TWMS J. App. and Eng. Math. V.13, N.1, 2023, pp. 189-201

RIESZ MRA OF DYADIC DILATIONS AND THE CORRESPONDING RIESZ WAVELET ON LCA GROUPS

R. KUMAR[1], SATYAPRIYA[2*], M. SINGH[3], §

Abstract. We have explored the concept of Riesz multiresolution analysis (Riesz MRA) on a locally compact Abelian group G, and have done a detailed study of the methods of construction of a Riesz wavelet from the given Riesz MRA. For simplicity, we have assumed the order of dilations to be two, i.e. we have worked with dyadic dilations. We have proved that precisely one function is required to construct a Riesz wavelet basis for the space $L^2(G)$.

Keywords: LCA Groups, Riesz Basis, Multiresolution Analysis, dyadic dilation, refinement equation.

AMS Subject Classification: 42C40, 22B05.

1. Introduction

Mallet presented the idea of multiresolution analysis (MRA) on the space $L^2(\mathbb{R})$ in 1986 [17], and since then it has become a tool of choice for investigation and construction of wavelet bases. After that, a number of studies have also pursued the application of MRA for Euclidean spaces [9, 19]. In the following years, Dahlke [8] generalized the concept of MRA for arbitrary locally compact Abelian (LCA) groups. Then on, many authors have contributed to the field of construction of wavelet bases on a variety of groups [4, 20, 23]. Gol and Tousi [13] have obtained equivalent multiresolution conditions using the theory of spectral function and shift-invariant spaces. Bownik and Jahan recently explored the wavelet theory in compact Abelian groups [4].

In all the mentioned works above, MRA has been used to construct a wavelet orthonormal basis for the underlying space. Here, we wish to generalize this notion and thus

---

[1] Department of Mathematics, Kirori Mal College, University of Delhi, Delhi, India.
e-mail: rajkmc@gmail.com; ORCID: https://orcid.org/0000-0003-0714-5045.
[2] Department of Mathematics, University of Delhi, Delhi, India.
e-mail: kmc.satyapriya@gmail.com; ORCID: https://orcid.org/0000-0001-5002-6709.
[*] Corresponding author.
[3] Department of Mathematics, SGND Khalsa College, University of Delhi, Delhi, India.
e-mail: mansimransin@gmail.com; ORCID: https://orcid.org/0000-0002-9071-9064.
§ Manuscript received: October 19, 2020; accepted: July 20, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.1 © Işık University, Department of Mathematics, 2023; all rights reserved.
The research of the second author is supported by the Senior Research Fellowship (SRF) of Human Resource Development Group, Council of Scientific and Industrial Research (HRDG-CSIR), India, (Grant No: 09/045(1653)/2019-EMR-I).

.

construct a wavelet Riesz basis through MRA. The primary motivation for studying Reisz bases is that these bases are found to be very handy for studying the sampling of band-limited signals (functions). It is well-known that, up to some transformation, Riesz bases are equivalent to the interpolation property. This makes them a robust tool in compress sensing and application to signal processing. There are a few works available in literature which deal with MRA and alike structures for Riesz bases [3, 11, 18, 21, 24]. The authors of these works have thoroughly explored the construction methods of Riesz wavelt bases through MRA.

We come across many practical problems in signal processing, data analysis, and time series problems where the information is in the form of discrete data. Taking this fact into cognizance, one realizes that a more generalized approach involving LCA groups is a way out. This also serves as our primary motivation for working in the setting of LCA groups. Some progress in this direction is made by a few researchers [5, 7, 15, 18, 21]. With an objective to further the exploration towards positive outcomes, in this work we have delved into a more generalized Riesz wavelet bases in the abstract settings of a LCA group.

In this paper, we have constructed a Riesz wavelet function and, thus, a Riesz wavelet basis applying Riesz MRA structure on $L^2(G)$. For simplicity, the case of dyadic dilations has been taken into consideration here. We have structured this paper in the following framework. Some preliminaries and notations have been given in section 2. A detailed method for the construction of Riesz wavelet from the given Riesz MRA is presented in section 3. Finally, we have concluded our work in section 4.

2. Preliminaries and Notations

2.1. **LCA Groups.** We briefly review some elementary concepts about LCA groups here. For a detailed study, we refer [12, 22].

We call a topological group G, an LCA group if

- along with being locally compact in its topology, it is also Hausdorff and metrizable; and,
- it can be written as a countable union of compact sets.

The symbols $'+'$ and $'0'$ are respectively used to denote the group composition and the identity element of G. The groups $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, $\mathbb{Z}_n$ are some of the frequently used LCA groups. These groups, along with their higher dimensional variants, are called *elementary LCA groups.*

If $\mathbb{T}$ denotes the circle group $\{z \in \mathbb{C} : |z| = 1\}$, then a *character* on G is a function $\gamma : G \to \mathbb{T}$ such that for any $x, y \in G$, $\gamma(x + y) = \gamma(x)\gamma(y)$. Let $\hat{G}$ denote the set of all continuous characters on G. This set $\hat{G}$, called the dual group of G, also forms an LCA group (see [22]) when equipped with the compact-open topology and the composition

$$(\gamma + \gamma')(x) = \gamma(x)\gamma'(x); \gamma, \gamma' \in \hat{G} \text{ and } x \in G.$$

