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# ON SOME NOVEL RESULTS FOR $\mathcal{C}$-WEAK-FUZZY CONTRACTIONS

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Abstract. This article implements the idea of introducing the concept of $\mathcal{C}$-weak-fuzzy contraction mapping in the framework of fuzzy cone metric spaces, and deriving some unique common fixed point results. A nontrivial example is enunciated to uphold our results. Our results unify, enrich and extend some pioneer results in the existing literature in the fuzzy setting. In the end, an application to Fredholm nonlinear integral equation is offered showing the superiority of our results, which is in turn supported by an example.

Keywords: Fuzzy cone metric spaces, fixed point, $\mathcal{C}$-weak-contraction, nonlinear integral equation.

AMS Subject Classification: 47H09, 47H10.

## 1. Introduction

Theory of contractive mappings is related to the area of metric fixed point theory. Kannan developed a series of results [18] states: the contractive mappings were not necessarily required to be continuous. A fixed point can be approximated by using the same convergence procedure without continuity of mapping, as by Banach [3]. This theory is utilized in many fields of study, including optimization theory, physics, chemistry, computer science, etc. (see e.g., [36, 37, 38]).

The notion of fuzzy metric space was introduced by Kramosil and Michalek [20]. In 2015, Oner et al. [24] introduced fuzzy cone metric space ($\mathcal{FCM}$) that extended the notion of fuzzy metric by George and Veeramani [8] and proved the corresponding theorem of fuzzy cone Banach contraction [25]. In 2009, Choudhury [7] provided a fixed point result for weak contraction mappings in cone metric spaces. In 2010, Vetro [35] extended the notion of $(\Phi, \Psi)$-weak contraction to fuzzy metric spaces and proved some common fixed

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point theorems for four self-mappings. Sintunavarat and Kumam [34] introduced the concept of $f$-contractions in 2012 and demonstrated a coincidence and common fixed point result for an $f$-weak contraction in cone metric space. In 2013, Razani et al. [28] obtained some fixed point results for weakly $T$-Chatterjea-contractive and generalized weakly $T$-Kannan-contractive mappings in the framework of complete metric spaces. In 2014, Gopal and Vetro [10] introduced the notions of $\alpha$-$\phi$ and $\beta$-$\psi$-fuzzy contractive mappings and proved some unique fixed point theorems in fuzzy metric spaces. A similar contractive condition introduced by Chatterjea [5] called $\mathcal{C}$-contraction. Shamoona et al. [15] proved some extended fixed point results on fuzzy cone Banach contractions with some weaker conditions.In 2021, Saleem et al. [22, 32] introduced fuzzy $(\alpha-\eta)$ and fuzzy $(\beta-\psi)-$generalized proximal contractions in complete $b$-fuzzy metric spaces and proved the existence of coincidence and best proximity point of such mappings.In the recent past, several authors proved various fixed point theorems employing more generalized conditions. For more results on contractive fixed point theory one can see; [4, 5, 13, 14, 17, 21, 22, 23].

In 2011, Harjani et al. [12] among others, studied weak contraction mappings with partially ordered metric spaces. In the present work our contraction condition is not restricted to pairs of points. Fuzzy metric theory has a significant influence on fuzzy integral equations. In the last few decades, fuzzy integration and fuzzy differential equations have motivated the researchers. In 2020, Mujahid et al. [1] defined the $\psi$-contraction and monotone $\psi$-contraction correspondence and obtained fixed point result in the fuzzy $b$-metric space. For more details about fuzzy metric spaces and fixed point theorems in these spaces, among a huge number of the papers regarding this topic, we suggest for reading the papers [26, 27, 31].

The above considerations became the motivation of present work. We generalize the notion of fuzzy cone metric spaces of George and Veeramani [8] by replacing $(0,+\infty)$ by $int(\mathcal{P})$, where $\mathcal{P}$ is a cone. We define *$\mathcal{C}$-weak-fuzzy* cone contraction conditions in the context of fuzzy cone metric spaces without a partial order. Our findings open up the scope of the latest research.

The paper is organized into four sections as follows. In Section 3, we present the unique fixed point results of $\mathcal{C}$-weak-fuzzy contractive type mappings in sense of $\mathcal{FCM}$-spaces. There is an illustrative example provided. In Section 4, we prove the existence of solution of Fredholm integral equation endorsed with an example. Finally, in Section 5, we discuss the conclusion and future work.

## 2. Preliminaries

For the sake of completeness, we recollect some basic definitions and elementary results. Let $U$ be a nonempty set and $T$ be a self-mapping on $U$. Throughout the paper we write fuzzy metric and fuzzy cone metric as $\mathcal{FM}$ and $\mathcal{FCM}$. The set of all fixed points of $T$ on $U$ is denoted by $Fix(T)_U$.

**Definition 2.1** ([33]). *An operation $*:[0,1]^2\to[0,1]$ is called continuous $\varsigma$-norm such that $\langle[0,1],*,1\rangle$ is a commutative totally ordered monoid, if the following axioms hold true:*
*($\varsigma$-1) $*$ is commutative, associative and continuous;*
*($\varsigma$-2) $1*a=a$ and $a*b\le c*d$, whenever, $a\le c$ and $b\le d$, for all $a,b,c,d\in[0,1]$.*

*There are three fundamental continuous $\varsigma$-norms: $\mathcal{T}_L,\mathcal{T}_P$ and $\mathcal{T}_M$ be defined as;*
*(i) The minimum operator $\mathcal{T}_M(a,b)=\min\{a,b\}$.*
*(ii) The product operator $\mathcal{T}_P(a,b)=ab$.*
*(iii) The Łukasiewicz's $\varsigma$-norm $\mathcal{T}_L(a,b)=\max\{a+b-1,0\}$.*

**Definition 2.2** ([8]). *Let $E$ be a real Banach space, a subset $\mathcal{P}$ of $E$ is called cone if:*

*(i)* $\mathcal{P} \neq \emptyset$, *closed and* $\mathcal{P} \neq \{\theta\}$, *where* $\theta$ *is the zero element of* $E$;
*(ii)* *if* $a, b \geq 0$ *and* $\mu, \nu \in \mathcal{P}$, *then* $a\mu + b\nu \in \mathcal{P}$;
*(iii)* *if both* $\mu, -\mu \in \mathcal{P}$, *then* $\mu = \theta$.

