *Contraction of an edge lessens the number of vertices exactly by one.*

Proof. By the definition, the new vertex set V' consists of $[V \setminus \{x, y\} \cup \{w\}]$ where $\sigma(G_f \setminus xy) = \sigma(G_f)\ \forall$ vertices $v \in V$ and $\sigma(w) = \wedge(\sigma(x), \sigma(y))$.

(ie) the two vertices x & y are merged into a single new vertex w.

Hence contraction of an edge lessens the number of vertices exactly by one. □

Theorem 3.4. *Contraction of an edge lessens the number of edges at least by one.*

Proof. Let xy be an edge which is to be contracted.

If all the vertices of $G_f \setminus xy$ is adjacent to only one vertex either x or y of the contracting edge then the contracted fuzzy graph will have exactly one edge less than the original graph G_f by (ii) of definition.

If some vertices of the contracted fuzzy graph are adjacent to both the vertices then the number of edges in the contracted fuzzy graph will be less than that of graph G_f by more than one.

Hence we have that contraction of an edge lessens the edgecount at least by one. □

Theorem 3.5. *Let* $e = xy$ *be an edge of a fuzzy graph* G_f $=(\sigma, \mu)$*, then we have* $\gamma(G_f) \geq \gamma(G_f \setminus e)$ *for any edge* $e \in G_f$.

Proof. We have $e = xy$ is an edge of a fuzzy graph G_f, the we have defined the vertex set of $G_f \setminus e$ as

$V' = [V \setminus \{u, v\} \cup \{w\}]$, where w is the new vertex.

Clearly $V' \subset V$. Also we have defined $\sigma(G_f \setminus e) = \sigma(G_f)$ for all vertices $v \in V$ & $\sigma(w) = \wedge(\sigma(x), \sigma(y))$

Therefore we have $\gamma_f(G_f) \geq \gamma_f(G_f \setminus e)$.

Hence $\gamma_f(G_f) \geq \gamma_f(G_f \setminus e)$ for any vertex $v \in V$. □

Theorem 3.6. *The edge contracted fuzzy graph of a complete fuzzy graph* (K_n) *is a complete fuzzy graph* K_{n-1}.

Proof. By the Theorem 2.3 the number of vertices of an edge contracted complete fuzzy graph $G_f(K_n)$ will be $n - 1$.

By the definition of complete fuzzy graph we have $\mu(xy) = \wedge(\sigma(x), \sigma(y))$ for all $xy \in V$ and also by (ii) of definition 2.1 we have edge contracted fuzzy graph with $n - 1$ vertices is also complete. □

Theorem 3.7. *The edge contracted fuzzy graph of a bipartite fuzzy graph is not a bipartite fuzzy graph.*

Proof. Since any edge in a bipartite fuzzy graph will consist of two vertices one from vertex set V_1 and one from vertex set V_2.

In the edge contracted fuzzy graph G'_e edges from all the vertices will be adjacent to the new contracted vertex.

Hence the resulting fuzzy graph will not be a bipartite graph. □

3.2. **Observation.**

(1) **Fuzzy Cycle:** The edge contracted graph of a fuzzy cycle C_n is a fuzzy cycle C_{n-1} .
(2) **Fuzzy Path:** The edge contracted graph of a fuzzy path P_n is a fuzzy path P_{n-1}.
(3) **Fuzzy Regular Graph:** The edge contracted graph of a fuzzy regular graph is not a regular fuzzy graph.

In edge contraction if the vertex obtained by contracting an edge fails the system will be in trouble. Hence we move further to rectify the same and introduce another new type of contraction called Neighborhood contraction.