The *Pontryagin duality theorem* allows us to identify the double dual group $\hat{\hat{G}}$ with the group G and hence we can write $\hat{\hat{G}} = G$. Therefore, $\gamma(x)$ can be interpreted as either the action of $\gamma \in \hat{G}$ on $x \in G$; or action of $x \in \hat{\hat{G}} = G$ on $\gamma \in \hat{G}$ and thus from now on we will use the notation

$$(\gamma, x) = \gamma(x); \gamma \in \hat{G}, x \in G.$$

Clearly, (γ, x) is a member of $\mathbb{T}$. Further, we will denote by $-x$, the inverse of the element $x \in G$, and by $-\gamma$, the inverse of the element $\gamma \in \hat{G}$. Moreover, these inverse elements satisfy the relation

$$(\gamma, -x) = (-\gamma, x) = \overline{(\gamma, x)}.$$

Remark 2.1. *Note that the usage of the symbol $'+'$ and $'-'$ will depend entirely on the context, whether it is being used as a group operation or as a usual sum. To be more precise, whenever we will be dealing with elements of G or $\hat{G}$, $+$ will be used as a group operation and in all other cases, $+$ will mean the usual sum.*

We now equip the group G with a translation invariant Radon measure μ_G i.e.

$$\int_G f(x+y)d\mu_G(x) = \int_G f(x)d\mu_G(x), \ \forall \ y \in G$$

and for all compactly supported functions f on G. This measure, which is unique up to a constant, is called the *Haar measure*. We refer [22] for the existence and uniqueness of Haar measure. The Haar measure μ_G has been kept fixed throughout this paper. Based on this Haar measure, we define the spaces $L^p(G)$ $(1 \leq p \leq \infty)$ in the usual way. Out of these spaces, only $L^2(G)$ is a Hilbert Space and in fact a separable Hilbert space due to our assumptions of G being metrizable and being a countable union of compact sets (see [12]).

We now define the operator of *Fourier transform* on $L^1(G)$ by:

$$\mathcal{F} : L^1(G) \to C_0(\hat{G}), \quad \mathcal{F}(f)(\gamma) = \int_G f(x)(\gamma, -x)d\mu_G(x). \tag{1}$$

Here, $C_0(\hat{G})$ is the space of all continuous functions on $\hat{G}$ vanishing at infinite.
The Haar measure $\mu_{\hat{G}}$ on $\hat{G}$ can be appropriately normalized so that for a specific class of functions, the following *inversion formula* holds, (see[22, Chapter 1]);

$$f(x) = \int_{\hat{G}} \hat{f}(\gamma)(\gamma, x)d\mu_{\hat{G}}\gamma, x \in G. \tag{2}$$

In this paper, we shall always choose a normalized Haar measure $\mu_{\hat{G}}$ for $\hat{G}$ so that the inversion formula holds. Once this is done, the Fourier transform can be extended to a surjective isometry $\mathcal{F} : L^2(G) \to L^2(\hat{G})$ exactly as in the classical case of $G = \mathbb{R}$. To simplify notations, from now onwards, in all integrals when the context is clear, we will write $d\mu_G(x) = dx$ and $d\mu_{\hat{G}}(\gamma) = d\gamma$.

Apart from the operator of Fourier transform, the operators of traslation, modulation and dilation will also be used frequently throughout this paper. So we need to have generalized versions of these three operators. The first two operators, i.e. the translation operator and the modulation operator can be extended to $L^2(G)$ without much difficulty. For instance, for any $y \in G$, the operators:

$$T_y f(x) = f(x-y) \quad \text{and} \quad \mathcal{E}_y f(\gamma) = (\gamma, y)f(\gamma); \ \forall \ x \in G, \forall \ \gamma \in \hat{G},$$

respectively define the generalized translation and modulation operators on $L^2(G)$ and $L^2(\hat{G})$. Also, for any $\xi \in \hat{G}$, $\mathcal{T}_\xi$ and E_ξ will be used to denote the generalized translation and modulation operators on $L^2(\hat{G})$ and $L^2(G)$ respectively.

To define a generalized version of dilation operator, we need a dilative automorphism (see [15]) on G. For if α is a dilative automorphism on G, then the dilation operator D on $L^2(G)$ is given by

$$D : L^2(G) \to L^2(G), \quad Df(x) = \delta_\alpha^{1/2} f(\alpha(x));$$

where the constant $\delta_\alpha > 0$ is such that

$$\int_G f(x)dx = \delta_\alpha \int_G f(\alpha(x))dx$$

for any appropriate function f on G.

This constant δ_α is called the *order of dilation* for the operator D. Further, using this dilation operator D, we can construct a dilation operator $\mathcal{D}$ for the space $L^2(\hat{G})$. The following lemma sums up the required information of the operator $\mathcal{D}$. We omit the striaghtforward proof. See [18] for similar constructions.

Lemma 2.1. *Let G be an LCA group and $\hat{G}$ be its dual group. Suppose $\alpha : G \to G$ is a dilative automorphism on G. Then the following hold:*

(*i*) *The map, $\hat{\alpha} : \hat{G} \to \hat{G}$ given by*

$$(\hat{\alpha}(\gamma), x) = (\gamma, \alpha(x));\ x \in G,$$

is a dilative automorphism (algebraic automorphism and topological homeomorphism) on $\hat{G}$.

(*ii*) $\int_{\hat{G}} F(\gamma)d\gamma = \delta_\alpha \int_{\hat{G}} F(\hat{\alpha}(\gamma))d\gamma$ *for any appropriately defined function F on $\hat{G}$.*

(*iii*) *The operator $\mathcal{D} : L^2(\hat{G}) \to L^2(\hat{G})$ given by $\mathcal{D}F(\gamma) = \delta_\alpha^{1/2} F(\hat{\alpha}(\gamma))$ is also a unitary operator on $L^2(\hat{G})$. This operator $\mathcal{D}$ works as dilation operator on $L^2(\hat{G})$.*

It is easy to note that all these generalized operators satisfy all the commutative relations amongst them and behave similarly under Fourier transform and inverse Fourier transform, as in the case of $G = \mathbb{R}$.