*If there is a number* $k > 0$, *the cone* $\mathcal{P}$ *is called normal, such that for all* $\mu, \nu \in E$.

$$0 \leq \mu \leq \nu \text{ yields } \|\mu\| \leq k\|\nu\|.$$

*The least positive number satisfies the above inequality is called the normal constant of* $\mathcal{P}$. *The cone* $\mathcal{P}$ *is regular if every increasing sequence bounded above is convergent. If* $\{\mu_n\}$ *is a sequence such that;*

$$\mu_1 \leq \mu_2 \leq \ldots\ldots \leq \mu_{\rm n} \leq \ldots\ldots \leq \nu,$$

*for some* $\nu \in E$, *then there is* $\mu \in E$ *such that* $\|\mu_n - \mu\| \to 0$ *as* $n \to \infty$.

The following definition of $\mathcal{FM}$ space was introduced by George and Veeramani [8].

**Definition 2.3** ([8])**.** *A 3-tuple* $(U, \mathcal{M}, *)$ *is called* $\mathcal{FM}$ *space, if* $U$ *is an arbitrary set,* $*$ *is continuous* $\varsigma$*-norm and* $\mathcal{M}$ *is a fuzzy set on* $U^2 \times (0, +\infty)$ *satisfies the following conditions: for all* $\mu, \nu, \omega \in U$ *and* $\hat{t}, \hat{s} > 0$,

*(Fm)-1* $\mathcal{M}(\mu, \nu, \hat{t}) > 0$;
*(Fm)-2* $\mathcal{M}(\mu, \nu, \hat{t}) = 1$ *iff* $\mu = \nu$;
*(Fm)-3* $\mathcal{M}(\mu, \nu, \hat{t}) = \mathcal{M}(\nu, \mu, \hat{t})$;
*(Fm)-4* $\mathcal{M}(\mu, \nu, \hat{t}) * \mathcal{M}(\nu, \omega, \hat{s}) \leq \mathcal{M}(\mu, \omega, \hat{t} + \hat{s})$;
*(Fm)-5* $\mathcal{M}(\mu, \nu, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ *is continuous.*

*We say that* $(\mathcal{M}, *)$, *or simply* $\mathcal{M}$, *is a fuzzy metric on* $U$. *Notice that condition* $(Fm) - 4$ *is a fuzzy version of the triangular inequality. The value* $\mathcal{M}(\mu, \nu, \hat{t})$ *is considered as the degree of nearness from* $\mu$ *to* $\nu$ *with respect to* $\hat{t}$.

**Definition 2.4.** *A fuzzy metric* $\mathcal{M}$ *on* $U$ *is said to be stationary if* $\mathcal{M}$ *does not depend on* $\hat{t}$, *i.e. if for each* $\mu, \nu \in U$, *the function* $\mathcal{M}_{\mu,\nu}(\hat{t}) = \mathcal{M}(\mu, \nu, \hat{t})$ *is constant.*

**Definition 2.5** ([11])**.** *A 3-tuple* $(U, \mathcal{M}_c, *)$ *is called* $\mathcal{FCM}$ *space, if* $\mathcal{P}$ *is a cone,* $U$ *is an arbitrary set,* $*$ *is continuous* $\varsigma$*-norm and* $\mathcal{M}$ *is a fuzzy set on* $U^2 \times int(\mathcal{P})$ *satisfies the following conditions: if for all* $\mu, \nu, \omega \in U$ *and* $\hat{t}, \hat{s} \in int(\mathcal{P})$,

*(Fcm)-1* $\mathcal{M}_c(\mu, \nu, \hat{t}) > 0$ *and* $\mathcal{M}_c(\mu, \nu, \hat{t}) = 1$ *if and only if* $\mu = \nu$;
*(Fcm)-2* $\mathcal{M}_c(\mu, \nu, \hat{t}) = \mathcal{M}_c(\nu, \mu, \hat{t})$;
*(Fcm)-3* $\mathcal{M}_c(\mu, \nu, \hat{t}) * \mathcal{M}_c(\nu, \omega, \hat{s}) \leq \mathcal{M}_c(\mu, \omega, \hat{t} + \hat{s})$;
*(Fcm)-4* $\mathcal{M}_c(\mu, \nu, .) : int(\mathcal{P}) \to [0, 1]$ *is continuous.*

*It is worth to note that* $0 < \mathcal{M}_c(\mu, \nu, \hat{t}) < 1$ *(for all* $\hat{t} > 0$*) provided* $\mu \neq \nu$. *If* $E = \Re$, $\mathcal{P} = [0, \infty)$, *and* $a * b = ab$, *then every* $\mathcal{FM}$ *space becomes* $\mathcal{FCM}$ *space.*

**Definition 2.6** ([24])**.** *(i) Let* $(U, \mathcal{M}_c, *)$ *be a* $\mathcal{FCM}$. *The sequence* $(\mu_j)$ *in* $U$ *converges to* $\mu \in U$ *if* $c \in (0, 1)$ *and* $\hat{t} \gg \theta$ *there exists* $j_1 \in \mathbf{N}$ *such that* $\mathcal{M}_c(\mu_j, \mu, \hat{t}) > 1 - c$, *for all* $j \geq j_1$. *We may write this* $\lim_{j \to \infty} \mu_j = \mu$ *or* $\mu_j \to \mu$ *as* $j \to \infty$.

*(ii)* *A sequence* $\{\mu_n\}$ *in* $U$ *is called Cauchy sequence if* $c \in (0, 1)$ *and* $\hat{t} \gg \theta$ *there exists* $j_1 \in \mathbf{N}$ *such that* $\mathcal{M}_c(\mu_j, \mu_k, \hat{t}) > 1 - c$, *for all* $j, k \geq j_1$.

*(iii)* *The* $\mathcal{FCM}$ *space* $(U, \mathcal{M}_c, *)$ *is called complete if every Cauchy sequence is convergent in* $U$.

*(iv)* *A sequence* $\{\mu_n\}$ *in* $U$ *called* $\mathcal{G}$*-Cauchy sequence iff* $\lim_{n\to\infty} \mathcal{M}_c\left(\mu_{n+p}, \mu_n, \hat{t}\right) = 1$ *for any* $p > 0$ *and* $\hat{t} > 0$.

