TWMS J. App. and Eng. Math. V.13, N.2, 2023, pp. 480-490

ON SOME TRIANGULAR INEQUALITIES AND APPLICATIONS IN 2-FUZZY METRIC SPACES

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Abstract. The aim of this paper is to study some level forms of triangular inequality of 2-fuzzy metric spaces which will be useful for application to fixed point problems. For this aim, we first define the concept of 2-fuzzy pre-metric spaces that have weaker axioms than 2-fuzzy metric spaces with the fundamental properties. Then, we investigate the level form inequalities in 2-fuzzy metric spaces equvalent to the triangular inequalities of 2-fuzzy metric spaces by also analyzing the conditions under in which these are provided. Finally, we prove a fixed point theorem for 2-fuzzy metric spaces by considering the obtained level forms of triangular inequalities.

Keywords: Fuzzy number, 2-fuzzy metric space, triangular inequality, fixed point theorem.

AMS Subject Classification: 03E72, 47H10, 54E35.

1. Introduction

In 1963, Gähler [10] introduced the structure of 2-metric spaces as a nonlinear expansion of a concept of crisp metric spaces. The value of three distinct points (at least one of these points is not on the same line) under a 2-metric function is geometrically interpreted as the value of the area of a triangle formed by these three points as vertices. In many generalizations of crisp metric space in the literature such as 2-metric space [10], partial metric space [25], soft metric space [8], G-metric space [28], b-metric space [16] and etc., only the concept of 2-metric space isn't topologically equivalent to a crisp metric space. So this structure is studied extensively with its different theoretical and applied aspects by many researchers [1, 7, 9, 10, 11].

As a different generalization of crisp metric space, the structure of fuzzy metric spaces (KM-fuzzy metric spaces) was given by Kramosil and Michálek [24] in 1975, by applying the concept of fuzziness to the axioms of crisp metric space. The conditions of this concept

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§ Manuscript received: January 14, 2021; accepted: August 05, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.2

express the fact that the properties of nonnegativity, identity and symmetry are generalized, but are not limited to some fuzziness or uncertainty, only the values of distances are thought as fuzzy notions. Then, George and Veeramani redefined the notion of KM-fuzzy metric space (named GV-fuzzy metric space) to induce a Hausdorff topology on a KM-fuzzy metric space. The study of the fundamental properties of GV-fuzzy metric spaces has received a lot of attention from researchers. Gregori and Romaguera [14] and Aygünoğlu et al. [5] investigated some topological properties of GV-fuzzy metric spaces, Gregori et al. [12, 13] discussed convergence problems and gave a characterization of a class of completable spaces in this spaces. Different kinds of studies both theoretical and applied in this spaces and their generalizations can be found in the papers [2, 3, 4, 17, 33, 26, 29, 30]. Then, by considering the 2-metric axioms and KM-fuzzy metric axioms, Sharma [31] gave the notion of fuzzy 2-metric spaces and obtained some fixed point results in these spaces.

Another different approach to the concept of fuzzy metric spaces (named KS-fuzzy metric spaces), where the distance between two different points is nonnegative, upper semi-continuous, convex and normal fuzzy number, was given by Kaleva and Seikkala [23] in 1984. They also studied the properties of KS-fuzzy metric spaces, some relationships with the existing structures and proved some fixed point theorems. Kaleva [19] studied different kinds of convergences and relationships between these convergences in KS-fuzzy metric spaces. In the papers [20, 21], Kaleva investigated the completions of KS-fuzzy metric spaces. Many authors [17, 18, 32] considered some nonlinear contractions and presented several fixed point theorems for these contractions in complete KS-fuzzy metric spaces.

With a different view to the fuzzy 2-metric space and similar consideration to the KS-fuzzy metric space, in [15], we investigated the concept of 2-fuzzy metric space which is an extension of 2-metric space in which the area of the triangular region formed by three different points is considered as the nonnegative, upper semi-continuous, convex and normal fuzzy number. Also, we studied some of their topological properties and presented the relationship of 2-Menger spaces given in [34] with 2-fuzzy metric spaces. But since the axioms related triangular inequality of the 2-fuzzy metric space is not easy to apply, we have needed to investigate, as a more useful concept, some level forms of these equalities. For this reason, in the present paper, we investigate some level forms of triangular inequality of 2-fuzzy metric spaces which satisfies easier application to fixed point problems. We first define the concept of 2-fuzzy pre-metric spaces that have weaker axioms than 2-fuzzy metric spaces. Then we investigate under which conditions triangular inequalities are equivalent to their levels forms in 2-fuzzy pre-metric spaces. Finally, we prove a fixed point theorem for 2-fuzzy metric spaces by considering the presented result.

2. Preliminaries

In this section, the notions of fuzzy numbers and 2-fuzzy metric spaces are given to use in the main section. Throughout this paper, U refers to the universal set.

Definition 2.1. [6] *(1) A fuzzy number is a mapping such that* $u : \mathbb{R} \to [0, 1]$.

(2) A fuzzy number u *is called convex if* $u(t_1) \geq min(u(t_2), u(t_3))$ *when* $t_2 \leq t_1 \leq t_3$.

(3) A fuzzy number u *is called normal if there exist a* $t_0 \in \mathbb{R}$ *such that* $u(t_0) = 1$.

(4) An α*-level set of* u *is defined by the set* $\{t|u(t) \geq \alpha\}$ *where* $\alpha \in (0, 1]$ *and denoted by* $[u]_\alpha$.

(5) A fuzzy number u *is said to be nonnegative if* $u(t) = 0$ *for all* $t < 0$.

We will denote the set of all upper semi-continuous, convex and normal fuzzy numbers by G *and the set of all nonnegative elements of* G *by* E.

Each real number $u \in \mathbb{R}$ may be handled as a fuzzy number $\bar{u}$ as the following way:

$$\bar{u}(t) = \begin{cases} 0, & t \neq u \\ 1, & t = u \end{cases} . \tag{1}$$

Lemma 2.1. [17] *Let $[u]_0 = \overline{\{t \in \mathbb{R} | u(t) > 0\}}$ for $u \in G$ and $\alpha \in (0,1]$. Then the following properties hold for all $\alpha \in [0,1]$:*

(i) $[u]_\alpha$ is a closed interval $[\lambda_\alpha(u), \rho_\alpha(u)]$,

(ii) $\lambda_0(u) = lim_{\alpha \to 0} \lambda_\alpha(u)$ and $\rho_0(u) = lim_{\alpha \to 0} \rho_\alpha(u)$,

(iii) $u(t)$ is nondecreasing when $t < \lambda_1$ and nonincreasing when $t \geq \rho_1$.

The values $\lambda_\alpha(u) = -\infty$ and $\rho_\alpha(u) = \infty$ are admissible. If $\lambda_\alpha(u) = -\infty$, then $[\lambda_\alpha(u), \rho_\alpha(u)]$ means $(-\infty, \rho_\alpha(u)]$.

