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PARALLEL CRITICAL GRAPHS

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ABSTRACT. Let G_1 and G_2 be two undirected graphs. Let u_1, $v_1 \in V(G_1)$ and u_2, $v_2 \in V(G_2)$. A parallel composition forms a new graph H that combines G_1 and G_2 by contracting the vertices u_1 with u_2 and v_1 with v_2. A new kind of graph called a parallel critical graph is introduced in this paper. We present the critical property using the domination number of G_1 and G_2 and provide a necessary and sufficient condition for parallel critical graphs. Few results relating to some class of graphs and parallel composition are discussed in this paper.

Keywords: Critical graph, domination number, graph operation, parallel composition graph, parallel critical graph.

AMS Subject Classification: 05C69.

1. INTRODUCTION

Graph operation is one of the most fascinating topics among the researcher in graph theory. The series-parallel composition is one of the binary operations in graph theory. Every branch of a network in series-parallel connection has characterized as a series-parallel network. It is tough to find the current flow if the resistors have a non - linear characteristic. In 1965, R. J. Duffin had provided the results for the network which has the series-parallel topology [2]. Takamizawa et al. presented a new method for series-parallel (SP) graphs. They had constructed the linear time algorithms for the same and provided the problems including the minimum vertex cover, minimum path cover, etc [7].

In this section, we present few results relating binary operations and domination number. Whenever it comes to binary operations, perhaps the most classic conjecture of graph theory is Vizing's conjecture. Vizing's conjecture concerns a relation between the domination number and the cartesian product of graphs [8]. Gravier and Khelladi had provided the domination number of tensor products of graphs [3]. In 2018, M. Yamuna et al. had provided the results on Hajos stable graphs [9]. In 1983, T. Kikuno et al. had provided a linear time algorithm for finding a minimum dominating set in a series-parallel graphs [5].

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Series-parallel composition plays a vital role in binary operations. We consider the parallel composition between any two connected graphs say G_1 and G_2. Using the domination number of G_1 and G_2, we characterize the parallel composition critical graphs in this paper.

2. Materials and Methods

We consider only simple connected undirected graphs $G = (V, E)$. The open neighborhood of vertex $v \in V(G)$ is denoted by $N(v) = \{ u \in V(G) \mid (u\ v) \in E(G) \}$ while it is closed neighborhood is the set $N[v] = N(v) \cup \{ v \}$. If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line and the graph is said to be an Euler graph. A Hamiltonian path of G is a path passing through every vertex of G. A Hamiltonian cycle is a closed Hamiltonian path. If a graph G has a Hamiltonian cycle, then G is called a Hamiltonian graph.

The vertex identification of a pair of vertices v_1 and v_2 of a graph produces a graph in which the vertices v_1 and v_2 are replaced with a single vertex v such that v is adjacent to the union of the vertices to which v_1 and v_2 were originally adjacents. An edge contraction is an operation which removes an edge from a graph while simultaneously contracting the two vertices that it was previously joined. For details on graph theory, we refer to [6].

A set of vertices D, in a graph $G = (V, E)$ is a dominating set if every vertex of $V - D$ is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set. The cardinality of any minimum dominating set for G is called the domination number of G and it is denoted by $\gamma(G)$. γ - set denotes a dominating set for G with minimum cardinality. A vertex v is said to be selfish in the γ - set D, if v is needed only to dominate itself. The private neighborhood of $v \in D$, denoted by $pn(v, D)$, is defined by $pn(v, D) = N(v) - N(D - \{ v \})$. A vertex in $V - D$ is 2 - dominated if it is dominated by at least 2 - vertices in D. For details on domination, we refer to [4].

Let G_1 and G_2 be two undirected graphs. Let $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$. A parallel composition forms a new graph H that combines the two graphs by contracting the vertices u_1 with u_2 and v_1 with v_2 [2]. The contracted vertices (u_1, u_2) and (v_1, v_2), denoted by u_{12} and v_{12} respectively as seen in Figure 1. Since we consider only simple graphs, we omit the parallel edges and self loops in a parallel composition graph H.

