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SOLUTION OF THE VOLTERRA-FREDHOLM INTEGRAL EQUATIONS VIA THE BERNSTEIN POLYNOMIALS AND LEAST SQUARES APPROACH

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Abstract. We develop a numerical scheme to solve a general category of Volterra-Fredholm integral equations. For this purpose, the Bernstein polynomials and their features have been used. We convert the main equation into a set of algebraic equations in which the coefficient matrix is obtained by the least squares approximation approach. The error analysis is given to corroborate the precision of the proposed method. Numerical results are presented to demonstrate the success of the scheme for solving integral equations.

Keywords: Volterra-Fredholm integral equation, least squares approximation, Bernstein polynomials, Chebyshev-Gauss-Lobatto points.

AMS Subject Classification: 45A05, 33C47, 93E24, 41A25.

1. Introduction

Despite the applications of integral equations in various scientific fields, many of them do not have analytical solution and to find their solutions, it is necessary to provide numerical methods [2, 5, 7–10, 13–18, 20, 21, 24]. Recently, spectral schemes such as the Taylor, Lagrange and Müntz–Legendre collocation methods have been proposed for solving integral and integro-differential equations [15, 17, 19, 20, 23].

We propose a scheme based on the Bernstein polynomials to solve a class of Volterra-Fredholm integral equations (VFIEs). Many researchers applied the Bernstein polynomials to solve different equations [3, 4]. For example, these polynomials are used to find an approximate solution of Fredholm integro-differential equation and integral equation of

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the second kind [1]. Our study relates to VFIE

$$\begin{aligned} A(x)y(x) + B(x)y(h(x)) =& f(x) + \gamma_1 \int_0^{h(x)} k_1(x,\rho)y(\rho)d\rho \\ &+ \gamma_2 \int_a^b k_2(x,\rho)y(h(\rho))d\rho, \end{aligned} \tag{1}$$

where $k_1(x,\rho), k_2(x,\rho), A(x), B(x), h(x)$ and $f(x)$ are known functions, a, b, γ_1, γ_2 are constants, and $y(x)$ is the unknown function.

In Section 2 some properties of the Bernstein polynomials are provided. In Section 3, these polynomials are applied to solve Eq. (1). In Section 4, we give an error estimation. In Section 5, two numerical examples presented to clarify the scheme.

2. Some properties of the Bernstein polynomials

The Bernstein polynomials are of great importance practically in the field of computer to aid geometric design as well as numerous other fields of mathematics because of their many useful properties. These polynomials have been frequently used in the solution of integral equations, differential equations and approximation theory [3, 4].

The Bernstein polynomial of degree n defined on $[\alpha, \beta]$ as:

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-\alpha)^i(\beta-x)^{n-i}}{(\beta-\alpha)^n}, \qquad i = 0, 1, ..., n. \tag{2}$$

For convenience, it sets $\alpha = 0$ and $\beta = 1$, so

$$B_{i,n}(x) = \binom{n}{i} x^i(1-x)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{i+k}. \tag{3}$$

Then, it defined $\phi(x) = [B_{0,n}(x), B_{1,n}(x), ..., B_{n,n}(x)]^T$, and can be

$$\phi(x) = ST_n(x), \tag{4}$$

where S is an upper triangular matrix as

$$S = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \cdots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n} \\ & \ddots & \vdots & \vdots \\ & (-1)^0\binom{n}{i} & \cdots & (-1)^{n-i}\binom{n}{i}\binom{n-i}{n-i} \\ & & \ddots & \vdots \\ 0 & & & (-1)^0\binom{n}{n} \end{bmatrix},$$

and $T_n(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}$.

For Bernstein polynomials we have $B_{i,n}(x) = -xB_{i,n-1}(x) + xB_{i-1,n-1}$, and these polynomials satisfy symmetric property $B_{i,n}(x) = B_{n-i,n}(1-x)$.