Lemma 2.2. [17] *A sequence $(u_n) \subseteq G$ is convergent to $u \in G$ if $\lim_{n\to\infty} \lambda_\alpha(u_n) = \lambda_\alpha(u)$ and $\lim_{n\to\infty} \rho_\alpha(u_n) = \rho_\alpha(u)$ for all $\alpha \in (0,1]$ where $[u]_\alpha = [\lambda_\alpha(u), \rho_\alpha(u)]$ and $[u_n]_\alpha = [\lambda_\alpha(u_n), \rho_\alpha(u_n)]$. If $(u_n) \subseteq E$ and $u = \bar{0}$, then the following is satisfied*

$$0 = \lambda_\alpha(u) \leq \rho_\alpha(u) \leq \lambda_\alpha(u_n) \leq \rho_\alpha(u_n). \tag{2}$$

for all $\alpha \in [0,1]$. Therefore, $\lim_{n\to\infty} u_n = \bar{0}$ if and only if $\lim_{n\to\infty} \rho_\alpha(u_n) = 0$.

Other properties and algebraic operations of fuzzy numbers can be found in [6, 27].

Definition 2.2. [15] *Let $C_L, C_R : [0,1]^3 \to [0,1]$ be two symmetric and nondecreasing mappings in its variables such that $C_L(0,0,0) = 0$ and $C_R(1,1,1) = 1$. A mapping $\mathbb{F} : U^3 \to E$ is said to be a 2-fuzzy metric if the followings are satisfied:*

(2FM1) For all pair of distinct points $x, u \in U$, there is a point $z \in U$ such that $\mathbb{F}(x,u,z) \neq \bar{0}$,

(2FM2) $\mathbb{F}(x,z,u) = \bar{0}$ when at least two of x, z, u are equal,

(2FM3) $\mathbb{F}(x,z,u) = \mathbb{F}(z,x,u) = \mathbb{F}(u,z,x) = \mathbb{F}(u,x,z) = \mathbb{F}(x,u,z) = \mathbb{F}(z,u,x)$ for all $x, u, z \in U$,

(2FM4) For all $x, u, z, w \in U$,

(i) $\mathbb{F}(x,z,u)(t_1+t_2+t_3) \geq C_L(\mathbb{F}(x,z,w)(t_1), \mathbb{F}(x,w,u)(t_2), \mathbb{F}(w,z,u)(t_3)$
whenever $t_1 \leq \lambda_1(x,z,w)$, $t_2 \leq \lambda_1(x,w,u)$, $t_3 \leq \lambda_1(w,z,u)$ and $t_1+t_2+t_3 \leq \lambda_1(x,z,u)$

(ii) $\mathbb{F}(x,z,u)(t_1+t_2+t_3) \leq C_R(\mathbb{F}(x,z,w)(t_1), \mathbb{F}(x,w,u)(t_2), \mathbb{F}(w,z,u)(t_3)$
whenever $t_1 \geq \lambda_1(x,z,w)$, $t_2 \geq \lambda_1(x,w,u)$, $t_3 \geq \lambda_1(w,z,u)$ and $t_1+t_2+t_3 \geq \lambda_1(x,z,u)$
where

$$[\mathbb{F}(x,z,u)]_\alpha = [\lambda_\alpha(x,z,u), \rho_\alpha(x,z,u)]$$

for all $\alpha \in [0,1]$. Then the 4-tuple pair $(U, \mathbb{F}, C_L, C_R)$ is said to be a 2-fuzzy metric space. The number $\mathbb{F}(x,z,u)(t)$ may be thought as the possibility that the area of the triangle which is formed by x, z and u in U (at least one of these points is not on the same line) as its vertices is t.

Definition 2.3. [15] *(x_n) be a sequence in a 2-fuzzy metric space $(X, \mathbb{F}, L, R)$.*

(i) (x_n) is called to be convergent to $x \in X$ if $\lim_{n\to\infty} \mathbb{F}(x_n, x, z) = \bar{0}$ for all $z \in X$.

(ii) (x_n) is called to be a Cauchy sequence if $\lim_{n\to\infty} \mathbb{F}(x_n, x_m, z) = \bar{0}$ for all $z \in X$.

(iii) $(X, \mathbb{F}, L, R)$ is called to be complete if every Cauchy sequence in X is convergent to some point $x \in X$.

Remark 2.1. *The property $\lim_{n\to\infty} \mathbb{F}(x_n, x, z) = \bar{0}$ for all $z \in X$ is equivalent to the property $\lim_{n\to\infty} \rho_0(x_n, x, z) = 0$ for all $z \in U$.*

3. Some inequalities in 2-fuzzy metric spaces

Lemma 3.1. *$C_L(0,1,1) = 0$ when $(U, \mathbb{F}, C_L, C_R)$ is a 2-fuzzy metric space.*

Proof Suppose that $C_L(0,1,1) > 0$. Let $x, u, w, z \in U$ and $x \neq u \neq w \neq z$. Take $t_1 = \lambda_1(x,z,w), t_2 = \lambda_1(z,w,u)$ and $t_3 < -t_1 - t_2$. We know from here that $\mathbb{F}(x,z,w)(t_1) = 1, \mathbb{F}(z,w,u)(t_2) = 1$ and $\mathbb{F}(x,w,u)(t_3) = 0$. So, $\mathbb{F}(x,z,u)(t_1+t_2+t_3) = 0$. From the condition (2FM4)(i), we obtain

$$\mathbb{F}(x,z,u)(t_1+t_2+t_3) \geq C_L(\mathbb{F}(x,z,w)(t_1), \mathbb{F}(x,w,u)(t_2), \mathbb{F}(w,z,u)(t_3) = C_L(1,0,1) > 0$$

which is a contradiction and it means that $C_L(0,1,1) = 0$.

Remark 3.1. *If C_L is a t-norm, then $C_L(0,1,1) = 0$ is obviously satisfied.*

Definition 3.1. *Let $C_L, C_R : [0,1]^3 \to [0,1]$ be two symmetric and nondecreasing mappings in its variables and $\mathbb{F} : U^3 \to E$ be a mapping. Then the 4-tuple $(U, \mathbb{F}, C_L, C_R)$ is called a 2-fuzzy pre-metric space provided that $C_L(0,1,1) = 0$, $C_R(1,1,1) = 1$ and $\mathbb{F}$ holds the conditions (2FM1)-(2FM3).*

It is obvious from Lemma 3.1 that a 2-fuzzy metric space is always 2-fuzzy pre-metric space.