4. Neighborhood Contraction

Definition 4.1. *Let* $G_f = (\sigma, \mu)$ *be a fuzzy graph and let* v *be any vertex of* G_f*. Then the neighbourhood contracted fuzzy graph of* G_f *with respect to the vertex* v *is denoted by*

$G_{f_{(v)}}$ and is defined as a graph with vertex set $V - N(v)$, where $\sigma(G_{f_{(v)}}) = \sigma(G_f)$ for all vertices $x \in V - N(v)$, and two vertices $u, w \in V - N(v)$ are adjacent in $G_{f_{(v)}}$ if any of the below mentioned condition holds:

(i) If $w = v$ and u is adjacent to any vertex of $N(v)$ in G_f and the edge membership is given by $\mu_v(uw) = \wedge\mu(ux)$ for all $x \in N(v)$.

(ii) If u, w does not belong to $N(v)$ and u, w are adjacent in G_f then $\mu_v(uw) = \mu(uw)$.

Example 4.1. *Consider the fuzzy graph given below*

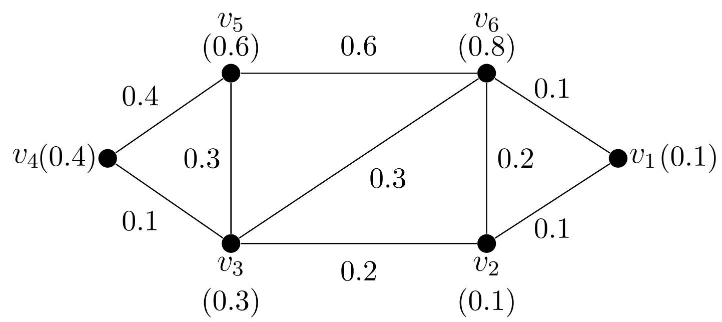

FIGURE 3

We have
Vertex dominating set is (VDS) is either $\{v_3, v_2\}$ or $\{v_3, v_1\}$
Domination number is $\gamma(G_f) = 0.4$
Edge dominating set (EDS): $\{v_1v_6, v_3v_4\}$
Edge domination number, $\gamma'(G_f) = 0.1 + 0.1 = 0.2$
The neighborhood contracted fuzzy graph of the above graph G with respect to the vertex v_2 is given by $G_f(v_2)$

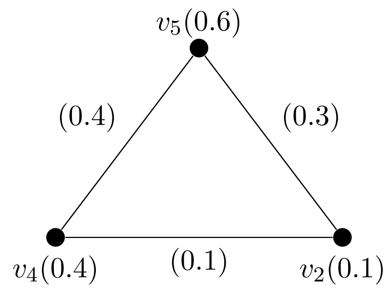

FIGURE 4

The domination number of $G_{f_{(v_2)}}$ is $\gamma_{(G_{v_2})} = 0.1$
The edge domination number of G_{v_7} is $\gamma'(G_{v_2}) = 0.1$.

Theorem 4.1. *Let v be any vertex of a fuzzy graph $G_f = (\sigma, \mu)$, then we have $\gamma(G_{f_{(v)}}) \geq \gamma(G_{f_{(v)}})$ for any vertex $v \in G_f$*

Proof. Let D be a dominating set of a fuzzy graph G_f and $v \in V(G_f)$. If v is an isolated vertex of G_f, then $G_{f_{(v)}} = G_f$.

Therefore $\gamma(G_f) = \gamma(G_{f_{(v)}})$.
Let v be any non-isolated vertex of G_f, then we have defined the vertex set of G_v as $V - N(v)$. Clearly $V - N(v) \subset V$.
Also we have defined $\sigma(G_{f_{(v)}}) = \sigma(G_f)$ for all vertices $v \in V - N(v)$.
Therefore we have $\gamma(G_f) > \gamma(G_{f_{(v)}})$. □

4.1. **Neighborhood contraction in some special classes of fuzzy graphs.**

Theorem 4.2. *If G_f is a complete fuzzy graph K_n then $G_{f_{(v)}}$ is a trivial graph for any vertex $v \in V$.*

Proof. Since G_f is a complete fuzzy graph K_n, any vertex will be adjacent to the remaining $n-1$ vertices. Hence neighborhood contraction with respect to any vertex yields a trivial graph.
Result:
The domination number of the neighborhood contracted fuzzy graph $((K_n))_v$ of a complete fuzzy graph K_n is $\sigma(v)$. □