We now introduce lattices, an important class of subgroups of LCA groups. A *lattice* Λ, (sometimes called a uniform lattice), in an LCA group G, is a countable, closed and discrete subgroup Λ of G for which the quotient group G/Λ is compact in the quotient topology. The *annihilator* $\Lambda^\perp$ of a lattice Λ is defined by

$$\Lambda^\perp = \{\gamma \in \hat{G} : (\gamma, \lambda) = 1,\ \ \forall \lambda \in \Lambda\}.$$

It follows from the definition of topology on $\hat{G}$ that $\Lambda^\perp$ is also a lattice in $\hat{G}$. Further, a lattice in G can be used to obtain a splitting of groups G and $\hat{G}$ into disjoint cosets (see [6], Chapter 21):

Lemma 2.2. *Let G be an LCA group and Λ a lattice in G. Then the following hold:*

(*i*) *There exists a Borel measurable relatively compact set $\mathcal{Q} \subseteq G$ such that*

$$G = \bigcup_{\lambda \in \Lambda} (\lambda + \mathcal{Q}),\ \ (\lambda + \mathcal{Q}) \cap (\lambda' + \mathcal{Q}) = \emptyset \text{ for } \lambda \neq \lambda';\ \ \lambda, \lambda' \in \Lambda. \tag{3}$$

(*ii*) *There exists a Borel measurable relatively compact set $\mathcal{S} \subseteq \hat{G}$ such that*

$$\hat{G} = \bigcup_{\omega \in \Lambda^\perp} (\omega + \mathcal{S}),\ \ (\omega + \mathcal{S}) \cap (\omega' + \mathcal{S}) = \emptyset \text{ for } \omega \neq \omega';\ \ \omega, \omega' \in \Lambda^\perp.$$

Moreover, the sets $\mathcal{Q}$ and $\mathcal{S}$ are respectively in one to one correspondance with the quotient groups G/Λ and $\hat{G}/\Lambda^\perp$.

Remark 2.2. *In this paper, the uniform lattice Λ and the dilative automorphism α are chosen such that $\alpha(\Lambda) \subseteq \Lambda$. Moreover, any pair (Λ, α) satisfying this relation is called a scaling system on G.*

The set $\mathcal{Q}$ which appears in equation (3) is called a *fundamental domain* associated with the lattice Λ. For our convenience, we will allow sets $\mathcal{Q}$ for which two conditions in (3) hold up to a set of measure zero. Also note that the sets of the form of $\mathcal{Q}$, which satisfies the two conditions of the equation (3), have been called *tiles* in [8, 23]. In this paper we will use the term *fundamental domain* for such sets. We now further wish to refine the fundamental domain $\mathcal{Q}$ and thus give the definition of a *self-similar fundamental domain.* The fundamental domain $\mathcal{Q}$ is said to be *self-similar* if for some finite subset Λ_0 of Λ, we have the following representation (see [8]):

$$\mathcal{Q} = \bigcup_{\lambda \in \Lambda_0} (\alpha^{-1}(\lambda) + \alpha^{-1}(\mathcal{Q})); \tag{4}$$

Throughout this paper, we will assume that the fundamental domain $\mathcal{Q}$ associated with the lattice Λ is self-similar. Thus for $\mathcal{Q}$, we have a representation of the form (4). Naturally, the immediate problem we face now is to find a precise representation of the set Λ_0 which appears in (4). The following lemma, proved in [23], gives us the required insight to this problem.

Lemma 2.3. *Let G be an LCA group with a uniform lattice Λ and an automorphism α. If $\mathcal{Q}$ is a self-similar fundamental domain associated to the lattice Λ, then the following hold:*

- (i) *The set Λ_0, which appears in* (4)*, is a complete set of coset representatives for $\alpha(\Lambda)$ in Λ.*
- (ii) $|\Lambda/\alpha(\Lambda)| = \delta_\alpha$.

From now on, we shall also assume that $\mathcal{S}$ is a self-similar fundamental domain of $\hat{G}$ associated to the lattice Λ. All the results, which we have stated for $\mathcal{Q}$, hold analogously for $\mathcal{S}$. Thus $\mathcal{S}$ has a representation of the form:

$$\mathcal{S} = \bigcup_{\lambda \in \Lambda_0^\perp} (\hat{\alpha}^{-1}(\omega) + \alpha^{-1}(\mathcal{S})); \tag{5}$$

where $\Lambda_0^\perp \subset \Lambda^\perp$ is finite.