*(v)* *The* $\mathcal{FM}$ *space* $(U, \mathcal{M}_c, *)$ *is called* $\mathcal{G}$*-complete if every* $\mathcal{G}$*-Cauchy sequence is convergent.*

*(vi) Let $(U, \mathcal{M}_c, *)$ be $\mathcal{FCM}$ space. Subset $A$ of $U$ is called fuzzy cone bounded if there exist $\hat{t} \gg \theta$ and $r \in (0,1)$ such that $\mathcal{M}_c(\mu, \nu, \hat{t}) \geq 1 - r$, if for all $\mu, \nu \in U$.*

**Definition 2.7.** [29] *Let $(U, \mathcal{M}_c, *)$ be a $\mathcal{FCM}$ space. The metric $\mathcal{M}_c$ is called triangular, if*

$$\frac{1}{\mathcal{M}_c(\mu, \omega, \hat{t})} - 1 \leq \left(\frac{1}{\mathcal{M}_c(\mu, \nu, \hat{t})} - 1\right) + \left(\frac{1}{\mathcal{M}_c(\nu, \omega, \hat{t})} - 1\right),$$

*for all $\mu, \nu, \omega \in U$ and $\hat{t} \gg \theta$.*

**Example 2.1** ([29])**.** *Let $(U, d)$ be a metric space. We define $a * b = ab$, for all $a, b \in [0,1]$,*

$$\mathcal{M}_c(\mu, \nu, \hat{t}) = \frac{\hat{t}}{\hat{t} + |\mu - \nu|}, \quad \text{for all } \mu, \nu \in U,\ \hat{t} > 0.$$

*Then $(U, \mathcal{M}_c, *)$ called standard $\mathcal{FM}$ space induced by $(U, d)$.*

**Definition 2.8** ([24])**.** *Let $(U, \mathcal{M}_c, *)$ be a $\mathcal{FCM}$ space. A mapping $T : U \to U$ is called fuzzy cone contractive, if there exists $a \in (0,1)$ such that*

$$\frac{1}{\mathcal{M}_c(T\mu, T\nu, \hat{t})} - 1 \leq a\left(\frac{1}{\mathcal{M}_c(\mu, \nu, \hat{t})} - 1\right), \quad \mu, \nu \in U,\ \hat{t} \gg \theta. \tag{1}$$

In 1972, the concept of $\mathcal{C}$-contraction was introduced by Chatterjea [5] as follows.

**Definition 2.9** ([7])**.** *A mapping $T : U \to U$ is called $\mathcal{C}$-contraction if there exists $0 < k < \frac{1}{2}$ such that for all $\mu, \nu \in U$,*

$$d(T\mu, T\nu) \leq k[d(\mu, T\nu) + d(\nu, T\mu)]. \tag{2}$$

**Definition 2.10** ([25])**.** *A function $\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function if $\psi(t)$ is monotone, nondecreasing, continuous and $\psi(t) = 0$ iff $t = 0$.*

Later, Choudhury [7] introduced the generalization of Chatterjea type construction as follows.

**Definition 2.11** ([7])**.** *A mapping $T : U \to U$ is called weakly contractive, if*

$$d(T\mu, T\nu) \leq d(\mu, \nu) - \psi(d(\mu, \nu)), \quad \mu, \nu \in U, \tag{3}$$

*where$\psi : [0, +\infty) \to [0, +\infty)$ is continuous and nondecreasing, $\psi(\mu) = 0$ iff $\mu = 0$ and $\lim_{\mu \to +\infty} \psi(\mu) = +\infty$.*

**Definition 2.12** ([7])**.** *A mapping $T : U \to U$ on a metric space $(U, d)$ is called $\mathcal{C}$-weak-contractive or a weak $\mathcal{C}$-contraction, if for all $\mu, \nu \in U$*

$$d(T\mu, T\nu) \leq \frac{1}{2}[d(\mu, T\nu) + d(\mu, T\nu)] - \psi(d(\mu, T\nu), d(\nu, T\mu)), \tag{4}$$

*where $\psi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous mapping such that $\psi(\mu, \nu) = 0$ iff $\mu = \nu = 0$. If we take $\psi(\mu, \nu) = k(\mu + \nu)$, where $0 < k < \frac{1}{2}$, then (4) reduces to (3) that is weak $\mathcal{C}$-contractions are generalizations of $\mathcal{C}$-contractions.*

**Definition 2.13** ([16])**.** *Let $U$ be a nonempty set and $S, T : U \to U$ be given mappings. The pair $(S, T)$ is said to be weakly compatible if $S$ and $T$ commute at their coincidence points (i.e. $ST\mu = TS\mu$ whenever $S\mu = T\mu$). A point $v \in U$ is called point of coincidence of $S$ and $T$ if there exists a point $u \in U$ such that $v = Su = Tu$.*

## 3. Fixed Point Results of $\mathcal{C}$-Weak-Fuzzy Contraction Mappings

**Definition 3.1.** *Let $(U, \mathcal{M}_c, *)$ be complete $\mathcal{FCM}$ space. A mapping $T : U \to U$ is called $\mathcal{C}$-weak-fuzzy contractive, if*

$$\begin{aligned}\frac{1}{\mathcal{M}_c(T\mu, T\nu, \hat{t})} - 1 \leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\mu, T\nu, \hat{t})} - 1 + \frac{1}{\mathcal{M}_c(\nu, T\mu, \hat{t})} - 1\right) \\ - \psi\left(\frac{1}{\mathcal{M}_c(\mu, T\nu, \hat{t})} - 1, \frac{1}{\mathcal{M}_c(\nu, T\mu, \hat{t})} - 1\right),\end{aligned} \tag{5}$$

*for all $\mu, \nu \in U$, $\hat{t} > 0$, and $\psi : [0, +\infty)\times[0, +\infty) \to [0, +\infty)$ is continuous and nondecreasing in each coordinate, $\psi(\mu, \nu) = 0$ iff $\mu = \nu$, and $\lim_{\mu\to+\infty} \psi(\mu, \nu) = k(\mu+\nu)$, $0 < k < \frac{1}{2}$.*

**Example 3.1.** *Let $(U, d)$ be a bounded metric space with $d(\mu, \nu) < \acute{k}$, for all $\mu, \nu \in U$, $g : \Re^+ \to (\acute{k}, +\infty)$ is an increasing continuous function. A function $\mathcal{M}_c$ be defined as:*

$$\mathcal{M}_c(\mu, \nu, \hat{t}) = 1 - \frac{d(\mu, \nu)}{g(\hat{t})}, \quad \textit{for all } \mu, \nu \in U,\ \hat{t} > 0.$$

*Then $(U, \mathcal{M}_c, *)$ is a $\mathcal{FCM}$ space in $U$, where $*$ is a $\mathcal{L}$ukasievicz $\varsigma$-norm, i.e., $\mathcal{T}_{\mathcal{L}}(a, b) = \max\{a + b - 1, 0\}$.*

Now, we provide that in a complete $\mathcal{FCM}$ space a *$\mathcal{C}$-weak-fuzzy contraction mapping* has a unique fixed point.