Theorem 4.3. *If G_f is a complete bipartite fuzzy graph $K(m,n)$, then the neighborhood contracted fuzzy graph $G_{f_{(v)}}$ is a bipartite fuzzy graph.*

Proof. If $K(m,n)$ is a complete bipartite fuzzy graph with partite sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$ then

$$(k_{m,n})_v = \begin{cases} k_{1,m-1} \; if \;\; v \in V_1 \\ k_{1,n-1} \; if \;\; v \in V_2 \end{cases}$$

Hence the resulting fuzzy contracted graph with respect to any vertex v is just a bipartite graph and it is not complete bipartite.
The following are the simple observations

(1) The neighborhood contracted fuzzy graph of a fuzzy path P_n is a fuzzy path given by
$$(P_n)_v = \begin{cases} P_{n-1}, \; if \;\; v \; is \; v_1 \; or \; v_n \\ P_{n-2}, \; otherwise. \end{cases}$$
(2) The neighborhood contracted fuzzy graph of a fuzzy cycle C_n is a fuzzy cycle C_{n-2}. □

Definition 4.2. *Let $G_f = (\sigma, \mu)$ be a fuzzy graph and let v be any vertex of G_f. Then the vertex v is said to be*
(i) Neighborhood contracted $\gamma-$ fixed vertex if v lies in every VDS of $G_{f_{(v)}}$.
(ii) Neighborhood contracted $\gamma-$ free vertex if v lies in some VDS of $G_{f_{(v)}}$ but not in all.
(iii) Neighborhood contracted $\gamma-$ totally free vertex if v does not lies in any VDS of $G_{f_{(v)}}$.

Example 4.2.

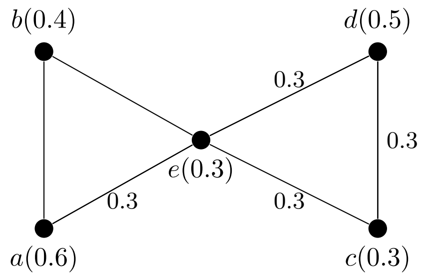

FIGURE 5

Neighborhood contracted graph with respect to vertex $'a'$, $G_{f_{(a)}}$ *is given below*

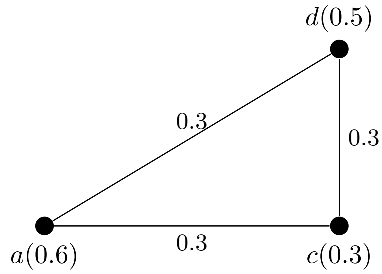

FIGURE 6

Dominating set of $G_{f_{(a)}}$ *is given by* $\{c\}$.*Here the vertex a is a neighbourhood contracted* $\gamma - totally\ free\ vertex.$
In the same graph the vertex $'e'$ *is a neighborhood contracted* $\gamma - fixed\ vertex.$

Result:

(1) If G_f is a complete fuzzy graph then every vertex of G_f will be a neighborhood contracted $\gamma-$ fixed vertex of $G_{f_{(v)}}$ for any vertex v.
(2) If G_f is a uniform vertex fuzzy graph, then every vertex v will be a neighborhood contracted $\gamma-$ fixed vertex of $G_{f_{(v)}}$ for any vertex v.

5. Application

A network is a group or system of interconnected people or things. There are varieties of networks available say computer networks, telecommunication networks, social network etc., Communication networks can be broadly classified into two types (i) wireless networks and (ii) wired network. A wired network is a system in which nodes are communicated through wires.

Any type of network can be modeled as a graph using nodes and links as vertices and edges. Fuzzy graph models will be more suitable to the real time problems to give a more effective solution for the factors like capacity; cost, distance, etc. can be included in the model. We will consider a wired telecommunication network problem. That is placing of signal towers in certain states or provisions within a country. The states can be represented by vertices and the edges represent the link between the states. The transmission range and the distance can be considered for assigning the membership values to the vertices and edges. Thus the system can be represented as fuzzy graph model.