Now, all the above information presented above and the fact $|\Lambda/\alpha(\Lambda)| = |\Lambda^\perp/\hat{\alpha}(\Lambda^\perp)|$ can be clubbed together to write

$$\Lambda/\alpha(\Lambda) = \{\lambda_0 + \alpha(\Lambda), \lambda_1 + \alpha(\Lambda), \cdots, \lambda_{\delta_\alpha - 1} + \alpha(\Lambda)\} \tag{6}$$

and

$$\Lambda^\perp/\hat{\alpha}(\Lambda^\perp) = \{\omega_0 + \hat{\alpha}(\Lambda^\perp), \omega_1 + \hat{\alpha}(\Lambda^\perp), \cdots, \omega_{\delta_\alpha - 1} + \hat{\alpha}(\Lambda^\perp)\}. \tag{7}$$

Remark 2.3. *To simplify the calculations in this paper, we make a further assumption that order of dilation is 2, i.e. $\delta_\alpha = 2$. The Riesz MRA obtained in this case is called the "MRA of dyadic dilations".*

Further, by using analogue of a result proved by K. Gröchenig and W. R. Madych in [10, Lemma 4], we can choose

$$\lambda_0 = 0 \in G \quad \text{and} \quad \omega_0 = 0 \in \hat{G},$$

and thus equations (6) and (7) reduce to

$$\Lambda/\alpha(\Lambda) = \{\alpha(\Lambda), \lambda_1 + \alpha(\Lambda)\} \quad \text{and} \quad \Lambda^\perp/\hat{\alpha}(\Lambda^\perp) = \{\hat{\alpha}(\Lambda^\perp), \omega_1 + \hat{\alpha}(\Lambda^\perp)\}. \tag{8}$$

The following lemma now asserts a relation between λ_1 and ω_1 as appearing in (8).

Lemma 2.4. *For $\gamma_1 = \hat{\alpha}^{-1}(\omega_1)$ and λ_1 as chosen in (8), we have that*

$$(\gamma_1, \lambda_1) = -1. \tag{9}$$

Proof. The proof follows once we note that the element $(\gamma_1, \lambda_1) \in \mathbb{T}$ has order 2. □

We now conclude the priliminary work on LCA groups by mentioning about the quotient groups $L^2(G/\Lambda)$ and $L^2(\hat{G}/\Lambda^\perp)$. Both these groups can be identified with the group $L^2(\mathbb{T})$ when we take $G = \mathbb{R}$. We refer [15, 5, 1] for a detailed information about both these groups.

Out of these two quotient groups, the latter will be used more frequently. So below, we give a lemma which helps us in explicitly representating the elements of the space $L^2(\hat{G}/\Lambda^\perp)$. Proof of this lemma follows without much calculations (see [15]).

Lemma 2.5. *If, for each $\lambda \in \Lambda$, the functions η_λ are defined by $\eta_\lambda(\gamma) = (\gamma, \lambda)\mathcal{X}_{\mathcal{S}}(\gamma)$, then the following are equivalent:*

(i) $F \in L^2(\hat{G}/\Lambda^\perp)$.

(ii) There exists a sequence $\{c_\lambda\}_{\lambda \in \Lambda} \in l^2(\Lambda)$ such that

$$F = \sum_{\lambda \in \Lambda} c_\lambda \varepsilon_\lambda;$$

where $\varepsilon_\lambda : \hat{G} \to \mathbb{C}$ is given by, $\varepsilon_\lambda(\gamma) = (\gamma, \lambda)$.

2.2. **Riesz Bases.** We will now have a brief discussion on Riesz bases in an arbitrary separable Hilbert space. For a detailed study on Riesz bases and their properties, we refer [6].

Definition 2.1. *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{I}$ be a countable index set. A sequence of elements $\{f_\beta\}_{\beta \in \mathbb{I}}$ is called a Riesz basis for $\mathcal{H}$ if there exist a bounded bijective operator $U : \mathcal{H} \to \mathcal{H}$ and an orthonormal basis $\{e_\beta\}_{\beta \in \mathbb{I}}$ of $\mathcal{H}$ such that, for each $\beta \in \mathbb{I}$, $f_\beta = Ue_\beta$.*

In the lemma below, we give one of the most used implications of the Riesz bases. For more details, we refer [6].

Lemma 2.6. *If $\{f_\beta\}_{\beta \in \mathbb{I}}$ is a Riesz basis for $\mathcal{H}$, then there exist constants $A, B > 0$ such that*

$$A||f||^2 \leq \sum_{\beta \in \mathbb{I}} |\langle f, f_\beta \rangle|^2 \leq B||f||^2. \tag{10}$$

The numbers A and B are called the Riesz bounds. Precisely, A is the lower Riesz bound and B is the upper Riesz bound. Moreover, the largest possible value of A is called the *optimal lower Riesz bound* and the smallest possible value of B is called the *optimal upper Riesz bound.*

The following lemma gives us one of the main characterizations of the Riesz bases in a separable Hilbert space. It does not involve any knowledge of the Riesz bounds. Proof of this lemma may be deduced by using various results in [6].

Lemma 2.7. *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{I}$ be a countable index set. Then, a sequence $\{f_\beta\}_{\beta \in \mathbb{I}}$ in $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if the map $T : l^2(\mathbb{I}) \to \mathcal{H}$, given by*

$$T(\{c_\beta\}) = \sum_{\beta \in \mathbb{I}} c_\beta f_\beta,$$

is well defined and bijective.

In the context of our paper, we mostly deal with a family of the type $\{T_\lambda \phi\}_{\lambda\in\Lambda}$ where $\phi \in L^2(G)$. So, we wish to see some alternate condtions under which such a family is a Riesz basis. For that, we first introduce a function Φ corresponding to this function ϕ.

Definition 2.2. *Let G be an LCA group with dual group $\hat{G}$ and let (Λ, α) be the scaling system defined on $\hat{G}$. If $\phi \in L^2(G)$ is given then corresponding to this function ϕ, the function Φ is given by*

$$\Phi(\gamma) = \sum_{\omega\in\Lambda^\perp} |\hat{\phi}(\gamma+\omega)|^2, \;\; \gamma \in \hat{G}. \tag{11}$$

It is easy to note that this function Φ is $\Lambda^\perp$-periodic and that $\Phi\mathcal{X}_{\mathcal{S}} \in L^1(G)$. So with the notations used previously in this paper, we can write $\Phi \in L^1(\hat{G}/\Lambda^\perp)$. We now give an equivalent condition for the family $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ to be a *Riesz sequence*, i.e. a Riesz basis for its closed linear span. A detailed proof of the following lemma can be found in [5].