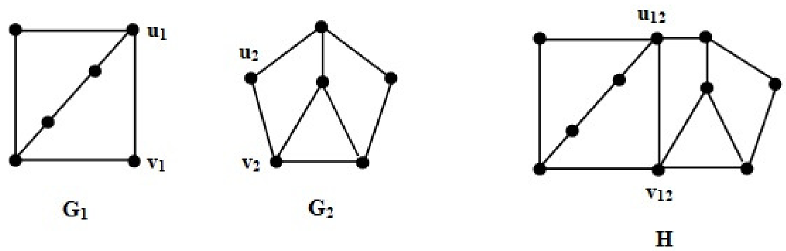

Figure 1. Parallel Composition

3. Results and Discussions

We define a parallel critical graph and provide a necessary and sufficient condition of parallel critical graphs in this section. Also, we discuss few results relating to some class of graphs with parallel composition.

Let G_1 and G_2 be any two connected graphs. In the process of identifying parallel critical graphs, we have to apply the parallel composition between every possible pair of vertices in G_1 and G_2.

(1) For any graph G with n vertices, the total number of unordered pair of vertices are n (n - 1). Let n_1 = | V(G_1) | and n_2 = | V(G_2) |. For our discussion the number of possible pair of vertices of G_i are n_i (n_i - 1), i = 1, 2.
(2) Let us construct the parallel composition graphs using G_1 and G_2, such graphs will be labelled as H_1, H_2, $\cdots$, H_k, $k = \prod_{i=1}^{2} n_i(n_i - 1)$.
(3) For example, we consider two undirected graphs G_1 and G_2 with $n_1 = 3$ and $n_2 = 4$. According to the above discussion, we have H_1, H_2, $\cdots$, H_k are the parallel composition graphs for G_1 and G_2, where k = (3 × 2) × (4 × 3) = 72.

Definition 3.1. *Two graphs G_1 and G_2 are said to be parallel critical graphs if $\gamma(H_i) < \gamma(G_1) + \gamma(G_2)$, i = 1, 2, $\cdots$, k.*

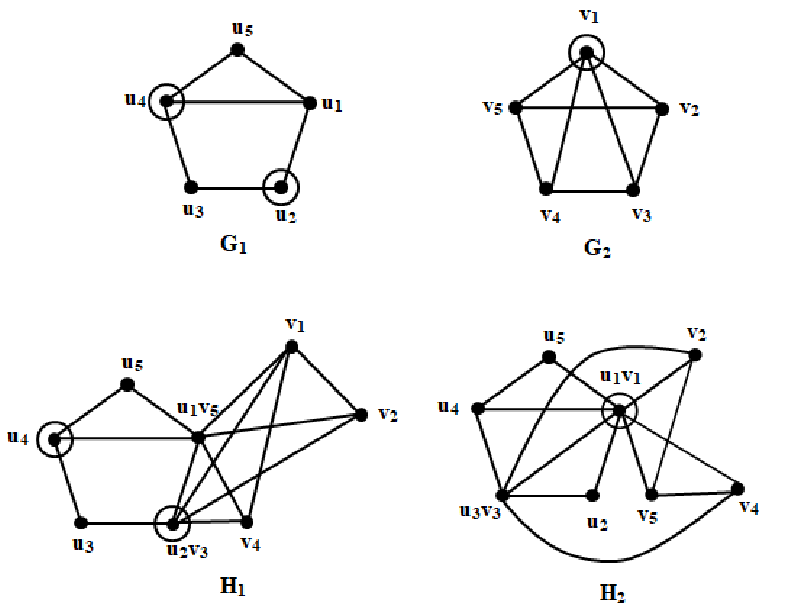

FIGURE 2. $\gamma(G_1) = 2$, $\gamma(G_2) = 1$ and $\gamma(H_1) = 2$ and $\gamma(H_2) = 1$. Note that $\gamma(H_1) = \gamma(G_1) + \gamma(G_2)$ - 1 and $\gamma(H_2) = \gamma(G_1) + (G_2)$ - 2. In general, $\gamma(H_i) < \gamma(G_1) + \gamma(G_2)$, where i = 1, 2, $\cdots$, k. implies G_1 and G_2 are said to be parallel critical graphs.

Throughout the discussion, we consider the following.