In the next theorem, we prove that the following level form inequality is equivalent to the triangle inequality (2FM4)(i) for the 2-fuzzy pre-metric space:

Theorem 3.1. *If $(U, \mathbb{F}, C_L, C_R)$ is a 2-fuzzy pre-metric space, then the triangle inequality (2FM4)(i) is hold if and only if*

$$\lambda_{C_L(\alpha,\beta,\gamma)}(x,z,u) \leq \lambda_\alpha(x,z,y) + \lambda_\beta(x,y,u) + \lambda_\gamma(y,z,u) \tag{3}$$

for all $\alpha, \beta, \gamma \in (0,1]$ and $x, u, y, z \in U$.

Proof Suppose that the triangle inequality (2FM4)(i) is satisfied. Let us take $x, u, y, z \in U$ and $\alpha, \beta, \gamma \in (0,1]$. Let $t_1 = \lambda_\alpha(x,z,y), t_2 = \lambda_\beta(x,y,u)$ and $t_3 = \lambda_\gamma(y,z,u)$. Then, we have that $\lambda_\alpha(x,z,y) \leq \lambda_1(x,z,y), \lambda_\beta(x,y,u) \leq \lambda_1, \lambda_\gamma(y,z,u), \alpha \leq \mathbb{F}(x,z,y)(t_1), \beta \leq \mathbb{F}(x,y,u)(t_2)$ and $\gamma \leq \mathbb{F}(y,z,u)(t_3)$. It is obvious that $\mathbb{F}(x,z,u)(t_1+t_2+t_3) \geq C_L(\alpha,\beta,\gamma)$ whenever $t_1+t_2+t_3 \leq \lambda_1(x,u,z)$. From here, we have that $\lambda_{C_L(\alpha,\beta,\gamma)} \leq t_1+t_2+t_3$. If $t_1+t_2+t_3 > \lambda_1(x,z,u)$, then the inequality $t_1+t_2+t_3 > \lambda_{C_L(\alpha,\beta,\gamma)}(x,z,u)$ holds. Hence, the necessary condition of theorem is proved.

To prove the sufficient condition, suppose that the inequality (3) is satisfied for all $\alpha, \beta, \gamma \in (0,1]$ and $x, u, y, z \in U$. If we take $t_1 \leq \lambda_1(x,z,y), t_2 \leq \lambda_1(x,y,u), t_3 \leq \lambda_1(y,z,u)$ and let $\alpha = \mathbb{F}(x,z,y)(t_1), \beta = \mathbb{F}(x,y,u)(t_2), \gamma = \mathbb{F}(y,z,u)(t_3)$, then we obtain that $\lambda_\alpha(x,z,y) \leq t_1, \lambda_\beta(x,y,u) \leq t_2$ and $\lambda_\gamma(y,z,u) \leq t_3$. Without loss of generality, we may suppose that $\alpha, \beta, \gamma > 0$. Otherwise triangle inequality (2FM4)(i) is directly satisfied. From (3), we have $\lambda_{C_L(\alpha,\beta,\gamma)}(x,z,u) \leq t_1+t_2+t_3$. Since $t_1+t_2+t_3 \leq \lambda_1(x,z,u)$, this implies that $\mathbb{F}(x,z,u)(t_1+t_2+t_3) \geq C_L(\alpha,\beta,\gamma) = C_L(\mathbb{F}(x,z,y)(t_1), \mathbb{F}(x,y,u)(t_2), \mathbb{F}(y,z,u)(t_3))$.

Definition 3.2. *Let $(U, \mathbb{F}, C_L, C_R)$ be a 2-fuzzy pre-metric space. Set*

$$\rho_\alpha{}'(x,z,u) = \begin{cases} lim_{\mu \to \alpha^+} \rho_\mu(x,z,u), & \alpha \in [0,1) \\ \lambda_1(x,z,u), & \alpha = 1 \end{cases}$$

for all $x, u, z \in U$ and $\alpha \in [0,1]$. It is straightforward to see that $\rho_\alpha{}'(x,z,u)$ is nondecreasing for all $\alpha \in [0,1]$ and the condition $\lambda_1(x,z,u) \leq \rho_\mu(x,z,u) \leq \rho_\alpha{}'(x,z,u) \leq \rho_\alpha(x,u,z)$ hold for all $\mu \in (\alpha, 1]$. It is also easily seen that $\rho_\alpha{}'(x,z,u) = lim_{\mu \to \alpha^+} \rho_\mu{}'(x,z,u)$. i.e., $\rho_\alpha{}'(x,z,u)$ is a right continuous mapping.

Lemma 3.2. *Let $(U, \mathbb{F}, C_L, C_R)$ be a 2-fuzzy pre-metric space. Then the following properties hold:*
(i) $\mathbb{F}(x,z,u)(\rho_\alpha{}'(x,z,u)) \geq \alpha$.
(ii) If $t > \rho_\alpha{}'(x,z,u)$, then $\mathbb{F}(x,z,u)(t) \leq \alpha$.
(iii) If $\alpha \neq 1$ and $\lambda_1(x,z,u) \leq t < \rho_\alpha{}'(x,z,u)$, then $\mathbb{F}(x,z,u)(t) < \alpha$.

Proof (i) If $\alpha = 1$, then the conclusion is directly true. Let $\alpha < 1$. Then, we obtain that $\mathbb{F}(x,z,u)(\rho_\alpha{}'(x,z,u)) \geq \mathbb{F}(x,z,u)(\rho_\alpha(x,z,u)) \geq \alpha$, since $\mathbb{F}(x,z,u)(t)$ is a nonincreasing mapping whenever $t > \rho_1(x,z,u)$.
(ii) Similar to the above situation let $\alpha < 1$. Suppose that $\mathbb{F}(x,z,u)(t) > \alpha$. Then, there is a $\mu > \alpha$ such that $\mathbb{F}(x,z,u)(t) > \mu$. So, we have that $t \leq \rho_\mu(x,z,u) \leq \rho'_\alpha(x,z,u)$ which contradicts to the hypothesis.
(iii) Let $\lambda_1(x,z,u) \leq t < \rho'_\alpha(x,z,u)$. Then, there exist a $\mu > \alpha$ such that $\lambda_1(x,z,u) \leq t < \rho_\mu(x,z,u)$, Hence, we obtain that $\mathbb{F}(x,z,u)(t) \geq \mu > \alpha$.

Remark 3.2. *$\rho'_\alpha(x,z,u)$ is the right endpoint of the α-level set $\{t|\mathbb{F}(x,z,u)(t) > \alpha\}$ whenever $\alpha \in [0,1)$.*

Lemma 3.3. *Let $(U, \mathbb{F}, C_L, C_R)$ be a 2-fuzzy pre-metric space. If triangle inequality (2FM4)(ii) is hold, then*

$$\rho'_{C_R(\alpha,\beta,\gamma)}(x,z,u) \leq \rho'_\alpha(x,z,y) + \rho'_\beta(x,y,u) + \rho'_\gamma(y,z,u) \tag{4}$$

for all $x,u,y,z \in U$ and $\alpha,\beta,\gamma \in [0,1]$ with $C_R(\alpha,\beta,\gamma) \neq 1$.