Lemma 2.8. *Let $\phi \in L^2(G)$ be given. Then the family $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a Riesz sequence with bounds A and B if and only if*

$$A \leq \Phi(\gamma) \leq B,$$

for a.e. $\gamma \in \hat{G}$.

We shall use above lemma to verify whether a family of the form $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a Riesz sequence or not.

3. Riesz MRA and the corresponding Riesz Wavelet

The theory of classical MRA for $L^2(G)$ has been presented in [8] and the concept of MRA with a Riesz basis structure for the space $L^2(\mathbb{R})$ has been given [21, 3]. We combine the definitions in these two papers to give the definition of Riesz MRA on the space $L^2(G)$. Note that we have already given this definition in our paper [16].

Definition 3.1. *A Riesz multiresolution analysis for $L^2(G)$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(G)$ and a function $\phi \in V_0$ such that*

(*i*) *the subspaces V_j are nested, i.e.*

$$\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots ;$$

(*ii*) *the subspaces V_j have a dense union and a trivial intersection, i.e.*

$$\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(G) \quad \text{and} \quad \underset{j\in\mathbb{Z}}{\cap} V_j = \emptyset;$$

(*iii*) *they are related by the dilation property: $V_j = D^j V_0$;*

(*iv*) *the subspaces V_j are translation invariant, i.e.*

$$f \in V_j \implies T_\lambda f \in V_j, \forall\ \lambda \in \Lambda \text{ and } \forall\ j \in \mathbb{Z};$$

(*v*) *$\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a Riesz basis for V_0.*

The subspaces V_j in the above definition are called the *multiresolution subspaces* and the function ϕ is called the scaling function.

Now all the conditions, which need to be imposed on the scaling function ϕ to generate a Riesz MRA for the space $L^2(G)$, have been summed up in the theorem below. We have thoroughly inverstigated all these conditions in our paper [16].

Theorem 3.1. *Let G be an LCA group with the dual group $\hat{G}$ and let (Λ, α) be a scaling system defined on G. Further, if the following conditions are satisfied:*

(i) *The family* $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ *is a Riesz sequence.*
(ii) *The subspaces* V_j *are defined by*

$$V_j = D^j(\overline{span}\{T_k\phi\}_{k\in\Lambda}) = \overline{span}\{D^jT_k\phi\}_{k\in\Lambda}, \quad j \in \mathbb{Z}. \tag{12}$$

(iii) *The function* $\hat{\phi}$ *is nonzero on a neighbourhood of* $0 \in \hat{G}$.
(iv) *There exists a function* $m_0 \in L^\infty(\hat{G}/\Lambda^\perp)$ *such that*

$$\hat{\phi}(\hat{\alpha}(\gamma)) = m_0(\gamma)\hat{\phi}(\gamma) \ \ \forall\, \gamma \in \hat{G}; \tag{13}$$

then the function ϕ *generates a frame multiresolution analysis.*

Equation (13) is called *the refinement equation* and if a given function ϕ satisfies this equation, then it is called *refinable*. Further, the function m_0 appearing in (13) is called the *refinement mask* or the *two-scale symbol* or the *low pass filter*. Also note that this function m_0 is unique.

Throughout this section, we will assume that we are given a function ϕ which generates a Riesz MRA, i.e. all the conditions of Theorem 3.1 are satisfied. Here, using Lemma 2.8, we also get existence of positive numbers $A, B > 0$ such that

$$0 < A \leq \Phi(\gamma) \leq B; \ \ \forall\, \gamma \in \hat{G}.$$

Further, using this given Riesz MRA, we will try to find a function ψ such that the family

$$\{D^jT_\lambda\psi : \lambda \in \Lambda, j \in \mathbb{Z}\} \tag{14}$$

is a Riesz basis for $L^2(G)$.

Remark 3.1. *Since we are delaing with only dyadic dilations ($\delta_\alpha = 2$) in this paper, so it is evident from some previous works on wavelets (see [17, 9]) that only one function $\psi \in L^2(G)$ is enough to generate a Riesz basis for $L^2(G)$. Such a function ψ is called a wavelet function.*

We now begin this process of construction of a wavelet function $\psi \in L^2(G)$ by writing the orthogonal decomposition of the space $L^2(G)$. For each $j \in \mathbb{Z}$, let W_j denote the orthogonal complement of V_j in V_{j+1}. Then it is easy to see that

$$V_{j+1} = V_j \oplus W_j$$

and hence

$$L^2(G) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Next, we show that the spaces W_j are related to each other by the same dilation property as the subspaces V_j. This property of the subspaces $W_j's$ reduces our work and now we only need to find some functions whose family of Λ-translates form a Riesz basis for W_0. All this information is presented in the following lemma.

Lemma 3.1. *Assume that $\phi \in L^2(G)$ generates a Riesz MRA. Then the following hold:*
(i) $W_j = D^jW_0, \quad \forall j \in \mathbb{Z}.$
(ii) *If a function $\psi \in W_0$ is such that the family $\{T_\lambda\psi : \lambda \in \Lambda\}$ is a Riesz basis for W_0, then for all $j \in \mathbb{Z}$, the family $\{D^jT_\lambda\psi : \lambda \in \Lambda,\}$ is a Riesz basis for W_j, and the family (14) is a Riesz basis for $L^2(G)$. Moreover, all these Riesz bases have exactly the same Riesz bounds.*

Proof. The proof follows from [6, Lemma 5.3.3], once we note that the operators T_λ and D are unitary. □

From the above lemma, we see that the space W_0 is of great importance to us and thus it becomes imperative for us to give its characterization. So, in the following lemma, we present an equivalent condition for a function $f \in V_1$ to be a member of W_0.