3. Solution method

We approximate the solution of Eq. (1) using the Bernstein polynomials on $[a, b]$ as follows. Assume that $y(x)$ to be the unique solution of Eq. (1) and $y_n(x)$ be its approximation such that

$$y_n(x) = \sum_{i=0}^{n} a_i B_{i,n}(x), \tag{5}$$

the coefficients a_is are unknown constants. It is approximately the same as Eq. (5) for $y(h(x))$,

$$y_n(h(x)) = \sum_{i=0}^{n} a_i B_{i,n}(h(x)). \tag{6}$$

Thus, Eq. (1) is written as follows:

$$\begin{aligned} &A(x)y_n(x) + B(x)y_n(h(x)) = f(x) + \gamma_1 W(x) + \gamma_2 H(x), \\ &W(x) := \int_a^{h(x)} k_1(x,\rho)y_n(\rho)d\rho, \\ &H(x) := \int_a^b k_2(x,\rho)y_n(h(\rho))d\rho. \end{aligned} \tag{7}$$

The Chebyshev-Gauss-Lobatto points are employed to compute coefficient index as

$$x_k = \frac{b-a}{2} - \frac{b-a}{2}\cos(\frac{\pi k}{n}), \qquad k = 0, 1, ..., n. \tag{8}$$

By relations (5), (6) and the collocation points (8), Eq. (7) is given,

$$f(x_k) + \gamma_1 W(x_k) + \gamma_2 H(x_k) = A(x_k)\sum_{i=0}^{n} a_i B_{i,n}(x_k) + B(x_k)\sum_{i=0}^{n} a_i B_{i,n}(h(x_k)). \tag{9}$$

With

$$\begin{aligned} W(x_k) &= \sum_{i=0}^{n} a_i \int_a^{h(x)} k_1(x,\rho)B_{i,n}(\rho)d\rho, \\ H(x_k) &= \sum_{i=0}^{n} a_i \int_a^b k_2(x,\rho)B_{i,n}(h(\rho))d\rho, \end{aligned} \tag{10}$$

and

$$Y(x_k) = \sum_{i=0}^{n} a_i[A(x_k)B_{i,n}(x_k) + B(x_k)B_{i,n}(h(x_k))], \tag{11}$$

we have

$$Y(x_k) - \gamma_1 W(x_k) - \gamma_2 H(x_k) = f(x_k), \tag{12}$$

hence

$$\begin{aligned} &\sum_{i=0}^{n} a_i[A(x_k)B_{i,n}(x_k) + B(x_k)B_{i,n}(h(x_k)) - \gamma_1 \int_a^{h(x_k)} k_1(x_k,\rho)B_{i,n}(\rho)d\rho - \\ &\gamma_2 \int_a^b k_2(x_k,\rho)B_{i,n}(h(\rho))d\rho] = f(x_k). \end{aligned} \tag{13}$$

Accordingly for $k = 0, 1, ..., n$,

$$\sum_{i=0}^{n} a_i Z_i(x_k) = f(x_k), \tag{14}$$

which has the following matrix form:

$$Z^T A = F, \tag{15}$$

where $F = [f(x_0), f(x_1), ..., f(x_n)]$ and $Z = Z_i(x_k)$.

Finally, we apply the least squares approximation to find unknown Bernstein coefficients of Eq. (15) and substituting in Eq. (5). Thus, the Bernstein polynomial solution of Eq. (1) is obtained. In the following, let

$$\begin{aligned} L(x, y_n(x)) = & -f(x_k) - \gamma_1 W(x_k) - \gamma_2 H(x_k) \\ & + A(x_k) \sum_{i=0}^{n} a_i B_{i,n}(x_k) + B(x_k) \sum_{i=0}^{n} a_i B_{i,n}(h(x_k)). \end{aligned} \tag{16}$$

To minimize I, as the square of the approximation error, we find the real coefficients $a_0, a_1, \ldots, a_n$ as $\frac{\partial I}{\partial a_i} = 0,\ i = 0, 1, ..., n$, i.e.
$\frac{\partial I}{\partial a_i} = 2 \int_a^b L(x, y_n(x)) \ \frac{\partial L(x, y_n(x))}{\partial a_i} dx = 0.$