**Theorem 3.1.** *Let $T : U \to U$ be a $\mathcal{C}$-weak-fuzzy contraction mapping on a complete $\mathcal{FCM}$ space $(U, \mathcal{M}_c, *)$ in which $\mathcal{M}_c$ is triangular. Then $T$ has a unique fixed point.*

*Proof.* Fix $\mu_\circ \in U$, we construct a sequence $\mu_{n+1} = T(\mu_n) = T^n\mu_\circ$, $n \geq 1$. If $T\mu_n = \mu_{n+1} = \mu_n$, then $\mu_n$ is a fixed point of $T$ and the proof is complete.
Otherwise, assume that $\mu_{n+1} \neq \mu_n$, $n \in \mathbf{N}$. Putting $\mu = \mu_{n-1}$, $\nu = \mu_n$ in (5), we get

$$\begin{aligned}\frac{1}{\mathcal{M}_c\left(\mu_n, \mu_{n+1}, \hat{t}\right)} - 1 &= \frac{1}{\mathcal{M}_c\left(T\mu_{n-1}, T\mu_n, \hat{t}\right)} - 1 \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, T\mu_n, \hat{t}\right)} - 1 + \frac{1}{\mathcal{M}_c\left(\mu_n, T\mu_{n-1}, \hat{t}\right)} - 1\right) \\ &\quad - \psi\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, T\mu_n, \hat{t}\right)} - 1, \frac{1}{\mathcal{M}_c\left(\mu_n, T\mu_{n-1}, \hat{t}\right)} - 1\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_{n+1}, \hat{t}\right)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_{n+1}, \hat{t}\right)} - 1, 0\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_{n+1}, \hat{t}\right)} - 1\right), \quad \hat{t} \gg \theta.\end{aligned} \tag{6}$$

Applying triangular property of $\mathcal{M}_c$ on (6), we infer

$$\begin{aligned}\frac{1}{\mathcal{M}_c\left(\mu_n, \mu_{n+1}, \hat{t}\right)} - 1 &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_{n+1}, \hat{t}\right)} - 1\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_n, \hat{t}\right)} - 1\right) + \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_n, \mu_{n+1}, \hat{t}\right)} - 1\right).\end{aligned}$$

Let $\xi_n = \frac{1}{\mathcal{M}_c(\mu_n,\mu_{n+1},\hat{t})} - 1$, then above inequality be written as

$$\xi_n \le \xi_{n-1}, \quad \text{for all } n \ge 1.$$

Thus, $\{\xi_n\}$ is a monotone decreasing sequence of $\Re^+$, and hence it should be convergent at some point. Let that point is $\xi$, i.e., $\xi_n \to \xi$ as $n \to \infty$. We shall show that $\xi = 0$. From (6), we obtain

$$\begin{aligned}
\xi_n &= \frac{1}{\mathcal{M}_c\left(\mu_n, \mu_{n+1}, \hat{t}\right)} - 1 \\
&\le \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_{n+1}, \hat{t}\right)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_{n+1}, \hat{t}\right)} - 1, 0\right) \\
&\le \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_n, \hat{t}\right)} - 1 + \frac{1}{\mathcal{M}_c\left(\mu_n, \mu_{n+1}, \hat{t}\right)} - 1\right) \\
&\quad - \psi\left(\frac{1}{\mathcal{M}_c\left(\mu_{n-1}, \mu_n, \hat{t}\right)} - 1 + \frac{1}{\mathcal{M}_c\left(\mu_n, \mu_{n+1}, \hat{t}\right)} - 1, 0\right).
\end{aligned}$$

Applying limit $n \to \infty$ on above inequality for $\hat{t} > 0$, we get

$$\xi \le \frac{1}{2}(2\xi) - \psi(2\xi, 0) \quad \text{ or } \quad \psi(2\xi, 0) \le 0,$$

which is contradiction unless $\xi = 0$. Hence, $\xi_n \to 0$ as $n \to \infty$ and it is obvious

$$\lim_{n\to\infty} \mathcal{M}_c\left(\mu_n, \mu_{n+1}, \hat{t}\right) = 1, \quad \text{ for } \hat{t} \gg \theta. \tag{7}$$

To prove that $\{\mu_n\}$ is a $C$auchy sequence, if not, then there exists $0 < \wp < 1$ for which we can find subsequences $\{\mu_{m(i)}\}$ and $\{\mu_{n(i)}\}$ of $\{\mu_n\}$ with $n(i) > m(i) > i$, such that

$$\mathcal{M}_c(\mu_{m(i)}, \mu_{n(i)}, \hat{t}) \le 1 - \wp, \tag{8}$$

and

$$\mathcal{M}_c(\mu_{m(i)}, \mu_{n(i)-1}, \hat{t}) > 1 - \wp. \tag{9}$$

Then,

$$\begin{aligned}
&\frac{1}{\mathcal{M}_c(\mu_{m(i)}, \mu_{n(i)}, \acute{\epsilon})} - 1 = \frac{1}{\mathcal{M}_c\left(T\mu_{m(i)-1}, T\mu_{n(i)-1}, \acute{\epsilon}\right)} - 1 \\
&\le \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{m(i)-1}, T\mu_{n(i)-1}, \acute{\epsilon}\right)} - 1 + \frac{1}{\mathcal{M}_c\left(\mu_{n(i)-1}, T\mu_{m(i)-1}, \acute{\epsilon}\right)} - 1\right) \\
&\quad - \psi\left(\frac{1}{\mathcal{M}_c\left(\mu_{m(i)-1}, T\mu_{n(i-1)}, \acute{\epsilon}\right)} - 1, \frac{1}{\mathcal{M}_c\left(\mu_{n(i)-1}, T\mu_{m(i)-1}, \acute{\epsilon}\right)} - 1\right) \qquad (10) \\
&= \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu_{m(i)-1}, \mu_{n(i)}, \acute{\epsilon}\right)} - 1 + \frac{1}{\mathcal{M}_c\left(\mu_{n(i)-1}, \mu_{m(i)}, \acute{\epsilon}\right)} - 1\right) \\
&\quad - \psi\left(\frac{1}{\mathcal{M}_c\left(\mu_{m(i)-1}, \mu_{n(i)}, \acute{\epsilon}\right)} - 1, \frac{1}{\mathcal{M}_c\left(\mu_{n(i)-1}, \mu_{m(i)}, \acute{\epsilon}\right)} - 1\right).
\end{aligned}$$