Lemma 3.2. *Assume that $\phi \in L^2(G)$ generates a Riesz MRA of dyadic dilation with two-scale symbol $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$. Let $F \in L^2(\hat{G}/\Lambda^\perp)$ and define $f \in V_1$ by:*

$$\hat{f}(\hat{\alpha}(\gamma)) = F(\gamma)\hat{\phi}(\gamma). \tag{15}$$

Then the following hold:

(*i*) *If we write $\mathcal{S}' = \hat{\alpha}^{-1}(\mathcal{S})$, then*

$$\langle f, T_\lambda \phi\rangle = \delta_\alpha \int_{\mathcal{S}'} \left((F\Phi\overline{H_0})(\gamma) + \mathcal{T}_{\gamma_1}(F\Phi\overline{H_0})(\gamma)\right)(\hat{\alpha}(\gamma), \lambda) d\gamma \tag{16}$$

(*ii*) *$f \in W_0$ if and only if*

$$F\overline{H_0}\Phi + \mathcal{T}_{\gamma_1}(F\overline{H_0}\Phi) = 0 \tag{17}$$

a.e. on $\mathcal{S}$.

Proof. The proof of (*i*) uses a generalized periodization trick (see [6, Theorem 21.2.2]) and relation between the measures of the sets $\hat{G}/\Lambda^\perp$ and $\mathcal{S}$ (see [5, Lemma 2.10]). Part (*ii*) follows from (*i*) when we take [1, Theorem 1.3 (5)] or [15, Remark 2.1] or into consideration. □

Remark 3.2. *Since the term $F\overline{H_0}\Phi + \mathcal{T}_{\gamma_1}(F\overline{H_0}\Phi)$ is $\gamma_1 + \Lambda^\perp$-periodic and $\mathcal{S} \subset \hat{G}$ is a fundamental domain associated with $\Lambda^\perp$, therefore if $F\overline{H_0}\Phi + \mathcal{T}_{\gamma_1}(F\overline{H_0}\Phi)$ is zero on $\mathcal{S}$, then it is zero on $\hat{G}$. Thus, by Lemma 3.2, we can also conclude that a function $f \in V_1$, defined via (15), is in W_0 if and only if $F\overline{H_0}\Phi + \mathcal{T}_{\gamma_1}(F\overline{H_0}\Phi) = 0$ a.e. on $\hat{G}$.*

We mention that, if in particular, the LCA group G is taken to be the Euclidean group $\mathbb{R}$, then the above Lemma reduces to one known result whose proof may be found in [2].

As mentioned earlier, we now only need to find a functions in $\psi \in W_0$ such that the family of its Λ-translates forms a Riesz basis for W_0. We intend to achieve this in two steps:

- We find a function $\psi \in W_0$ such that the family of its Λ-transalates generates the space W_0, i.e.

$$W_0 = \overline{\text{span}}\{T_\lambda \psi_i : \lambda \in \Lambda, 1 \le i \le \delta_\alpha - 1\}. \tag{18}$$

- We will then show that the family (14) forms a Riesz basis for the space W_0.

In the lemma below, we give a sufficient condition, in terms of solvability of a system of linear equation, for the family $\{T_\lambda \psi : \lambda \in \Lambda\}$ to generate the space W_0. This alternate characterization will be of much use to us.

Lemma 3.3. *Let G be an LCA group and let $\phi \in L^2(G)$ generates a Riesz MRA of dyadic dilations and with two scale symbol $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$. Suppose there exist a function $F_1 \in L^\infty(\hat{G}/\Lambda^\perp)$ and the function ψ is defined via*

$$\hat{\psi}(\hat{\alpha}(\gamma)) = F(\gamma)\hat{\phi}(\gamma). \tag{19}$$

If there exist functions $G_0, G_1 \in L^\infty(\hat{G}/\Lambda^\perp)$ such that the following equations

$$\overline{H_0(\gamma)}\Phi(\gamma)F(\gamma) + \overline{H_0(\gamma-\gamma_1)}\Phi(\gamma-\gamma_1)F(\gamma-\gamma_1) = 0, \tag{20}$$

$$H_0(\gamma)G_0(\gamma) + F(\gamma)G_1(\gamma) = 1 \tag{21}$$

$$H_0(\gamma-\gamma_1)G_0(\gamma) + F(\gamma-\gamma_1)G_1(\gamma) = 0 \tag{22}$$

are satisfied for a.e. $\gamma \in \hat{G}$*, then*

$$W_0 = \overline{span}\{T_\lambda \psi : \lambda \in \Lambda\}.$$

Proof. Note that, for $\{c_\lambda\}_{\lambda\in\Lambda}$, $\{d_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$, any $f \in V_1$ has an expression of the form:

$$f(x) = \sum_{\lambda\in\Lambda} c_\lambda \phi(x-\lambda) + \sum_{\lambda\in\Lambda} d_\lambda \psi(x-\lambda) \tag{23}$$

The proof is just a simple manipulation of the above equation. Proof follows when we take inverse Fourier transformation of equations (20),(21) and (22). We skip the straighforward manipulations and calculations involved. □

Making use of the above lemma, we now explicitly construct a function ψ such that it generates W_0 in the sense of (18). We will not give a detailed proof of the following theorem, but will briefly give the directions for the same.