- Let H be the parallel composition graph by combining the two graphs by contracting the vertices u_1 with u_2 and v_1 with v_2. The contracted vertices (u_1, u_2) and (v_1, v_2), denoted by u_{12} and v_{12} respectively.
- We use H, instead of writing H_i for notation convenient.
- Let D_1, D_2 and D be γ - sets for G_1, G_2 and H respectively.
- Partitioning D into subsets X and Y such that $D = X \cup Y$, where $X \in V (G_1)$ and $Y \in V (G_2)$. If $u_{12} \in D$, then either u_1 or u_2 is considered in X or Y. Similar condition holds for v_{12} also.

- Splitting a graph H into two components H_{11} and H_{12}. This means that, while spliting a graph H, V (H_{1i}) = V (G_i) ∩ V (H) ∪ u_i and E (H_{1i}) = E (G_i) ∩ E (H), where i = 1, 2.

As we know that the domination number will not increase while contracting the two vertices in G. The domination number of H will retain the same or will decrease by either 1 or 2, will discuss in detail in Theorem 3.1.

Theorem 3.1. *Let G_1 and G_2 be any two graphs. Let H be the parallel composition graph. If $\gamma(H) < \gamma(G_1) + \gamma(G_2)$, then $\gamma(H) = \gamma(G_1) + \gamma(G_2) - k$, where k = 1 or 2.*

Proof. Let G_1 and G_2 be the two graphs. The discussion is true for all possible pairs of vertices in G_1 and G_2. Consider an arbitrary pair of vertices (u_1, v_1) ∈ V(G_1) and (u_2, v_2) ∈ V(G_2) and construct the parallel composition graph H. Assume that $\gamma(H) < \gamma(G_1) + (G_2)$. If possible assume that $\gamma(H) = \gamma(G_1) + \gamma(G_2) - k, k > 2$.

Consider any γ - set D of H. Splitting a graph H by H_{11} and H_{12}. Let X and Y be γ - sets for H_{11} and H_{12} respectively. Since $\gamma(H) < \gamma(G_1) + \gamma(G_2)$, either $| X | < | D_1 |$ or $| Y | < | D_2 |$. Suppose that X is a dominating set for G_1 if $| X | < | D_1 |$, which is a contradiction to our assumption that D_1 is a γ - set for G_1. Similarly, we get a contradiction when $| Y | < | D_2 |$. So, if $| X | < | D_1 |$ or $| Y | < | D_2 |$, then X, Y can not be a γ - set for G_1, G_2 respectively.

While splitting a graph H, if $u_{12} \in D$, then either $u_1 \in X$ or $u_2 \in Y$. So, we focus mainly on the following cases which are relating u_{12} and v_{12} with D and V - D.

(1) $u_{12}, v_{12} \in V - D$.
(2) $u_{12} \in D$ and $v_{12} \in V - D$.
(3) $u_{12} \in V - D$ and $v_{12} \in D$.
(4) $u_{12}, v_{12} \in D$.

Case 1. $u_{12}, v_{12} \in V - D$

Assume that u_{12}, v_{12} are dominated by some x or { x, y }, where x, y ∈ D. We have the following subcases.

1. x ∈ V(G_i), or
2. (x, y) ∈ V(G_i), or
3. x ∈ V(G_i), y ∈ V(G_j)

where i, j = 1, 2 and i ≠ j.

Subcase 1 x ∈ V (G_i)

Let x ∈ X, since x ∈ D. u_{12}, v_{12} dominated by x in H.

- Consider $| X | < | D_1 |$. Since u_{12}, v_{12} dominated by x in H, x dominates u_1, v_1 in G_1, implies X is a dominating set for G_1, which is a contradiction to our assumption that D_1 is a γ - set for G_1. Therefore $| X | = | D_1 |$.
- So, it is clear that $|Y | < | D_2 |$. Y dominates atleast G_2 - { u_2, v_2 }. This implies, the domination number of G_2 may increase atmost by two, that is $D_3 = Y \cup \{ u_2 \} \cup \{ v_2 \}$ is a dominating set for G_2 and $| Y | \leq | D_3 | - 2$.

So, $\gamma(H) = | X | + | Y | \leq | D_1 | + | D_3 | - 2 = \gamma(G_1) + \gamma(G_2) - k$, where k = 2, a contradiction to our assumption that k > 2.

Similarly, we get a contradiction when x ∈ Y.

Subcase 2 x, y ∈ V (G_i)

The proof is similar to Subcase 1.