By using (8) and (9), we obtain

$$1-\wp \geq \mathcal{M}_c(\mu_{m(i)},\mu_{n(i)},\acute{\epsilon}) \geq \mathcal{M}_c(\mu_{m(i)},\mu_{n(i)-1},\frac{\acute{\epsilon}}{2}) * \mathcal{M}_c(\mu_{n(i)-1},\mu_{n(i)},\frac{\acute{\epsilon}}{2})$$
$$> (1-\wp) * \mathcal{M}_c(\mu_{n(i)-1},\mu_{n(i)},\frac{\acute{\epsilon}}{2}). \quad (11)$$

Taking limit on (11) and using (7), we get

$$\lim_{i\to\infty} \mathcal{M}_c(\mu_{m(i)},\mu_{n(i)},\acute{\epsilon}) = 1-\wp. \quad (12)$$

Again for all $i \geq 1$,

$$\mathcal{M}_c(\mu_{m(i)},\mu_{n(i)},\acute{\epsilon}) \geq \mathcal{M}_c(\mu_{m(i)},\mu_{m(i)-1},\frac{\acute{\epsilon}}{2}) * \mathcal{M}_c(\mu_{m(i)-1},\mu_{n(i)},\frac{\acute{\epsilon}}{2}), \quad (13)$$

and

$$\mathcal{M}_c(\mu_{m(i)-1},\mu_{n(i)},\acute{\epsilon}) \geq \mathcal{M}_c(\mu_{m(i)-1},\mu_{m(i)},\frac{\acute{\epsilon}}{2}) * \mathcal{M}_c(\mu_{m(i)},\mu_{n(i)},\frac{\acute{\epsilon}}{2}). \quad (14)$$

Letting $i \to \infty$ in (13) and (14), and using (7), we have

$$\lim_{i\to\infty} \mathcal{M}_c(\mu_{m(i)-1},\mu_{n(i)},\acute{\epsilon}) = 1-\wp. \quad (15)$$

Similarly, one can easily show that

$$\lim_{i\to\infty} \mathcal{M}_c(\mu_{m(i)},\mu_{n(i)-1},\acute{\epsilon}) = 1-\wp. \quad (16)$$

Putting (12), (15) and (16) in the inequality (10), we get

$$\frac{1}{1-\wp} - 1 \leq \frac{1}{2}\left(\frac{1}{1-\wp} - 1 + \frac{1}{1-\wp} - 1\right)$$
$$- \psi\left(\frac{1}{1-\wp} - 1, \frac{1}{1-\wp} - 1\right),$$

and so

$$\frac{\wp}{1-\wp} \leq \left(\frac{2\wp}{2(1-\wp)}\right) - \psi\left(\frac{\wp}{1-\wp}, \frac{\wp}{1-\wp}\right).$$

This implies $\psi\left(\frac{\wp}{1-\wp}, \frac{\wp}{1-\wp}\right) \leq 0$, which is contradiction because $\frac{\wp}{1-\wp} > 0$. Hence $\{\mu_n\}$ is $C$auchy sequence in complete $\mathcal{FCM}$ space $U$. Then, there exists $\mu \in U$ such that $\mu_n \to \mu$ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \mathcal{M}_c\left(\mu_n, \mu, \hat{t}\right) = 1, \quad \text{for } \hat{t} \gg \theta.$$

Since $\mathcal{M}_c$ is triangular, from (5) for $\hat{t} \gg \theta$, we have

$$\begin{aligned}\frac{1}{\mathcal{M}_c\left(\mu, T\mu, \hat{t}\right)} - 1 &\leq \left(\frac{1}{\mathcal{M}_c\left(\mu, \mu_{n+1}, \hat{t}\right)} - 1\right) + \left(\frac{1}{\mathcal{M}_c\left(\mu_{n+1}, T\mu, \hat{t}\right)} - 1\right)\\
&= \left(\frac{1}{\mathcal{M}_c\left(\mu, \mu_{n+1}, \hat{t}\right)} - 1\right) + \left(\frac{1}{\mathcal{M}_c\left(T\mu_n, T\mu, \hat{t}\right)} - 1\right)\\
&\leq \left(\frac{1}{\mathcal{M}_c\left(\mu, \mu_{n+1}, \hat{t}\right)} - 1\right) + \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu, T\mu_n, \hat{t}\right)} - 1 + \frac{1}{\mathcal{M}_c\left(\mu_n, T\mu, \hat{t}\right)} - 1\right)\\
&\quad - \psi\left(\frac{1}{\mathcal{M}_c\left(\mu, T\mu_n, \hat{t}\right)} - 1, \frac{1}{\mathcal{M}_c\left(\mu_n, T\mu, \hat{t}\right)} - 1\right)\\
&= \left(\frac{1}{\mathcal{M}_c\left(\mu, \mu_{n+1}, \hat{t}\right)} - 1\right) + \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu, \mu_{n+1}, \hat{t}\right)} - 1 + \frac{1}{\mathcal{M}_c\left(\mu_n, T\mu, \hat{t}\right)} - 1\right)\\
&\quad - \psi\left(\frac{1}{\mathcal{M}_c\left(\mu, \mu_{n+1}, \hat{t}\right)} - 1, \frac{1}{\mathcal{M}_c\left(\mu_n, T\mu, \hat{t}\right)} - 1\right).\end{aligned}$$