Theorem 3.2. *Let G be an LCA group and let $\phi \in L^2(G)$ generate a Riesz MRA of dyadic dilations and two scale symbol $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$. Then there always exist a function ψ in W_0 generating W_0.*

Proof. First note that it is enough to prove that the three equations (20), (21) and (22) are satisfied a.e. on $\mathcal{S}$, the fundamental domain associated to the lattice $\Lambda^\perp$ in $\hat{G}$. We further devide this set $\mathcal{S}$ into two disjoint parts:

$$\mathcal{S}_1 = \{\gamma \in \mathcal{S} : 0 \neq |H_0(\gamma)| \geq |H_0(\gamma-\gamma_1)|\}$$
$$\mathcal{S}_2 = \{\gamma \in \mathcal{S} : 0 \neq |H_0(\gamma-\gamma_1)| \geq |H_0(\gamma)|\}$$

We skip the easy calculations involved and we directly move to the table below which gives us one set of solution functions F, G_0 and G_1 satisfying (20), (21) and (22).

	$\mathcal{S}_1$	$\mathcal{S}_2$
F	$-\frac{\overline{H_0(\gamma-\gamma_1)}\Phi(\gamma-\gamma_1)}{\overline{H_0(\gamma)}\Phi(\gamma)}$	1
G_0	$\frac{\overline{H_0(\gamma)}\Phi(\gamma)}{\Phi(\hat{\alpha}(\gamma))}$	$\frac{\overline{H_0(\gamma)}\Phi(\gamma)}{\Phi(\hat{\alpha}(\gamma))}$
G_1	$-\frac{H_0(\gamma-\gamma_1)\overline{H_0(\gamma)}\Phi(\gamma)}{\Phi(\hat{\alpha}(\gamma))}$	$\frac{\lvert H_0(\gamma-\gamma_1)\rvert^2\Phi(\gamma-\gamma_1)}{\Phi(\hat{\alpha}(\gamma))}$

TABLE 1. Table for variables F, G_0 and G_1

Clearly all the three functions F, G_0 and G_1 are in $L^\infty(\hat{G}/\Lambda^\perp)$. This completes the proof. □

This completes our quest of a function ψ which generate the space W_0. We now show that the family of the type (14), constructed using the function obtained in above theorem, is indeed a Riesz basis for W_0.

Theorem 3.3. *Assume that $\phi \in L^2(G)$ generates a Riesz MRA of dyadic dilations and two scale symbol $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$. Further assume that the functions ψ is defined by* (19) *and the functions F is assumed to be as it appears in Theorem 3.2. Then the family* (14) *generates a Riesz basis for the space $L^2(G)$.*

Proof. Analogous to the function Φ as defined in Definition 2.2, we define the function Ψ by

$$\Psi(\gamma) = \sum_{\omega \in \Lambda^\perp} |\hat{\psi}(\gamma + \omega)|^2.$$

It is easy to see that

$$\Psi(\hat{\alpha}(\gamma)) = |F(\gamma)|^2 \Phi(\gamma) + |F(\gamma - \gamma_1)|^2 \Phi(\gamma - \gamma_1).$$

We now make use of Lemma 2.8 to show that the family $\{T_\lambda \psi\}_{\lambda \in \Lambda}$ is a Riesz basis for W_0, i.e. we wish to find constants $C, D > 0$ such that

$$C \leq \Psi(\gamma) \leq D, \ \ \forall\, \gamma \in \mathcal{S}.$$

But as it is more convinient for us to deal with the expression $\Psi(\hat{\alpha}(\gamma))$, so we need to ensure that the bounds $C, D > 0$ which exist are such that

$$C \leq \Psi(\hat{\alpha}(\gamma)) \leq D, \ \ \forall\, \gamma \in \hat{\alpha}^{-1}(\mathcal{S}).$$

Analogous to the previous theorem, we devide the set $\hat{\alpha}^{-1}(\mathcal{S})$ into two disjoint parts:

$$\tilde{\mathcal{S}_1} = \{\gamma \in \hat{\alpha}^{-1}(\mathcal{S}) : 0 \neq |H_0(\gamma)| \geq |H_0(\gamma - \gamma_1)|\}$$
$$\tilde{\mathcal{S}_2} = \{\gamma \in \hat{\alpha}^{-1}(\mathcal{S}) : 0 \neq |H_0(\gamma - \gamma_1)| \geq |H_0(\gamma)|\}.$$

Clearly, $\tilde{\mathcal{S}_1} \subset \mathcal{S}_1$ and $\tilde{\mathcal{S}_2} \subset \mathcal{S}_2$. First if we let $\gamma \in \tilde{\mathcal{S}_1}$, then

$$\Psi(\hat{\alpha}(\gamma)) = \frac{|H_0(\gamma - \gamma_1)|^2 \Phi(\gamma - \gamma_1)^2}{|H_0(\gamma)|^2 \Phi(\gamma)} + \Phi(\gamma).$$

It is then easy to see that

$$A \leq \Psi(\hat{\alpha}(\gamma)) \leq \frac{B}{A}(A + B).$$

The above inequality also holds when $\gamma \in \tilde{\mathcal{S}_2}$.
Thus, we conclude that the family $\{T_\lambda \psi : \lambda \in \Lambda\}$ generates a Riesz basis for W_0 and hence the family (14) generates a Riesz basis for $L^2(G)$. □

4. Conclusions (mandatory)

We developed a theory for the construction of a Riesz wavelet from a given Riesz MRA on an LCA group. For simplicity, we assumed that the given Riesz MRA was constructed using dyadic dilations. It has been shown that, in the case of dyadic dilations, there always exist a functions ψ such that the family

$$\{D^j T_\lambda \psi : \lambda \in \Lambda, j \in \mathbb{Z}\} \tag{14}$$

generates a Riesz basis for $L^2(G)$.