Subcase 3 x ∈ V(G_1) and y ∈ V(G_2).

Let x ∈ X, y ∈ Y. u_{12}, v_{12} dominated by x, y respectively. Assume that u_1 dominated by x in G_1 and v_2 dominated by y in G_2.

- If $| X | < | D_1 |$ and $| Y | = | D_2 |$, then $D_3 = X \cup \{ v_1 \}$ is a dominating set for G_1, implies $| X | \leq | D_3 | - 1$. So, $\gamma(H) = | X | + | Y | \leq | D_2 | + | D_3 | - 1 = \gamma(G_1) + (G_2) - k$, where $k = 1$.
- If $| X | = | D_1 |$ and $| Y | < | D_2 |$, then $\gamma(H) \leq \gamma(G_1) + (G_2) - k$, where $k = 1$ (proof is similar to the above discussion).
- If $| X | < | D_1 |$ and $| Y | < | D_2 |$, then $D_3 = X \cup \{ v_1 \}$, $D_4 = Y \cup \{ u_2 \}$ are dominating sets for G_1 and G_2 respectively such that $| X | \leq | D_3 | - 1$, $| Y | \leq | D_4 | - 1$. So, $\gamma(H) = | X | + | Y | \leq | D_3 | + | D_4 | - 2 = \gamma(G_1) + \gamma(G_2) - k$, where $k = 2$ (since x, y dominates u_{12}, v_{12} respectively.).

In all cases, we get a contradiction to our assumption that $k > 2$.
Case 2 $u_{12} \in D$ and $v_{12} \in V - D$
Consider $u_1 \in X$ or $u_2 \in Y$.
Let $u_1 \in X$. Since $v_{12} \in V - D$, there is some y dominates v_{12} in H. We have the following subcases.

1. $y \in V(G_1)$ (y may be u_1 also), or
2. $y \in V(G_2)$.

Subcase 1 $y \in V(G_1)$.
Assume that $y \neq u_1 \in V(G_1)$

- Consider $| X | < | D_1 |$. X is a dominating set for G_1, which is a contradiction to our assumption that D_1 is a γ - set for G_1. Therefore $| X | = | D_1 |$.
- So, it is clear that $|Y | < | D_2 |$. Y dominates atleast $G_2 - N [u_2] - v_2$, implies the domination number of G_2 may increase atmost by two, that is $D_3 = Y \cup \{ u_2 \} \cup \{ v_2 \}$ is a dominating set for G_2 and $| Y | | \leq D_3 | - 2$.

So, $\gamma(H) = | X | + | Y | \leq | D_1 | + | D_3 | - 2 = \gamma(G_1) + \gamma(G_2) - k$, where $k = 2$, a contradiction to our assumption that $k > 2$.

A similar discussion will be true, when $y = u_1 \in V(G_1)$ also.
Subcase 2 $y \in V(G_2)$

- If $| X | < | D_1 |$ and $| Y | = | D_2 |$, then X dominates atleast $G - \{ v_1 \}$, that is $D_3 = X \cup \{ v_1 \}$ is a dominating set for G_1 such that $| X | \leq | D_3 | - 1$. So, $\gamma(H) = | X | + | D_2 | \leq | D_2 | + | D_3 | - 1 = \gamma(G_1) + (G_2) - k$, where $k = 1$.
- If $| X | = | D_1 |$ and $| Y | < | D_2 |$, then Y dominates atleast $G - \{ u_2 \}$, that is $D_4 = Y \cup \{ u_2 \}$ is a dominating set for G_2 such that $| Y | \leq | D_4 | - 1$. So, $\gamma(H) = | D_1 | + | Y | \leq | D_1 | + | D_4 | - 1 = \gamma(G_1) + (G_2) - k$, where $k = 1$.
- If $| X | < | D_1 |$ and $| Y | < | D_2 |$, then using the above discussion $\gamma(H) = | X | + | Y | \leq | D_3 | + | D_4 | - 2 = \gamma(G_1) + \gamma(G_2) - k$, where $k = 2$.

In all cases, we get a contradiction to our assumption that $k > 2$. Similarly, we get a contradiction when $u_2 \in Y$.
Case 3 $u_{12} \in V - D$, $v_{12} \in D$
The proof is similar to Case 2.
Case 4 u_{12}, $v_{12} \in D$
We have the following subcases.