Applying limit $n \to \infty$ on above inequality, we have

$$\begin{aligned}\frac{1}{\mathcal{M}_c\left(\mu, T\mu, \hat{t}\right)} - 1 &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu, T\mu, \hat{t}\right)} - 1\right) - \psi\left(0, \frac{1}{\mathcal{M}_c\left(\mu, T\mu, \hat{t}\right)} - 1\right)\\
&\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(\mu, T\mu, \hat{t}\right)} - 1\right),\end{aligned}$$

which is a contradiction unless $\psi(0,0) = 0$ and $\mathcal{M}_c\left(\mu, T\mu, \hat{t}\right) = 1$, i.e., $T\mu = \mu$.
If $\mu_1$ and $\mu_2$ be two fixed points of $T$, then

$$\begin{aligned}&\frac{1}{\mathcal{M}_c\left(\mu_1, \mu_2, \hat{t}\right)} - 1 = \frac{1}{\mathcal{M}_c\left(T\mu_1, T\mu_2, \hat{t}\right)} - 1\\
&\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\mu_1, T\mu_2, \hat{t})} - 1 + \frac{1}{\mathcal{M}_c(\mu_2, T\mu_1, \hat{t})} - 1\right)\\
&\quad - \psi\left(\frac{1}{\mathcal{M}_c(\mu_1, T\mu_2, \hat{t})} - 1, \frac{1}{\mathcal{M}_c(\mu_2, T\mu_1, \hat{t})} - 1\right)\\
&\leq \left(\frac{1}{\mathcal{M}_c\left(\mu_1, \mu_2, \hat{t}\right)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_c(\mu_1, \mu_2, \hat{t})} - 1, \frac{1}{\mathcal{M}_c(\mu_2, \mu_1, \hat{t})} - 1\right).\end{aligned}$$

From the definition of $\psi$, it is contradiction unless $\left(\frac{1}{\mathcal{M}_c(\mu_1,\mu_2,\hat{t})} - 1\right) = 0$ or $\mathcal{M}_c\left(\mu_1, \mu_2, \hat{t}\right) = 1$, that is $\mu_1 = \mu_2$. This completes the proof of uniqueness of the fixed point. □

**Example 3.2.** *Let $U = \Re$ and $*$ be a minimum norm. Define $f : U \to U$ by*

$$T\mu = \frac{1+\mu}{2}, \quad \text{for all } \mu \in U.$$

*Define fuzzy metric $\mathcal{M}_c$ as*

$$\mathcal{M}_c(\mu, \nu, \hat{t}) = \frac{\hat{t}}{\hat{t} + |\mu - \nu|}, \quad \text{for all } \mu, \nu \in U,\ \hat{t} > 0.$$

*Then, $(U, \mathcal{M}_c, *)$ be a complete fuzzy cone metric space. A function $\psi : [0, +\infty)^2 \to [0, +\infty)$ be defined as*

$$\psi(a, b) = \frac{a+b}{4}, \quad k < \frac{1}{2}$$

*We show that $T$ is a $\mathcal{C}$-weak-fuzzy contraction. For all $\mu, \nu \in U$,*

$$\left(\frac{1}{\mathcal{M}_c(T\mu, T\nu, \hat{t})} - 1\right) = \frac{|\mu - \nu|}{2\hat{t}} \geq \frac{1}{\hat{t}}\left(\frac{3\mu - 3\nu}{4}\right)$$
$$= \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\mu, T\nu, \hat{t})} - 1 + \frac{1}{\mathcal{M}_c(\nu, T\mu, \hat{t})} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_c(\mu, T\nu, \hat{t})} - 1, \frac{1}{\mathcal{M}_c(\nu, T\mu, \hat{t})} - 1\right).$$

*From Theorem 3.1, $T$ has a unique fixed point i.e., $\mu = 1$.*

**Theorem 3.2.** *Let $(U, \mathcal{M}_c, *)$ be $\mathcal{FCM}$ space in which $\mathcal{M}_c$ is triangular and $S, T : U \to U$ be two maps such that*

$$\begin{aligned}\frac{1}{\mathcal{M}_c(S\mu, S\nu, \hat{t})} - 1 \leq \frac{1}{2}&\left(\frac{1}{\mathcal{M}_c(T\mu, S\nu, \hat{t})} - 1 + \frac{1}{\mathcal{M}_c(T\nu, S\mu, \hat{t})} - 1\right) \\ &- \psi\left(\frac{1}{\mathcal{M}_c(T\mu, S\nu, \hat{t})} - 1, \frac{1}{\mathcal{M}_c(T\nu, S\mu, \hat{t})} - 1\right),\end{aligned} \tag{17}$$

*for all $\mu, \nu \in U$, $\hat{t} > 0$, and $\psi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous and nondecreasing in each coordinate, $\psi(\mu, \nu) = 0$ iff $\mu = \nu$, $\lim_{\mu \to +\infty} \psi(\mu, \nu) = k(\mu + \nu)$, $0 < k < \frac{1}{2}$. If $S(U) \subseteq T(U)$ and $T(U)$ is a complete subset of $U$, then $S$ and $T$ have a unique point of coincidence in $U$. Moreover, if $(S, T)$ is weakly compatible, then $S$ and $T$ have a unique common fixed point.*

*Proof.* Let $\mu_0 \in U$ be an arbitrary point. Since $S(U) \subseteq T(U)$, there exists a point $\mu_1 \in U$ such that $S\mu_0 = T\mu_1$. Continuing this process, we obtain a sequence $\{\mu_n\}$ in $U$ such that $\nu_n = S\mu_n = T\mu_{n+1}$.
Assume that $\nu_{n+1} \neq \nu_n$ for all $n \in \mathbb{N}$, otherwise, $T$ and $S$ have a coincidence point. Then, by (17) and triangular property of $\mathcal{M}_c$, we have

$$\begin{aligned}&\left(\frac{1}{\mathcal{M}_c(\nu_n, \nu_{n+1}, \hat{t})} - 1\right) = \left(\frac{1}{\mathcal{M}_c(S\mu_n, S\mu_{n+1}, \hat{t})} - 1\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(T\mu_n, S\mu_{n+1}, \hat{t})} - 1 + \frac{1}{\mathcal{M}_c(T\mu_{n+1}, S\mu_n, \hat{t})} - 1\right) \\ &\quad - \psi\left(\frac{1}{\mathcal{M}_c(T\mu_n, S\mu_{n+1}, \hat{t})} - 1, \frac{1}{\mathcal{M}_c(T\mu_{n+1}, S\mu_n, \hat{t})} - 1\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\nu_{n-1}, \nu_{n+1}, \hat{t})} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_c(\nu_{n-1}, \nu_{n+1}, \hat{t})} - 1, 0\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\nu_{n-1}, \nu_{n+1}, \hat{t})} - 1\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\nu_{n-1}, \nu_n, \hat{t})} - 1\right) + \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\nu_n, \nu_{n+1}, \hat{t})} - 1\right),\end{aligned} \tag{18}$$

and so

$$\frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\nu_n, \nu_{n+1}, \hat{t})} - 1\right) \leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\nu_{n-1}, \nu_n, \hat{t})} - 1\right).$$