For $i, j = 0, 1, ..., n$ we have

$\sum_{i=0}^{n} a_i \int_a^b z_j(x) z_i(x) dx = \int_a^b f(x) z_i(x) dx.$

So,

$$\begin{aligned} z_i(x) = & A(x) B_{i,n}(x) + B(x) B_{i,n}(h(x)) - \gamma_1 \int_a^{h(x)} k_1(x, \rho) B_{i,n}(\rho) d\rho \\ & - \gamma_2 \int_a^b k_2(x, \rho) B_{i,n}(h(\rho)) d\rho, \end{aligned}$$

and in the matrix form

$$\mathcal{K}\mathcal{A} = \mathcal{G}, \tag{17}$$

where

$$\mathcal{K} = \begin{bmatrix} (z_0, z_0) & (z_0, z_1) & \ldots & (z_0, z_n) \\ (z_1, z_0) & (z_1, z_1) & \ldots & (z_1, z_n) \\ \vdots & \vdots & \ldots & \vdots \\ (z_n, z_0) & (z_n, z_1) & \ldots & (z_n, z_n) \end{bmatrix},$$

$A = \begin{bmatrix} a_0, & a_1, & \ldots, & a_n \end{bmatrix}^T$ and $G = [\ (z_0, f), \ \ (z_1, f), \ \ \ldots, \ \ (z_n, f)\ ]^T$.
If matrix $\mathcal{K}$ be a nonsingular matrix, then $\mathcal{A} = \mathcal{K}^{-1}\mathcal{G}$ will be a unique solution of Eq. (17).

4. Convergence analysis

We give an error estimation for solution of Eq. (1) based on the Bernstein polynomials. Degree of polynomial approximation for function f described in term of its modulus of continuity that defined in the below form:
For each $\delta > 0$, $\omega(\delta) = \sup |f(x_1) - f(x_2)|$ for all $x_1, x_2 \in [a, b]$ such that $|x_1 - x_2| \leq \delta$.

Assume $p_n(x)$ is an approximation polynomial of continuous function $f(x)$, then we have

$$|f(x) - p_n(x)| \leq \frac{5}{4} \omega(\frac{1}{\sqrt{n}}). \tag{18}$$

Let $\tilde{y}(x)$ and $y(x)$ are the approximate and exact solutions of the integral equation (1), respectively, so

$$\begin{aligned} A(x)\tilde{y}(x) + B(x)\tilde{y}(h(x)) - \gamma_1 \textstyle\int_a^{h(x)} k_1(x,\rho)\tilde{y}(\rho)d\rho \\ -\gamma_2 \textstyle\int_a^b k_2(x,\rho)\tilde{y}(h(\rho))d\rho = f(x) + r_n(x), \end{aligned} \tag{19}$$

where $r_n(x)$ is the perturbation function.
Consider $M_1 = \sup_{a<x,\rho<b} |k_1(x,\rho)| < \infty$, $M_2 = \sup_{a<x,\rho<b} |k_2(x,\rho)| < \infty$, $M_3 = \sup_{a<x<b} |A(x)| < \infty$ and $M_4 = \sup_{a<x<b} |B(x)| < \infty$. Also, we suppose $e_1(x) = y(x) - \tilde{y}(x)$ and $e_2(x) = y(h(x)) - \tilde{y}(h(x))$ to be the error functions of this method then leads to

$$e(x) = \max\{e_1(x), e_2(x)\}. \tag{20}$$

By subtracting Eq. (19) from Eq. (1) and using Eq. (20), we have

$$\begin{aligned} |r(x)| &\leq M_3\,|e(x)| + M_4\,|e(x)| + |\gamma_1|\ M_1\, e(x) + |\gamma_2|\ M_2\, e(x) \\ &= (M_3 + M_4 + |\gamma_1|\ M_1 + |\gamma_2|\ M_2)\, e(x), \end{aligned} \tag{21}$$

where by substituting Eq. (18) into Eq. (21), an error bound is obtained for $r(x)$ as:

$$|r(x)| \leq (M_3 + M_4 + |\gamma_1|\ M_1 + |\gamma_2|\ M_2)\ \frac{5}{4}\omega(\frac{1}{\sqrt{n}}). \tag{22}$$