Proof Let $x,u,y,z \in U$ and $\alpha,\beta,\gamma \in [0,1]$ with $C_R(\alpha,\beta,\gamma) \neq 1$. Suppose that $\rho'_\alpha(x,z,y) < \infty, \rho'_\beta(x,y,u) < \infty$ and $\rho'_\gamma(y,z,u) < \infty$. Otherwise (4) is trivially hold. By Lemma (3.2)(ii), we have $\mathbb{F}(x,z,y)(t_1) \leq \alpha$, $\mathbb{F}(x,y,u)(t_2) \leq \beta$ and $\mathbb{F}(y,z,u)(t_3) \leq \gamma$ for any t_1,t_2,t_3 with $\rho_alpha'(x,z,y) < t_1, \rho'_\beta(x,y,u) < t_2$ and $\rho'_\gamma(y,z,u) < t_3$. Now, we show that $t_1 + t_2 + t_3 \geq \lambda_1(x,z,u)$. Suppose that $t_1, +t_2 + t_3 < \lambda_1(x,z,u)$. Take t'_1, t'_2, t'_3 satisfying $t'_1 \geq t_1 > \lambda_1(x,z,y), t'_2 \geq t_2 > \lambda_1(x,y,u), t'_3 \geq t_3 > \lambda_1(y,z,u)$ and $t'_1 + t'_2 + t'_3 = \lambda_1(x,z,u)$. Then by the triangle inequality (2FM4)(ii), we obtain

$$\begin{aligned} 1 = \mathbb{F}(x,z,u)(\lambda_1(x,z,u)) &\leq C_R(\mathbb{F}(x,z,u)(t'_1), \mathbb{F}(x,y,u)(t'_2), \mathbb{F}(y,z,u)(t'_3)) \\ &\leq C_R(\alpha,\beta,\gamma) < 1 \end{aligned}$$

which is a contradiction. Hence, we get $\lambda_1(x,z,u) \leq t_1, +t_2 + t_3$. Since $t_1 > \lambda_1(x,z,y), t_2 > \lambda_1(x,y,u), t_3 > \lambda_1(y,z,u)$ and $t_1 + t_2 + t_3 \geq \lambda_1(x,z,u)$, we obtain from triangle inequality (2FM4)(ii)

$$\begin{aligned} \mathbb{F}(x,z,u)(t_1, +t_2 + t_3) &\leq C_R(\mathbb{F}(x,z,y)(t_1), \mathbb{F}(x,y,u)(t_2), \mathbb{F}(y,z,u)(t_3)) \\ &\leq C_R(\alpha,\beta,\gamma) \end{aligned}$$

which means that $\rho'_{C_R(\alpha,\beta,\gamma)}(x,z,u) \leq t_1, +t_2 + t_3$. Therefore inequality (4) is satisfied since t_1,t_2,t_3 are chosen arbitrarily.

In the following, we show that the triangle inequality (2FM4)(ii) is equivalent to the level form (4) described by $\rho'_\alpha(x,z,u)$ for the 2-fuzzy pre-metric spaces when C_R is a right continuous mapping.

Theorem 3.2. *Let $(U, \mathbb{F}, C_L, C_R)$ be a 2-fuzzy pre-metric space such that C_R is right continuous. Then the following conditions are equivalent:*
(i) The triangle inequality (2FM4)(ii) is satisfied.
(ii) (4) is hold for all $x,u,y,z \in U$ and $\alpha,\beta,\gamma \in [0,1]$ with $C_R(\alpha,\beta,\gamma) \neq 1$.
(iii) (4) is hold for all $x,u,y,z \in U$ and $\alpha,\beta,\gamma \in (0,1]$ with $C_R(\alpha,\beta,\gamma) \neq 1$.

Proof The implication $(i) \Rightarrow (iii)$ is satisfied directly from Lemma 3.3. Now, we show that (iii) implies (ii). If there exist $\alpha, \beta > 0$ with $C_R(\alpha, \beta, 0) \neq 1$, then we can find a sequence (γ_n) such that $\gamma_n \to 0^+$, $C_R(\alpha, \beta, \gamma_n) \neq 1$ and $\rho'_{C_R(\alpha,\beta,\gamma_n)}(x, z, u) \leq \rho'_\alpha(x, z, y) + \rho'_\beta(x, y, u) + \rho'_{\gamma_n}(y, z, u)$ from the right continuity of C_R. Again by the right continuity of C_R and $\rho'_{\gamma_n}(y, z, u) \to \rho'_0(y, z, u)$, we have

$$\rho'_{C_R(\alpha,\beta,0)}(x, z, u) \leq \rho'_\alpha(x, z, y) + \rho'_\beta(x, y, u) + \rho'_0(y, z, u)$$

as $n \to \infty$. If there exist $\alpha > 0$ with $C_R(\alpha, 0, 0) \neq 1$ or if $C_R(0, 0, 0) \neq 1$, it can be shown similar to the above situation that $\rho'_{C_R(\alpha,0,0)}(x, z, u) \leq \rho'_\alpha(x, z, y) + \rho'_0(x, y, u) + \rho'_0(y, z, u)$ and $\rho'_{C_R(0,0,0)}(x, z, u) \leq \rho'_0(x, z, y) + \rho'_0(x, y, u) + \rho'_0(y, z, u)$, respectively.
Now, suppose that (4) is hold for all $x, u, y, z \in U$ and $\alpha, \beta, \gamma \in [0, 1]$ with $C_R(\alpha, \beta, \gamma) \neq 1$. Take $t_1 \geq \lambda_1(x, z, y), t_2 \geq \lambda_1(x, y, u)$ and $t_3 \geq \lambda_1(y, z, u)$ with $t_1 + t_2 + t_3 \geq \lambda_1(x, z, u)$. Let $\alpha_0 = \mathbb{F}(x, z, y)(t_1), \beta_0 = \mathbb{F}(x, y, u)(t_2)$ and $\gamma_0 = \mathbb{F}(y, z, u)(t_3)$. If $C_R(\alpha_0, \beta_0, \gamma_0) = 1$, then the triangle inequality (4) is obviously hold. If $C_R(\alpha_0, \beta_0, \gamma_0) \neq 1$, then from (ii) and Lemma 3.2(iii), we have

$$\begin{aligned} t_1 \geq \rho'_{\alpha_0}(x, z, y), t_2 \geq \rho'_{\beta_0}(x, y, u), t_3 \geq \rho_{\gamma_0}(y, z, u) \\ \text{and } t_1 + t_2 + t_3 \geq \rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u) \end{aligned} \tag{5}$$

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Consider the following three cases:

Case I: If $\mathbb{F}(x, z, u)(\rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u)) = C_R(\alpha_0, \beta_0, \gamma_0)$, then we have

$$\mathbb{F}(x, z, u)(t_1 + t_2 + t_3) \leq \mathbb{F}(x, z, u)(\rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u)) = C_R(\alpha_0, \beta_0, \gamma_0).$$

Case II: If $\mathbb{F}(x, z, y)(\rho'_{\alpha_0}(x, z, y)) > \alpha_0$ or $\mathbb{F}(x, y, u)(\rho'_{\beta_0}(x, y, u)) > \beta_0$ or $\mathbb{F}(y, z, u)(\rho'_{\gamma_0}(y, z, u)) > \gamma_0$, then by (refe3) $t_1 > \rho'_{\alpha_0}(x, z, y)$ or $t_2 > \rho'_{\beta_0}(x, y, u)$ or $t_3 > \rho'_{\gamma_0}(y, z, u)$. Thus, we have

$$\rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u) \leq \rho'_{\alpha_0}(x, z, y) + \rho'_{\beta_0}(x, y, u) + \rho'_{\gamma_0}(y, z, u) < t_1 + t_2 + t_3$$

From Lemma 3.2(ii), we obtain that $\mathbb{F}(x, z, u)(t_1 + t_2 + t_3) \leq C_R(\alpha_0, \beta_0, \gamma_0)$.

Case III: If neither of the above situations hold, then by Lemma 3.2(i), the followings obtained:

$$\mathbb{F}(x, z, u)(\rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u)) > C_R(\alpha_0, \beta_0, \gamma_0) \tag{6}$$

$$\mathbb{F}(x, z, y)(\rho'_{\alpha_0}(x, z, y)) = \alpha_0 \tag{7}$$

$$\mathbb{F}(x, y, u)(\rho'_{\beta_0}(x, y, u)) = \beta_0 \tag{8}$$

$$\mathbb{F}(y, z, u)(\rho'_{\gamma_0}(y, z, u)) = \gamma_0. \tag{9}$$

If we consider the right continuity of C_R in (6), there are $\alpha_1 > \alpha_0, \beta_1 > \beta_0$ and $\gamma_1 > \gamma_0$ such that $C_R(\alpha_1, \beta_1, \gamma_1) < 1$ and $\mathbb{F}(x, z, u)(\rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u)) > C_R(\alpha_1, \beta_1, \gamma_1)$. By Lemma 3.2(ii), we obtain

$$\rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u) \leq \rho'_{C_R(\alpha_1,\beta_1,\gamma_1)}(x, z, u).$$

On the other hand, it is obviously seen that $\rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u) \geq \rho'_{C_R(\alpha_1,\beta_1,\gamma_1)}(x, z, u)$ since $C_R(\alpha_1, \beta_1, \gamma_1) \geq C_R(\alpha_0, \beta_0, \gamma_0)$. Therefore, it follows that $C_R(\alpha_1, \beta_1, \gamma_1) = C_R(\alpha_0, \beta_0, \gamma_0)$. By (7)-(9) and Lemma 3.2(i), we get $\rho'_{\alpha_0}(x, z, y) > \rho'_{\alpha_1}(x, z, y)$, $\rho'_{\beta_0}(x, y, u) > \rho'_{\beta_1}(x, y, u)$ and $\rho'_{\gamma_0}(y, z, u) > \rho'_{\gamma_1}(y, z, u)$. Since $C_R(\alpha_1, \beta_1, \gamma_1) < 1$, we also have $\rho'_{C_R(\alpha_1,\beta_1,\gamma_1)}(x, z, u) \leq \rho'_{\alpha_1}(x, z, y) + \rho'_{\beta_1}(x, y, u) + \rho'_{\gamma_1}(y, z, u)$ by (4). Considering all cases, we obtain

$$\rho'_{C_R(\alpha_0,\beta_0,\gamma_0)}(x, z, u) < \rho'_{\alpha_0}(x, z, y) + \rho'_{\beta_0}(x, y, u) + \rho'_{\gamma_0}(y, z, u) \leq t_1 + t_2 + t_3.$$

Again from Lemma 3.2(ii), this implies the following inequality

$$\begin{aligned}\mathbb{F}(x,z,u)(t_1+t_2+t_3) &\le C_R(\alpha_0,\beta_0,\gamma_0)\\ &= C_R(\mathbb{F}(x,z,y)(t_1),\mathbb{F}(x,y,u)(t_2),\mathbb{F}(y,z,u)(t_3)).\end{aligned}$$

In the following definition, we consider the set $\{t|\mathbb{F}(x,z,u)(t)\ge\alpha\}$ which is another form of the α-level set.

Definition 3.3. *Let $(U,\mathbb{F},C_L,C_R)$ be a 2-fuzzy pre-metric space. Define*

$$\rho^*_\alpha(x,z,u)=\begin{cases}\rho_\alpha(x,z,u), & \alpha\in[0,1)\\ \lambda_1(x,z,u), & \alpha=1\end{cases}$$

for all $x,z,u\in U$ and $\alpha\in[0,1]$.

Lemma 3.4. *Let $(U,\mathbb{F},C_L,C_R)$ be a 2-fuzzy pre-metric space, $x,u,z\in U$ and $\alpha\in[0,1]$. Then the following properties hold:*
(i) If $\alpha=0$ or $\alpha=1$, then $\rho^*_\alpha(x,z,u)=\rho'_\alpha(x,z,u)$.
(ii) If $\alpha\in(0,1)$, then $\rho^*_\alpha(x,z,u)=lim_{\mu\to\alpha^-}\rho'_\mu(x,z,u)$ and $\rho'_\alpha(x,z,u)=lim_{\mu\to\alpha^+}\rho^*_\mu(x,z,u)$.

Proof (i) It is clear from the Definition 3.2 and Definition 3.3.
(ii) Since $\rho_\alpha(x,z,u)\le\rho'_\mu(x,z,u)\le\rho_\mu(x,z,u)$ when $\mu<\alpha$, we have

$$\rho^*_\alpha(x,z,u)=\rho_\alpha(x,z,u)=lim_{\mu\to\alpha^-}\rho_\mu(x,z,u)=lim_{\mu\to\alpha^-}\rho'_\mu(x,z,u)$$
$$\rho'_\alpha(x,z,u)=lim_{\mu\to\alpha^+}\rho_\mu(x,z,u)=lim_{\mu\to\alpha^+}\rho^*_\mu(x,z,u).$$

Let $\mathfrak{C}$ denote the set of all mappings $C_R:[0,1]\times[0,1]\times[0,1]\to[0,1]$ satisfying the following conditions for all $\alpha,\beta,\gamma\in[0,1]$ with $C_R(0,0,0)<C_R(\alpha,\beta,\gamma)<1$:
(i) If $\alpha,\beta,\gamma<1$, then $C_R(\alpha',\beta',\gamma')<C_R(\alpha,\beta,\gamma)$ for all $\alpha'\in[0,\alpha)$, $\beta'\in[0,\beta)$ and $\gamma'\in[0,\gamma)$.
(ii) If $\alpha=1$, then $C_R(\alpha,\beta',\gamma')<C_R(\alpha,\beta,\gamma)$ for all $\beta'\in[0,\beta)$ and $\gamma'\in[0,\gamma)$.
(ii) If $\alpha=1$ and $\beta=1$, then $C_R(\alpha,\beta,\gamma')<C_R(\alpha,\beta,\gamma)$ for all $\gamma'\in[0,\gamma)$.