This implies that $\mathcal{M}_c\left(\nu_n, \nu_{n+1}, \hat{t}\right) \geq \mathcal{M}_c\left(\nu_{n-1}, \nu_n, \hat{t}\right)$ for all $n \in \mathbb{N}$ and hence $\left\{\mathcal{M}_c\left(\nu_n, \nu_{n+1}, \hat{t}\right)\right\}$ is a nondecreasing sequence of positive real numbers in $(0, 1]$.

Let $\acute{U}(\hat{t}) = \lim_{n\to\infty} \mathcal{M}_c\left(\nu_n, \nu_{n+1}, \hat{t}\right)$, we show that $\acute{U}(\hat{t}) = 1$ for all $\hat{t} \gg \theta$. If not, there exists $\hat{t} \gg \theta$ such that $\acute{U}(\hat{t}) < 1$, then taking $n \to \infty$ on (18), we obtain

$$\left(\frac{1}{\acute{U}(\hat{t})} - 1\right) \leq \frac{1}{2}\left(\frac{1}{\acute{U}(\hat{t})} - 1\right) - \psi\left(\frac{1}{\acute{U}(\hat{t})} - 1, 0\right),$$

which is a contradiction. Therefore, $\mathcal{M}_c\left(\nu_n, \nu_{n+1}, \hat{t}\right) \to 1$ as $n \to \infty$. Now, for each positive integer $p$, we have

$$\mathcal{M}_c\left(\nu_n, \nu_{n+p}, \hat{t}\right) \geq \mathcal{M}_c\left(\nu_n, \nu_{n+1}, \frac{\hat{t}}{p}\right) * \mathcal{M}_c\left(\nu_{n+1}, \nu_{n+2}, \frac{\hat{t}}{p}\right) * \cdots * \mathcal{M}_c\left(\nu_{n+p-1}, \nu_{n+p}, \frac{\hat{t}}{p}\right)$$

follows that

$$\lim_{n\to\infty} \mathcal{M}_c\left(\nu_n, \nu_{n+p}, \hat{t}\right) \geq 1 * 1 * \cdots * 1 = 1.$$

Hence $\{\nu_n\}$ is a $C$auchy sequence. Since $T(U)$ is complete, then there exists $q \in T(U)$ such that $\nu_n \to q$ as $n \to \infty$.

Let $p \in U$ such that $Tp = q$. We shall show that $p$ is a coincidence point of $T$ and $S$. For every $\hat{t} \gg \theta$,

$$\begin{aligned}\left(\frac{1}{\mathcal{M}_c\left(Sp, T\mu_{n+1}, \hat{t}\right)} - 1\right) &= \left(\frac{1}{\mathcal{M}_c\left(Sp, S\mu_n, \hat{t}\right)} - 1\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(Tp, S\mu_n, \hat{t}\right)} - 1 + \frac{1}{\mathcal{M}_c\left(T\mu_n, Sp, \hat{t}\right)} - 1\right) \\ &- \psi\left(\frac{1}{\mathcal{M}_c\left(Tp, S\mu_n, \hat{t}\right)} - 1, \frac{1}{\mathcal{M}_c\left(T\mu_n, Sp, \hat{t}\right)} - 1\right).\end{aligned}$$

Taking $n \to \infty$ gives that

$$\begin{aligned}\frac{1}{\mathcal{M}_c\left(Sp, q, \hat{t}\right)} - 1 &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(Sp, q, \hat{t}\right)} - 1\right) - \psi\left(0, \frac{1}{\mathcal{M}_c\left(Sp, q, \hat{t}\right)} - 1\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_c\left(Sp, q, \hat{t}\right)} - 1\right),\end{aligned}$$

which is a contradiction unless $\psi(0,0) = 0$ and $\mathcal{M}_c\left(Sp, q, \hat{t}\right) = 1$, i.e., $Sp = q$. Therefore, $Tp = Sp = q$ and so $q$ is a point of coincidence for $S$ and $T$.

The uniqueness of the point of coincidence is a consequence of the condition (17), so we omit the details. By using weakly compatibility of $S$ and $T$, we obtain

$$Sq = STp = TSp = Tq \tag{19}$$

and so $Sq = Tq$. Uniqueness of the point of coincidence implies $q = Sq = Tq$. Consequently, $q$ is a unique common fixed point of $S$ and $T$. □

## 4. Application to Fredholm Integral Equations

In this section, we prove the existence of unique solution of the following Fredholm integral equation

$$\omega(t) = \int_a^b \c{K}(t, s) h(s, \omega(s)) ds + g(t), \tag{20}$$

for all $t \in [a,b]$. Let $U = C([a,b],\Re)$ be the space of all $\Re$-valued continuous functions on the interval $[a,b]$. Observe that $U$ is a complete metric space with respect to sup-metric

$$d(\omega,\nu) = \sup_{t\in[a,b]} |\omega(t)-\nu(t)|.$$

Also, the space $(U,\mathcal{M}_c,*)$ with

$$\mathcal{M}_c(\omega,\nu,\hat{t}) = \frac{\hat{t}}{\hat{t}+d(\omega,\nu)},$$

for all $\omega,\nu \in U$ and $\hat{t}>0$ and taking $p*q=pq,$ for all $p,q\in[0,1]$, is a complete $\mathcal{FCM}$ space in which $\mathcal{M}_c$ is triangular.