Acknowledgement. The authors are deeply indebted to the anonymous referees for their valuable comments and suggestions on an earlier version of the manuscript.

References

[1] Barbieri D., Hernandez E., Mayeli A., (2017), Tiling by lattices for locally compact abelian groups, C. R. Math. 355, (2), pp.193–199.

[2] Benedetto J. J., Li S., (1998), The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal. 5, no. 4, 389–427.

[3] Bownik M., (2003), Riesz wavelets and generalized multiresolution analyses, Appl. Comput. Harmon. Anal., 14, no. 3, 181–194.

[4] Bownik M., Jahan Q., (2020), Wavelets on compact abelian groups, Appl. Comput. Harmon. Anal., 49, no. 2, 471–494.

[5] Cabrelli C., Paternostro V., (2010), Shift-invariant spaces on LCA groups, J. Funct. Anal., 258, no. 6, 2034–2059.

[6] Christensen O., (2015), An introduction to frames and Riesz bases, second edition, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Inc., Boston, MA.

[7] Christensen O., Goh S. S., (2019), The unitary extension principle on locally compact abelian groups, Appl. Comput. Harmon. Anal., 47, no. 1, 1–29.

[8] Dahlke S., (1993), Multiresolution analysis and wavelets on locally compact abelian groups, Wavelets, images, and surface fitting, (Chamonix-Mont-Blanc), 141–156, A K Peters, Wellesley, MA, 1994.

[9] Daubechies I., (1992), Ten lectures on wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[10] Gröchenig K., Madych W. R., (1992), Multiresolution analysis, Haar bases, and self-similar tilings of $\mathbb{R}^n$, IEEE Trans. Inform. Theory, 38, no. 2, part 2, 556–568.

[11] Han B., Jia R.-Q., (2007), Characterization of Riesz bases of wavelets generated from multiresolution analysis, Appl. Comput. Harmon. Anal., 23, no. 3, 321–345.

[12] Hewitt E., Ross K., (1963), Abstract Harmonic Analysis, vols. 1 and 2., Springer, Berlin.

[13] Kamyabi-Gol R. A., Tousi R. Raisi, (2010), Some equivalent multiresolution conditions on locally compact abelian groups, Proc. Indian Acad. Sci. Math. Sci., 120, no. 3, 317–331.

[14] Kim H. O. et al., (2002), On Riesz wavelets associated with multiresolution analyses, Appl. Comput. Harmon. Anal., 13, no. 2, 138–150.

[15] Kumar R., Satyapriya, (2021), Construction of a frame multiresolution analysis on locally compact Abelian groups, Aust. J. Math. Anal. Appl., 18, no. 1, art.5, 19 pp.

[16] Kumar R., Satyapriya, Construction of a Riesz Multiresolution analysis on Locally Compact Abelian Groups, (communicated).

[17] Mallat S. G., (1989), Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$, Trans. Amer. Math. Soc, 315, no. 1, 69—87.

[18] Mayeli A., (2019), Riesz wavelets, tiling and spectral sets in LCA groups, Complex Anal. Oper. Theory, 13, no. 3, 1177–1195.

[19] Meyer Y., (1990), Ondelettes et opérateurs. I, Actualités Mathématiques., Hermann, Paris.

[20] Rahimi A., Seddighi N., (2018), A constructive approach to the finite wavelet frames over prime fields, Proc. Indian Acad. Sci. Math. Sci., 128, no. 4, Paper No. 51, 11 pp.

[21] Ron A., Shen Z., (1995), Frames and stable bases for shift-invariant subspaces of $L_2(R^d)$, Canad. J. Math., 47, no. 5, 1051–1094.

[22] Rudin W., (1962), Fourier analysis on groups, Interscience Tracts in Pure and Applied Mathematics, No. 12, Interscience Publishers (a division of John Wiley and Sons), New York-London.

[23] Yang Q., Taylor K. F., (2012), Multiresolution analysis and Harr-like wavelet bases on locally compact groups, J. Appl. Funct. Anal, 7, no. 4, 413—439.

[24] Zalik R. A., (2007), On MRA Riesz wavelets, Proc. Amer. Math. Soc., 135, no. 3, 787–793.

Raj Kumar, an alumnus of IIT Delhi, has been teaching as an Associate Professor in the Department of Mathematics, Kirori Mal College, University of Delhi since 2001. He is actively involved in research and has been supervising Ph.D. scholars since 2012. His area of research are Theory of Frames, Wavelets and Signal Processing.

Satyapriya completed his graduation (B.Sc. Hons.) and post-graduation (M.Sc.) at Kirori Mal College, University of Delhi, India. He is a gold-meadalist at college level in masters. Currently, he is pursuing his Ph.D. under the supervision of Dr. Raj Kumar. His main research interests are the theory of frames, wavelets and signal processing.

Mansimran Singh has completed his graduation and post-graduation from St.Stephen's College, University of Delhi. He recieved his M.Phil. at the same university. Currently, he is working as an assistant professor in Mathematics at Sri Guru Nanak Dev Khalsa College. His research interest include several topics of functional and harmonic analysis.