Theorem 3.3. *Let $(U,\mathbb{F},C_L,C_R)$ be a 2-fuzzy pre-metric space. If C_R is continuous and $C_R\in\mathfrak{C}$, then the triangle inequality (2FM4)(ii) is hold if and only if*

$$\rho^*_{C_R(\alpha,\beta,\gamma)}(x,z,u)\le\rho^*_\alpha(x,z,y)+\rho^*_\beta(x,y,u)+\rho^*_\gamma(y,z,u) \tag{10}$$

for all $x,u,y,z\in U$ and $\alpha,\beta,\gamma\in[0,1]$ with $C_R(0,0,0)<C_R(\alpha,\beta,\gamma)<1$.

Proof We may assume that $C_R(0,0,0)\ne1$. Respect to the Theorem 3.2, it is sufficient to show that (4) is hold for all $x,y,z,u\in U$ and $\alpha,\beta,\gamma\in(0,1]$ with $C_R(\alpha,\beta,\gamma)\ne1$ if and only if (10) is hold for all $x,y,z,u\in U$ and $\alpha,\beta,\gamma\in(0,1]$ with $C_R(0,0,0)<C_R(\alpha,\beta,\gamma)<1$.

Assume that (4) is hold for all $x,y,z,u\in U$ and $\alpha,\beta,\gamma\in(0,1]$ with $C_R(\alpha,\beta,\gamma)\ne1$. Let $C_R(0,0,0)<C_R(\alpha,\beta,\gamma)<1$ for all $\alpha,\beta,\gamma\in(0,1]$. It is clear that $\alpha,\beta,\gamma)\ne(1,1,1)$. Since $C_R\in\mathfrak{C}$, there are nondecreasing sequence $(\alpha_n)\subset(0,\alpha],(\beta_n)\subset(0,\beta],(\gamma_n)\subset(0,\gamma]$ such that $\alpha_n\to\alpha,\beta_n\to\beta,\gamma_n\to\gamma$ and $C_R(\alpha_n,\beta_n,\gamma_n)<C_R(\alpha,\beta,\gamma)$. Then by (4), it follows that $\rho'_{C_R(\alpha_n,\beta_n,\gamma_n)}(x,z,u)\le\rho'_{\alpha_n}(x,z,y)+\rho'_{\beta_n}(x,y,u)+\rho'_{\gamma_n}(y,z,u)$. By Lemma 3.4, we have that $\rho^*_{C_R(\alpha,\beta,\gamma)}(x,z,u)\le\rho^*_\alpha(x,z,y)+\rho^*_\beta(x,y,u)+\rho^*_\gamma(y,z,u)$ as $n\to\infty$.

Now, suppose that (10) is hold for all $x,y,z,u\in U$ and $\alpha,\beta,\gamma\in(0,1]$ with $C_R(0,0,0)<C_R(\alpha,\beta,\gamma)<1$. We must show the following cases:

Case I: Let $C_R(\alpha,\beta,\gamma)\ne C_R(0,0,0)$. Since $C_R(\alpha,\beta,\gamma)\ne1$, we have $(\alpha,\beta,\gamma)\ne(1,1,1)$. By continuity of C_R, there are nondecreasing sequence $(\alpha_n)\subset[\alpha,1],(\beta_n)\subset[\beta,1],(\gamma_n)\subset[\gamma,1]$ such that $\alpha_n\to\alpha,\beta_n\to\beta,\gamma_n\to\gamma$ and $C_R(0,0,0)<C_R(\alpha_n,\beta_n,\gamma_n)<1$. From (10), we get $\rho^*_{C_R(\alpha_n,\beta_n,\gamma_n)}(x,z,u)\le\rho^*_{\alpha_n}(x,z,y)+\rho^*_{\beta_n}(x,y,u)+\rho^*_{\gamma_n}(y,z,u)$.

If $C_R(\alpha,\beta,\gamma) < C_R(\alpha_n,\beta_n,\gamma_n)$ for all $n \in \mathbb{N}$, then from Lemma 3.4, we obtain that $\rho'_{C_R(\alpha,\beta,\gamma)}(x,u,z) \leq \rho'_\alpha(x,u,y) + \rho'_\beta(x,y,z) + \rho'_\gamma(y,u,z)$.

If there is a $n_0 \in \mathbb{N}$ satisfying $C_R(\alpha_n,\beta_n,\gamma_n) = C_R(\alpha,\beta,\gamma)$ for all $n > n_0$, then from Lemma 3.4, we have that $\rho'_{C_R(\alpha,\beta,\gamma)}(x,z,u) \leq \rho'_\alpha(x,z,y) + \rho'_\beta(x,y,u) + \rho'_\gamma(y,z,u)$.

Case II: Let $C_R(\alpha,\beta,\gamma) = C_R(0,0,0)$. Choose

$$\begin{aligned}
\alpha_1 &= sup\{\xi \in [\alpha,1] | C_R(\xi,\beta,\gamma) = C_R(0,0,0)\} \\
\beta_1 &= sup\{\nu \in [\beta,1] | C_R(\alpha,\nu,\gamma) = C_R(0,0,0)\} \\
\gamma_1 &= sup\{\eta \in [\gamma,1] | C_R(\alpha,\beta,\eta) = C_R(0,0,0)\}.
\end{aligned}$$