**Theorem 4.1.** *Let $T: U \to U$ be an integral operator defined by*

$$T(\omega(t)) = \int_a^b Ķ(t,s)h(s,\omega(s))ds + g(t), \quad \textit{for all } t\in[a,b],$$

*where $g \in C([a,b],\Re)$, $Ķ \in C([a,b]\times[a,b],\Re)$, and $h \in C([a,b]\times\Re,\Re)$. If there exists $\gamma>0$ such that $\gamma \sup_{t\in[a,b]} \int_a^b |Ķ(t,s)|ds = \gamma_1,\ \gamma_1\in[0,1)$, and for all $\omega,\nu\in U$, we get*

$$|h(s,\omega(s)) - h(s,\nu(s))| \le \gamma \left|\omega(s)-\nu(s)\right|,$$

*Then, the integral equation* (20) *has a unique solution in $C([a,b],\Re)$.*

*Proof.* Let $\omega,\nu \in U$ and consider

$$\begin{aligned}|T(\omega(t)) - T(\nu(t))| &\le \int_a^b |Ķ(t,s)(h(s,\omega(s)) - h(s,\nu(s)))|ds\\ &\le \int_a^b |Ķ(t,s)||h(s,\omega(s)) - h(s,\nu(s))|ds\\ &\le \gamma\int_a^b |Ķ(t,s)|\left|\omega(s)-\nu(s)\right| ds\\ &\le \gamma d(\omega,\nu)\sup_{t\in[a,b]}\int_a^b |Ķ(t,s)|ds,\end{aligned}$$

which follows that

$$\begin{aligned}d(T\omega,T\nu) &\le \gamma_1 d(\omega,\nu)\\ &\le \gamma_1[d(\omega,T\nu)+d(T\nu,T\omega)+d(T\omega,\nu)]\end{aligned}$$

and so

$$d(T\omega,T\nu) \le \frac{\gamma_1}{1-\gamma_1}[d(\omega,T\nu)+d(\nu,T\omega)].$$

Define $\psi: [0,+\infty)^2 \to [0,+\infty)$ as $\psi(a,b) = (1-2\Lambda)\left(\frac{a+b}{2}\right)$, where $\Lambda = \frac{\gamma_1}{1-\gamma_1}$ and so $\psi\in\Psi$. Then, the above inequality can be rewritten as

$$d(T\omega,T\nu) \le \frac{1}{2}[d(\omega,T\nu)+d(\nu,T\omega)] - \psi(d(\omega,T\nu),d(\nu,T\omega)),$$

and hence

$$\begin{aligned}\frac{1}{\mathcal{M}_c(T\omega,T\nu,\hat{t})}-1&=\frac{d(T\omega,T\nu)}{\hat{t}}\\&\leq\frac{\frac{1}{2}[d(\omega,T\nu)+d(\nu,T\omega)]-\psi(d(\omega,T\nu),d(\nu,T\omega))}{\hat{t}}\\&=\frac{1}{2}\left(\frac{1}{\mathcal{M}_c(\omega,T\nu,\hat{t})}-1+\frac{1}{\mathcal{M}_c(\nu,T\omega,\hat{t})}-1\right)\\&\quad-\psi\left(\frac{1}{\mathcal{M}_c(\omega,T\nu,\hat{t})}-1,\frac{1}{\mathcal{M}_c(\nu,T\omega,\hat{t})}-1\right),\end{aligned}$$

for all $\omega,\nu\in U,\hat{t}>0$. Therefore, all the conditions of Theorem 3.1 are satisfied and so the integral equation (20) has a unique solution in $C([a,b],\Re)$. □

**Example 4.1.** *Solve the Fredholm Integral Equation*

$$f(\mu)=1+\int_0^\alpha 2\mu f(\nu)d\nu,$$

*for* $0<\alpha<1$. *Note that, now*

$$\sup_{a\leq\mu\leq b}\int_a^b|\c{K}(\mu,\nu)|d\nu=\sup_{0\leq\mu\leq\alpha}\int_0^\alpha 2\mu d\nu=2\mu\alpha<1.$$

*Furthermore,* $g(\mu)=1$ *and* $\c{K}(\mu,\nu)=2\mu$ *are continuous functions on* $[0,1]$. *The conditions of Theorem 3.1 are fullfilled. So, we may start with any* $f_0\in\mathcal{C}([0,1])$ *and repeatedly apply* $T$ *where* $Tf(\mu)=1+\int_0^\alpha 2\mu f(\nu)d\nu$. *Let* $f_0(\mu)=1$. *Then,*

$$f_1(\mu)=Tf_0(\mu)=1+\int_0^\alpha 2\mu f_0(\nu)d\nu=1+\int_0^\alpha 2\mu d\nu=1+2\alpha\mu,$$

*and*

$$\begin{aligned}f_2(\mu)=Tf_1(\mu)&=1+\int_0^\alpha 2\mu f_1(\nu)d\nu=1+\int_0^\alpha 2\mu(1+2\alpha\nu)d\nu\\&=1+2\mu\left[\alpha+\alpha^3\right],\end{aligned}$$

*and*

$$\begin{aligned}f_3(\mu)=Tf_2(\mu)&=1+\int_0^\alpha 2\mu f_2(\nu)d\nu=1+\int_0^\alpha 2\mu\left[1+(2\nu)\left(\alpha+\alpha^3\right)\right]d\nu\\&=1+2\mu\left[\alpha+\alpha^3+\alpha^5\right].\end{aligned}$$

*Continuing, we get*

$$f_n(\mu)=1+2\mu\left[\alpha+\alpha^3+\alpha^5+\cdots+\alpha^{2^{n+1}}\right].$$

*Therefore,*

$$f(\mu)=\lim_{n\to\infty}f_n(\mu)=1+2\mu\sum_{n=0}^{\infty}\alpha^{2n+1}$$

*It is easy to check that this sum is convergent for* $\alpha\in(0,1)$.

## 5. Conclusion

In the present work, we have introduced a new notion called *$\mathcal{C}$-weak-fuzzy contraction mapping*, which is a generalization of fuzzy cone contraction in the sense of Gregori and Sapena. With the help of *$\mathcal{C}$-weak-fuzzy contraction* conditions, we have proved some new results and common fixed point theorems in fuzzy cone metric spaces. Moreover, we provided some examples and applications in support of the introduced new concepts and presented results.

Fuzzy fixed point theory continues to grow in most branches of mathematics. Some new theorems are proven and new mathematical methods are developed to generalize known results in some situations. We would like to discuss the following key questions for future studies.

- Is it possible to develop theories or methods to generalize new types of mappings, such as hybrid mappings or general spaces (eg., [36]) to the current fixed point theorems?
- Can the results derived in this article be utilized to solve some nonlinear problems arising from engineering and science, for example the practical applications discussed in [37, 38].

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