1. u_1, $v_1 \in X$, or
2. u_2, $v_2 \in Y$, or
3. $u_1 \in X$ and $v_2 \in Y$, or
4. $u_2 \in Y$ and $v_1 \in X$.

Subcase 1 u_1, $v_1 \in X$

- Consider | X | < | D_1 |. X is a dominating set for G_1, which is a contradiction to our assumption that D_1 is a γ - set for G_1. Therefore | X | = | D_1 |.
- So, it is clear that |Y | < | D_2 |. Y dominates atleast G_2 - N [u_2] - N[v_2], implies the domination number of G_2 may increase atmost by two, that is D_3 = Y ∪ { u_2 } ∪ { v_2 } is a dominating set for G_2 and | Y | | ≤ D_3 | - 2.

So, γ(H) = | X | + | Y | ≤ | D_1 | + | D_3 | - 2 = γ(G_1) + γ(G_2) - k, where k = 2, a contradiction to our assumption that k > 2.

Subcase 2 u_2, v_2 ∈ Y
The proof is similar to subcase 1 of Case 4.

Subcase 3 u_1 ∈ X and v_2 ∈ Y
The proof is similar to subcase 2 of case 2.

Subcase 4 u_2 ∈ Y and v_1 ∈ X
The proof is similar to subcase 3 of Case 4.

We get a contradiction in all possible cases. So, we conclude that if γ(H) < γ(G_1) + γ(G_2), then γ(H) = γ(G_1) + γ(G_2) - k, k = 1 or 2. □

We provide a necessary and sufficient condition of parallel critical graphs in Theorem 3.2.

Theorem 3.2. *Let G_1 and G_2 be any two connected graphs. Let D_1 and D_2 be γ - sets for G_1 and G_2. Let H be the parallel composition graph and D be a γ - set for H. γ(H) < γ(G_1) + γ(G_2) if and only if either*

(1) *$u_i \in D_i$ or $v_i \in D_i$, or*
(2) *there is a selfish vertex in G_i, or*
(3) *$\gamma(G_i - \{u_i, v_i\}) < \gamma(G_i)$, or*
(4) *$\gamma(G_i - N [u_i]) < \gamma(G_i)$ and $u_j \in D_j$, or $\gamma(G_i - N [u_i] - v_i) < \gamma(G_i)$ and $u_j \in D_j$, or*
(5) *$\gamma(G_i - N [v_i]) < \gamma(G_i)$ and $v_j \in D_j$, or $\gamma(G_i - u_i - N [v_i]) < \gamma(G_i)$ and $v_j \in D_j$ and u_j, v_j, or*
(6) *$\gamma(G_i - N [u_i] - N [v_i]) < \gamma(G_i)$ and $u_j, v_j \in D_j$.*

where i, j = 1, 2, i ≠ j.

Proof. Assume that γ(H) < γ(G_1) + γ(G_2). Let D be a γ - set for H. Split the parallel graph H into H_{11} and H_{12}. Let X and Y be γ - sets for H_{11} and H_{12} respectively. If possible assume that conditions 1 - 6 are not satisfied.
As discussed in Theorem 3.1, we have the following cases.

(1) u_{12}, v_{12} ∈ V - D.
(2) u_{12} ∈ D and v_{12} ∈ V - D.
(3) u_{12} ∈ V - D and v_{12} ∈ D.
(4) u_{12}, v_{12} ∈ D.

Case 1 u_{12}, v_{12} ∈ V - D
Assume that u_{12}, v_{12} are dominated by some x or { x, y }, where x, y ∈ D. We have the following subcases.

1. x ∈ V(G_i), or
2. (x, y) ∈ V(G_i), or
3. x ∈ V(G_i), y ∈ V(G_j),

where i, j = 1, 2 and i ≠ j.

Subcase 1 x ∈ V (G_i)
Let x ∈ X, since x ∈ D. u_{12}, v_{12} dominated by x in H.

- Consider | X | < | D_1 |. Since u_{12}, v_{12} dominated by x in H, x dominates u_1, v_1 in G_1, implies X is a dominating set for G_1. Which is a contradiction to our assumption that D_1 is a γ - set for G_1. Therefore | X | = | D_1 |.
- So, it is clear that |Y | < | D_2 |. Y dominates atleast G_2 - { u_2, v_2 }, a contradiction to our assumption that Condition 3 is not satisfied.