Then $C_R(\alpha_1,\beta_1,\gamma_1) = C_R(0,0,0)$ from continuity of C_R. So, $C_R(0,0,0) < C_R(\xi',\nu',\eta')$ for all $(\xi',\nu',\eta') \in [\alpha_1,1] \times [\beta_1,1] \times [\gamma_1,1] \setminus \{(\alpha_1,\beta_1,\gamma_1)\}$. Also, there is (ξ',ν',η') such that $C_R(0,0,0) < C_R(\xi'',\nu'',\eta'') < 1$ from continuity of C_R. It is obvious that $(\xi'',\nu'',\eta'') \neq (\alpha,\beta,\gamma)$. Hence $C_R(0,0,0) < C_R(\xi',\nu',\eta') < 1$ for all $(\xi',\nu',\eta') \in [\alpha_1,\xi''] \times [\beta_1,\nu''] \times [\gamma_1,\eta''] \setminus \{(\alpha_1,\beta_1,\gamma_1)\}$. From the above case, we also have that $\rho'_{C_R(\alpha'',\beta'',\gamma'')}(x,u,z) \leq \rho'_{\alpha''}(x,u,y) + \rho'_{\beta''}(x,y,z) + \rho'_{\gamma''}(y,u,z)$. If we let $\xi'' \to \alpha_1, \nu'' \to \beta_1$ and $\eta'' \to \gamma_1$, then we obtain

$$\begin{aligned}
\rho'_{C_R(\alpha,\beta,\gamma)}(x,z,u) &= \rho'_{C_R(\alpha_1,\beta_1,\gamma_1)}(x,z,u) \\
&\leq \rho'_{\alpha_1}(x,z,y) + \rho'_{\beta_1}(x,y,u) + \rho'_{\gamma_1}(y,z,u) \\
&\leq \rho'_\alpha(x,z,y) + \rho'_\beta(x,y,u) + \rho'_\gamma(y,z,u).
\end{aligned}$$

Remark 3.3. *If we let $C_R = max$, then $C_R \in \mathfrak{C}$. From Theorem 3.3, we deduce that triangle inequality (2FM4)(ii) is hold if and only if*

$$\rho_\alpha(x,z,u) \leq \rho_\alpha(x,z,y) + \rho_\alpha(x,y,u) + \rho_\alpha(y,z,u)$$

for all $x,y,z,u \in U$ and $\alpha \in [0,1]$.

4. An application to fixed point theory

In this section, we assume that $(U,\mathbb{F},C_L,C_R)$ is a 2-fuzzy metric space satisfying $\rho_\alpha(x,y,z) < \infty$ and $C_R(0,0,0) = 0$.

Theorem 4.1. *Let f be a mapping of a complete 2-fuzzy metric space $(U,\mathbb{F},C_L,C_R)$ from itself such that there exists a real number k with $0 < k < 1$ satisfying*

$$\rho_0(f(x),f(y),z) \leq k\rho_0(x,y,z)$$

for all $x,y,z \in U$. Then f has a unique fixed point in U.

Proof Let $x_0 \in U$ and define the sequences of iterates (x_n) with $x_n = f^n(x_0)$. For all $n \in (N)$ and $z \in U$,

$$\rho_0(f^{n+1}(x_0),f^n(x_0),z) \leq k\rho_0(f^n(x_0),f^{n-1}(x_0),z) < \rho_0(f^n(x_0),f^{n-1}(x_0),z).$$

It follows that $a_n(z) = \rho_0(f^n(x_0),f^{n-1}(x_0),z)$ is both strictly decreasing and bounded below for all $z \in U$. So, $(a_n(z))$ converges to some function of z, let $a(z)$ denote this function. Since $a(z) \leq ka(z) < a(z)$ for all $z \in U$, we have $a(z) = 0$. Now, we claim that (x_n) is a Cauchy sequence in U. To show this assertion suppose (x_n) is not a Cauchy sequence in U. Then there exist $\varepsilon > 0$, $u \in U$ and strictly increasing sequences (m_k), (n_k) of positive integers such that

$$\rho_0(f^{n_k}(x_0),f^{m_k}(x_0),u) \geq \varepsilon \text{ and } \rho_0(f^{n_k-1}(x_0),f^{m_k}(x_0),u) < \varepsilon$$

for all $k \in \mathbb{N}$ whenever $m_k > n_k \geq k$. Since $R(0,0,0) = 0$ and $\rho_0'(x,y,z) = \rho_0(x,y,z)$, we obtain

$$\begin{aligned}
&\rho_0(f^{n_k}(x_0), f^{m_k}(x_0), u) - \rho_0(f^{n_k-1}(x_0), f^{m_k}(x_0), u) \\
&\leq \rho_0(f^{m_k}(x_0), f^{m_k-1}(x_0), u) + \rho_0(f^{m_k}(x_0), f^{m_k-1}(x_0), f^{n_k}(x_0)) \\
&\leq \rho_0(f^{m_k}(x_0), f^{m_k-1}(x_0), u) + k\rho_0(f^{m_k-1}(x_0), f^{m_k-1}(x_0), f^{n_k-1}(x_0)) \\
&= \rho_0(f^{m_k}(x_0), f^{m_k-1}(x_0), u)
\end{aligned}$$

by Lemma 3.3. Since $(\rho_0(f^{n_k}(x_0), f^{m_k}(x_0), u) - \varepsilon)$ and $\varepsilon - \rho_0(f^{n_k-1}(x_0), f^{m_k}(x_0), u)$ are sequences of nonnegative real numbers and $a_n(u) \to 0$ as $n \to \infty$, from the above inequality it follow that

$$\lim_{k\to\infty} \rho_0(f^{n_k}(x_0), f^{m_k}(x_0), u) = \varepsilon$$

and

$$\lim_{k\to\infty} \rho_0(f^{n_k-1}(x_0), f^{m_k}(x_0), u) = \varepsilon.$$

Since the following inequality satisfies for all $x, y, a, b \in U$

$$|\rho_0(x,y,a) - \rho_0(x,y,b)| \leq \rho_0(a,b,x) + \rho_0(a,b,y),$$

we have

$$\begin{aligned}
&|\rho_0(f^{m_k-1}(x_0), u, f^{n_k-1}(x_0)) - \rho_0(f^{m_k-1}(x_0), u, f^{n_k}(x_0))| \\
&\leq \rho_0(f^{n_k-1}(x_0), f^{n_k}(x_0), f^{m_k-1}(x_0)) + \rho_0(f^{n_k-1}(x_0), f^{n_k}(x_0), u) \\
&< \rho_0(f^{n_k}(x_0), f^{n_k-1}(x_0), u).
\end{aligned}$$

It follows from taking limits of this inequality that $\lim_{k\to\infty} \rho_0(f^{m_k-1}(x_0), f^{n_k-1}(x_0), u) = \varepsilon$. But now, from the following

$$\rho_0(f^{m_k}(x_0), f^{n_k}(x_0), u) \leq k\rho_0(f^{m_k-1}(x_0), f^{n_k-1}(x_0), u)$$

we have $\varepsilon \leq k\varepsilon < \varepsilon$ which contradicts $\varepsilon > 0$. Thus, (x_n) is a Cauchy sequence in U and convergent to a point $x \in U$. i.e., $\rho_0(f^n(x_0), x, z) \to 0$ as $n \to \infty$ for all $z \in U$. Since

$$\begin{aligned}
&|\rho_0(f(x), z, f^n(x_0)) - \rho_0(f(x), z, x)| \leq \rho_0(f^n(x_0), x, f(x)) + \rho_0(f^n(x_0), x, z) \\
&\leq k\rho_0(f^{n-1}(x_0), x, x) + \rho_0(f^n(x_0), x, z) = \rho_0(f^n(x_0), x, z),
\end{aligned}$$

by taking limits we obtain $\lim_{k\to\infty} \rho_0(f^n(x_0), f(x), z) = \rho_0(f(x), x, z) = 0$ for all $z \in U$ and thus $f(x) = x$.