It explained the error bound to another form as follows: It supposes that $\{B_{0,n}, B_{1,n}, \ldots, B_{n,n}\} \subset L^2[0,1]$, $n \in N \cup \{0\}$ to be the set of Bernstein polynomials, and $S = span\,\{B_{0,n}, B_{1,n}, \ldots, B_{n,n}\}$. Since S is a finite dimensional vector space, f has the unique best approximation $p^* \in S$ as [11, 12]:

$$\forall p \in S,\ \exists p^* \in S; \quad \|f - p^*\|_2 \leq \|f - p\|_2,$$

where $\|f\|_2 = \sqrt{(f,f)}$. Since $p^* \in S$, there exist coefficients $a_i, i = 0, 1, ..., n$ such that

$$f \approx p^* = \sum_{i=0}^{n} a_i B_{i,n}(x) = A^T\phi.$$

Remark 4.1. *It supposes that p^* is the best approximation $f \in L^2[0,1]$ out of $S = span\,\{B_{0,n}, B_{1,n}, \ldots, B_{n,n}\}$, and $p^* = \sum_{i=0}^{n} a_i B_{i,n}(x) = A^T\phi$, then $\lim_{n\to\infty}\|f - p^*\|_2 = 0$.*

Let $\tilde{y}(x) = A^T\phi$ be the best approximation of f [6], then

$$\begin{aligned} \|y - \tilde{y}\|_2^2 \leq \|y - y_{TP}\|_2^2 &= \int_0^l (y(x) - y_{TP}(x))^2 dx = \int_0^l (\left|y^{(n+1)}(\varepsilon)\right| \frac{(x-x_0)^{n+1}}{(n+1)!})^2 dx \\ &\leq \frac{M^2}{((n+1)!)^2} \int_0^l (x-x_0)^{2n+2} dx \leq \frac{2M^2 S^{2n+3}}{((n+1)!)^2(2n+3)}, \end{aligned}$$

where y_{TP} is Taylor's expansion of order n, $M = \max\left|y^{(n+1)}(x)\right|, x \in [0,l]$, and $S = \max\{l - x_0, x_0\}$.
Finally, we obtain [22]

$$\frac{\|y(x) - \tilde{y}(x)\|_2}{\|y(h(x)) - \tilde{y}(h(x))\|_2} \leq \frac{q + \beta}{p - \gamma},$$

where

$$\gamma = \sup_{a\leq x\leq b} |\gamma_1| \int_a^{h(x)} |k_1(x,\rho)| d\rho, \quad \beta = \sup_{a\leq x\leq b} |\gamma_2| \int_a^{b} |k_2(x,\rho)| d\rho,$$
$$p = \min_{a\leq x\leq b} |A(x)|, \quad q = \max_{a\leq x\leq b} |B(x)|,$$

and $p - \gamma > 0$.

Table 1. Numerical results of Example 5.1

x	$n=3$	$n=5$	$n=6$	$n=7$	$n=8$	*Exact solutions*
0.0	0.001124	0.000040	0.000009	0.000004	0.000014	0.000000
0.1	0.095193	0.095302	0.095309	0.095310	0.095309	0.095310
0.2	0.182056	0.182322	0.182322	0.182322	0.182322	0.182322
0.3	0.262350	0.262368	0.262364	0.262364	0.262364	0.262364
0.4	0.336711	0.336471	0.336472	0.336472	0.336472	0.336472
0.5	0.405776	0.405462	0.405465	0.405465	0.405466	0.405465
0.6	0.470181	0.470004	0.470004	0.470004	0.470004	0.470004
0.7	0.530562	0.530633	0.530628	0.530628	0.530628	0.530628
0.8	0.587557	0.587788	0.587787	0.587787	0.587788	0.587787
0.9	0.641802	0.641849	0.641855	0.641854	0.641853	0.641854
1.0	0.693933	0.693174	0.693142	0.693148	0.693159	0.693147

5. Numerical examples

We show the efficiency of our method for approximating the solution of VFIEs through two examples.

Example 5.1. *Consider the VFIE*

$$y(x) = f(x) + \int_0^{\frac{x}{3}} x\rho \, y(\rho)\, d\rho + \int_0^1 (x-\rho)\, y(\frac{\rho}{3})d\rho, \tag{23}$$

with exact solution $y(x) = \ln(x+1)$, *where*

$$\begin{aligned} f(x) =& \ln(x+1) - (\frac{x^2}{6} + \frac{x}{2}\ln 3 - \frac{x^3}{36}(1+\ln 9) + \frac{x}{18}(x^2-9)\ln(3+x)) \\ &+ (\frac{5}{4} + x - (4\ln\frac{4}{3})(x+1)). \end{aligned}$$

Table 1 gives the numerical results of our scheme with $n = 3,5,6,7,8$ *and Fig. 1 shows errors of the proposed method with* $n = 8$.