Subcase 2 x, y ∈ V (G_i)
The proof is similar to Subcase 1.
Subcase 3 x ∈ V (G_1) and y ∈ V (G_2).
Let x ∈ X, y ∈ Y. u_{12}, v_{12} dominated by x, y respectively. Assume that u_1 dominated by x in G_1 and v_2 dominated by y in G_2.

- If | X | < | D_1 | and | Y | = | D_2 |, then D_3 = X ∪ { v_1 } is a dominating set for G_1.
- If | X | = | D_1 | and | Y | < | D_2 |, then D_4 = Y ∪ { u_2 } is a dominating set for G_2.
- If | X | < | D_1 | and | Y | < | D_2 |, then D_3 and D_4 is a dominating set for G_1 and G_2 respectively.

In all cases, we get a contradiction to our assumption that Condition 2 is not satisfied.
Case 2 u_{12} ∈ D and v_{12} ∈ V - D
Consider u_1 ∈ X or u_2 ∈ Y.
Let u_1 ∈ X. Since v_{12} ∈ V - D, there is some y dominates v_{12} in H. We have the following subcases.

1. y ∈ V(G_1) (y may be u_1 also), or
2. y ∈ V(G_2).

Subcase 1 y ∈ V(G_1)
Assume that y ≠ u_1 ∈ V (G_1).

- Consider | X | < | D_1 |. Since u_{12} and v_{12} dominated by x in H, x dominates both u_1 and v_1 in G_1, implies X is a dominating set for G_1. Which is a contradiction to our assumption that D_1 is a γ - set for G_1. Therefore | X | = | D_1 |.
- So, it is clear that |Y | < | D_2 |. Y dominates atleast G_2 - N [u_2] - v_2, a contradiction to our assumption that Condition 4 is not satisfied.

A similar discussion will be true, when y = u_1 ∈ V (G_1) also.
Subcase 2 y ∈ V(G_2)

- If | X | < | D_1 | and | Y | = | D_2 |, then X dominates atleast G - { v_1 }, that is D_3 = X ∪ { v_1 } is a dominating set for G_1, a contradiction to our assumption that Condition 2 is not satisfied.
- If | X | = | D_1 | and | Y | < | D_2 |, then Y dominates atleast G - { u_2 }, that is D_4 = Y ∪ { u_2 } is a dominating set for G_2, a contradiction to our assumption that Condition 1 and 2 are not satisfied.
- If | X | < | D_1 | and | Y | < | D_2 |, then X dominates atleast G - { v_1 } and Y dominates atleast G - { u_2 }. This implies that, D_3 = X ∪ { v_1 }, D_4 = Y ∪ { u_2 } are dominating sets for G_1 and G_2 respectively, a contradiction to our assumption that Condition 1 and 2 are not satisfied.

Similarly, we get a contradiction when u_2 ∈ Y.
Case 3 u_{12} ∈ V - D, v_{12} ∈ D
The proof is similar to Case 2.
Case 4 u_{12}, v_{12} ∈ D

1. u $_1$, v_1 ∈ X, or

2. u_2, $v_2 \in$ Y, or
3. $u_1 \in$ X and $v_2 \in$ Y, or
4. $u_2 \in$ Y and $v_1 \in$ X.

Assume that (u_1, v_1) $\in$ X or (u_2, v_2) $\in$ Y.
Subcase 1 u_1, $v_1 \in$ X

- Consider | X | < | D_1 |. X is a dominating set for G_1, which is a contradiction to our assumption that D_1 is a γ - set for G_1. Therefore | X | = | D_1 |.
- So, it is clear that | Y | < | D_2 |. Y dominates atleast G_2 - N [u_2] - N[v_2], a contradiction to our assumption that Condition 6 is not satisfied.

Similarly, we get a contradiction when (u_2, v_2) $\in$ Y.
Subcase 2 $u_1 \in$ X and $v_2 \in$ Y
The proof is similar to subcase 2 of case 2. Similarly, we get a contradiction when $u_2 \in$ Y and $v_1 \in$ X.

In all cases, we get a contradiction. We conclude that, if γ(H) < γ(G_1) + γ(G_2), then conditions 1 to 6 are satisfied.