5. Conclusions

In this paper, we study some level forms of triangular inequality of 2-fuzzy metric spaces and solve a fixed point problem of self-mappings in 2-fuzzy metric spaces by using these level forms. For future work, we aim to study topological properties of 2-fuzzy metric spaces such as completeness, compactness and countability and etc. by using these level forms. Also, we plan to carry some well-known fixed point theorems to 2-fuzzy metric spaces.

Acknowledgement. The authors are thankful to the anonymous referees for their valuable suggestions.

References

[1] Aliouche, A. and Simpson, C., (2012), Fixed points and lines in 2-metric spaces, Advances in Mathematics, 229(1), pp. 668-690.

[2] Aldemir, B., Güner, E., Aydoğdu, E. and Aygün, H., (2020), Some fixed point theorems in partial fuzzy metric spaces, Journal of the Institute of Science and Technology, 10(4), pp. 2889-2900.

[3] Aydoğdu, E.., Aldemir, B., Güner, E. and Aygün, H., (2020). Some properties of partial fuzzy metric topology, Advances in Intelligent Systems and Computing, Springer, Cham, pp. 1267-1275.

[4] Aydoğdu, E., Aygünoğlu, A. and Aygün, H., (2020), The space of continuous function between fuzzy metric spaces, Erzincan University Journal of Science and Technology, 13(3), pp. 1132-1137.

[5] Aygünoğlu, A., Aydoğdu, E. and Aygün, H., (2020), Construction of fuzzy topology by using fuzzy metric, Filomat, 34(2), pp. 433-441.

[6] Chai, Y. and Zhang, D., (2016), A representation of fuzzy numbers, Fuzzy Sets and Systems 295, pp. 1-18.

[7] Çetkin, V., Güner, E. and Aygün, H., (2020), On 2s-metric spaces, Soft Computing, 24(17), pp. 12731-12742.

[8] Das, S. and Samanta, S. K., (2013), Soft metric, Annals of Fuzzy Mathematics and Informatics, 6(1), pp. 77-94.

[9] Farhangdoost, M. R., (2012), Metrizable and 2-metrizable topological spaces, Journal of Dynamical Systems and Geometric Theories, 10(1), pp. 61-69.

[10] Gähler, S., (1963), 2-Metrische Räume und ihre topologische struktur, Mathematische Nachrichten, 26, pp. 115-118.

[11] Golet, I., (1995), Fixed point theorems for multivalued mapping in probabilistic 2-metric spaces, An. St. Univ. Ovidius Constanta, 3, pp. 44-51.

[12] Gregori, V., Lopez-Crevillen, A., Morillas, S. and Sapena, A., (2009), On convergence in fuzzy metric spaces, Topology and its Applications, 156, pp. 3002-3006.

[13] Gregori, V., Miñana, J. J., Morillas, S. and Sapena, A., (2016), Characterizing a class of completable fuzzy metric spaces, Topology and its Applications, 203, pp. 3-11.

[14] Gregori, V. and Romaguera, S., (2000), Some properties of fuzzy metric spaces, Fuzzy Sets and Systems, 115, pp. 485-489.

[15] Güner, E. and Aygün, H., (2020), On 2-fuzzy metric spaces, Advances in Intelligent Systems and Computing, Springer, Cham, pp. 1258-1266.

[16] Güner, E. and Aygün, H., (2021), On b_2-metric spaces, Konuralp Journal of Mathematics, 9(1), pp. 33-39.

[17] Huang, H. and Wu, C., (2007), On the triangle inequalities in fuzzy metric spaces, Information Sciences, 177(4), pp. 1063-1072.

[18] Jleli, M., Karapınar, E. and Samet, B., (2014), On cyclic $(\psi,)$-contractions in Kaleva-Seikkala's type fuzzy metric spaces, Journal of Intelligent and Fuzzy Systems, 27(4), pp. 2045-2053.

[19] Kaleva, O., (1985), On the convergence of fuzzy sets, Fuzzy sets and Systems, 17(1), pp. 53-65.

[20] Kaleva, O., (1985), The completion of fuzzy metric spaces, Journal of Mathematical Analysis and Applications, 109(1), pp. 194-198.

[21] Kaleva, O, (2008), A comment on the completion of fuzzy metric spaces, Fuzzy Sets and Systems, 159(16), pp. 2190-2192.

[22] Kaleva, O. and Kauhanen, J., (2014), A note on compactness in a fuzzy metric space, Fuzzy Sets and Systems, 238, pp. 135-139.

[23] Kaleva, O. and Seikkala, S., (1984), On fuzzy metric spaces, Fuzzy Sets and Systems, 12(3), pp. 215-229.

[24] Kramosil, I. and Michálek, J., (1975), Fuzzy metrics and statistical metric spaces, Kybernetika, 11(5), pp. 336-344.

[25] Matthews, S., (1994), Partial metric topology, Annals of the New York Academy of Sciences, 728, pp. 1830-197.

[26] Miñana, J. J. and Valero, O., (2018), A duality relationship between fuzzy metrics and metrics, International Journal of General Systems, 47(6), pp. 593-612.

[27] Mizumoto, M. and Tanaka J., (1979), Some properties of fuzzy numbers, Advances in Fuzzy Set Theory and Applications, pp. 153-164.

[28] Mustafa, Z. and Sims, B., (2006), A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis, 7(2), pp. 289–297.

[29] Grigorenko, O., Miñana, J. J., Sostak, A. and Valero, O., (2020), On t-Conorm based fuzzy (pseudo)metrics, Axioms, 9(3), https://doi.org/10.3390/axioms9030078.
[30] Pedraza, T., Rodríguez-López, J. and Valero, O., (2020), Aggregation of fuzzy quasi-metrics, Information Sciences, https://doi.org/10.1016/j.ins.2020.08.045.
[31] Sharma, S., (2003), On fuzzy metric space, Southeast Asian Bulletin of Mathematics, 26(1), pp. 133-145.
[32] Xiao, J. Z., Zhu, X. H. and Jin, X., (2012), Fixed point theorems for nonlinear contractions in Kaleva–Seikkala's type fuzzy metric spaces, Fuzzy Sets and Systems, 200, pp. 65-83.
[33] Varol, B. P. and Aygün, H., (2012), Intuitionistic fuzzy metric groups, International Journal of Fuzzy Systems, 14(3), pp. 454-461.
[34] Zeng, W. Z., (1987), Probabilistic 2-metric spaces, J. Math. Res. Exposition, 7, pp. 241-245.

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