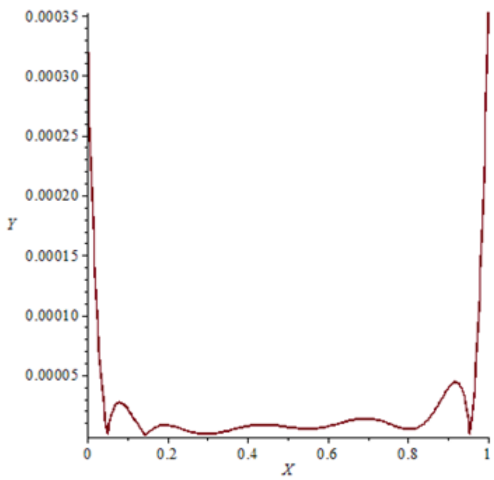

Figure 1. The absolute errors of numerical solution for Eq. (23) with $n = 8$

TABLE 2. Numerical results of Example 5.2

x	$n=3$	$n=5$	$n=6$	$n=7$	$n=8$	*Exact solutions*
0.0	-0.036036	0.005435	0.000000	0.000000	0.000000	0.000000
0.1	0.012630	-0.001975	0.000020	0.000020	0.000020	0.000020
0.2	0.013424	0.001182	0.000640	0.000640	0.000640	0.000640
0.3	-0.003397	0.007749	0.004860	0.004860	0.004860	0.004860
0.4	-0.007574	0.022205	0.020480	0.020480	0.020480	0.020480
0.5	0.031150	0.060666	0.062500	0.062500	0.062500	0.062500
0.6	0.143032	0.150888	0.155520	0.155520	0.155520	0.155520
0.7	0.358330	0.332261	0.336140	0.336140	0.336140	0.336140
0.8	0.707301	0.655813	0.655360	0.655360	0.655360	0.655360
0.9	1.220202	1.184212	1.180980	1.180980	1.180980	1.180980
1.0	1.927291	1.991761	2.000000	2.000000	2.000000	2.000000

Example 5.2. *Consider the VFIE*

$$y(x)+y(h(x))=f(x)+\gamma_1\int_0^{h(x)}(x-\frac{\rho}{2})\ y(\rho)d\rho+\gamma_2\int_0^1 x\rho\ y(h(\rho))d\rho, \tag{24}$$

with the exact solution $y(x)=2x^5$, *where*
$h(x)=\frac{x}{2},\ \gamma_1=1,\ \gamma_2=1,\ f(x)=-\frac{1}{112}x+\frac{33}{16}x^5-\frac{11}{2688}x^7.$
Table 2 gives the exact and approximate solutions with $n=3,5,6,7,8$ *and Fig. 2 shows the absolute errors of our scheme with* $n=7$.

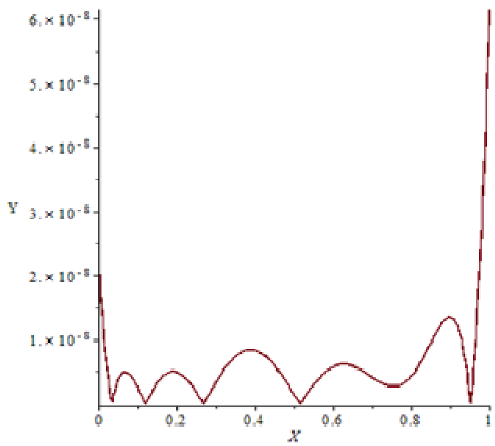

FIGURE 2. The absolute errors of numerical solution for Eq. (24) with $n=7$

6. CONCLUSION

We established an efficient scheme based on the Bernstein polynomials and least squares approach to study VFIEs. The performance of our scheme is illustrated through the several experiments. Considering these examples, the method has stable properties, and when the number of the Bernstein bases functions used for approximation increases, the errors reduce. The obtained rapid convergence shows the ability of our method to solve VFIEs.

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