Vertex domination is a fault tolerant design that helps to handle the situation when a node fails. That is if a node in a network fails the communication to the other nodes can be still made with the help of vertices in the vertex dominating set. Likewise the idea of EDS can be used to design a fault tolerant network which will handle the situation when a link fails. That is if a link (edge) fails then the communication can reach the other nodes through the edges which are in the EDS. In other words if the edges identified in the edge dominating set fails communication will be interrupted.

Now let us work our application by considering the states of some country and framing it into a fuzzy graph model.

We will consider the problem of placing towers at state heads in few selected states of a country say, State 1, State 2, State 3, State 4, State 5, State 6, State 7. Let us assign the vertices a,b,c,d,e,f,g to represent the states 1,2,3,4,5,6,7 respectively. Then we will have the fuzzy graph model as below

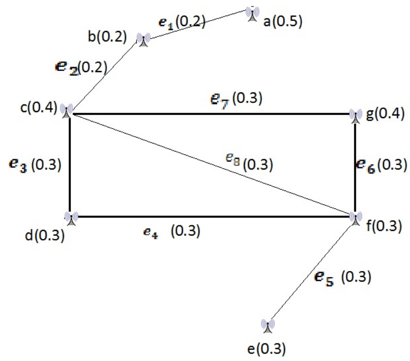

FIGURE 7

In this model the edge dominating set is $\{e_1, e_8\}$ or $\{e_2, e_8\}$ and the $\gamma'(G_f)$ is $0.2+0.3 = 0.5$.

If we remove the edges in the EDS, then it will either interrupt the communication between the states or there will be a delay in the reception of the communication. That is to provide a un- interrupted communication transfer among all the states the edges e_1, e_8 and e_2 should be properly maintained. Removing of an edge by merging the two end vertices is called as edge contraction. Now if we remove the edge e_8 by merging its end vertices c and f then the resulting edge contracted fuzzy graph will be as below

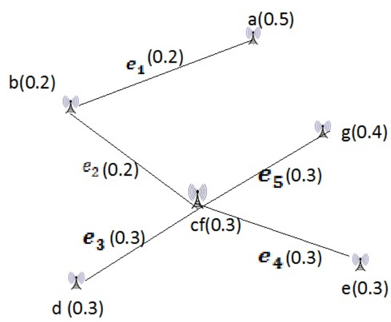

FIGURE 8

Here the edge dominating set consists of only one edge e_2 , and the edge domination-number is 0.2.That is in Physical terms instead of placing the nodes at c (State 3) and f

(State 6), we can place a node at an intermediate point cf (in between those two states), which will lead to the design of an efficient system. Thus edge contraction helps us to design efficient models in network systems.

6. Conclusions

We have defined and studied two types of contractions, namely edge and neighborhood contraction in fuzzy graphs. We have discussed few basic results on the same and investigated these new topics on some special classes of fuzzy graphs. We have applied the big idea of domination to the networks which has been explained through an example. Our future work is to further extend this concept of contraction to other variants of domination and also to apply it for different types of Fuzzy graphs.

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Dr. S. Ramya is an assistant professor in the Department of Mathematics, R. M. K. Engineering College, Chennai, India. She obtained her BSc and MSc in Mathematics from the University of Madras in Mathematics from Alagappa University, Bachelor of Education from Tamilnadu Teachers Education University and Doctoral Degree in Mathematics from the Bharathiar University, Coimbatore in 2019 in Fuzzy Graph theory. Her research interests include Fuzzy Graph theory and Graph theory.

Dr. S. Lavanya is an assistant professor in the Department of Mathematics, Bharathi Women's College, Chennai, India. She obtained her BSc, MSc and M.Phil degrees in Mathematics from the University of Madras and her PhD from SCSVMV University. Her area of research are Graph theory, Fuzzy Graph theory and Fuzzy optimization.