Conversely assume that the conditions of the theorem are satisfied. If possible assume that γ(H) $\geq \gamma$(G_1) + γ(G_2).

(1) $u_i \in D_i$ or $v_i \in D_i$.
If $u_1 \in D_1$ and $u_2 \in D_2$, then $D_3 = D_1 \cup D_2 \cup \{ u_{12} \} - \{ u_1, u_2 \}$ is a dominating set for H such that | D_3 | < | D_1 | + | D_2 |, a contradiction to our assumption that γ(H) $\geq \gamma$(G_1) + γ(G_2).
Similarly, we get a contradiction when $v_1 \in D_1$ and $v_2 \in D_2$.

(2) There is a selfish vertex in G_i.
If u_1 is a selfish vertex in G, then we know that γ(G_1 - u_1) < γ(G_1). Let D_3 be a γ - set for G_1 - u_1.
- If $u_1 \in D_1$, $u_2 \in D_2$, implies γ(H) < γ(G_1) + γ(G_2), by Condition 1 of the hypothesis.
- If $u_1 \in D_1$, $u_2 \in$ V(G_2) - D_2, $D_4 = D_3 \cup D_2$ is a dominating set for H (since there is some x which dominates u_2 in G_2, dominate u_{12} in H), implies | D | < | D_3 | + | D_2 |.
- If $u_1 \in$ V(G_2) - D_1, $u_2 \in D_2$, then | D | < | D_3 | + | D_2 | (proof is similar to the above discussion).

Hence γ(H) < γ(G_1) + γ(G_2), a contradiction to our assumption that γ(H) $\geq \gamma$(G_1) + γ(G_2).

(3) γ(G_i - $\{u_i, v_i\}$) < γ(G_i)
Consider γ(G_2 - $\{u_2, v_2\}$) < γ(G_2). Let D_3 be a γ - set of G_2 - $\{u_2, v_2\}$ such that | D_3 | < | D_2 |.
- If (u_1, v_1) $\in D_1$, then $D_4 = D_1 \cup D_3 \cup \{ u_{12} \} \cup \{ v_{12} \} - \{ u_1, u_2, v_1, v_2 \}$ is a dominating set for H such that | D_4 | < | D |.
- If $u_1 \in D_1$, $v_1 \in$ V(G_1) - D_1, then $D_4 = D_1 \cup D_3 \cup \{ u_{12} \} - \{ u_1, u_2 \}$ is a dominating set for H such that | D_4 | < | D |.
- If $u_1 \in$ V(G_1) - D_1, $v_1 \in D_1$, then | D_4 | < | D | (proof is similar to the above discussion).
- If u_1, $v_1 \in$ V(G_1) - D_1, then $D_4 = D_1 \cup D_3$ is a dominating set for H such that | D_4 | < | D |.

In all cases, we get a contradiction to our assumption that γ(H) $\geq \gamma$(G_1) + γ(G_2).

(4) $\gamma(G_i - N [u_i]) < \gamma(G_i)$ and $u_j \in D_j$, or $\gamma(G_i - N [u_i] - v_i) < \gamma(G_i)$ and $u_j \in D_j$.
- Consider $\gamma(G_2 - N [u_2]) < \gamma(G_2)$ and $u_1 \in D_1$. Let D_3 be a γ - set of $G_2 - N [u_2]$ such that $| D_3 | < | D_2 |$. Let $D_4 = D_1 \cup D_3 \cup \{ u_{12} \} - \{ u_1 \}$, implies $| D_4 | < | D_1 | + | D_3 |$.
- Consider $\gamma(G_2 - N [u_2] - v_2) < \gamma(G_2)$ and $u_1 \in D_1$. Let D_3 be a γ set of $G_2 - N [u_2] - v_2$ such that $| D_3 | < | D_2 |$. $D_4 = D_1 \cup D_3 \cup \{ u_{12} \} - \{ u_1 \}$, implies $| D_4 | < | D_1 | + | D_3 |$.

In all cases, we get a contradiction to our assumption that $\gamma(H) \geq \gamma(G_1) + \gamma(G_2)$.

(5) $\gamma(G_i - N [v_i]) < \gamma(G_i)$ and $v_j \in D_j$, or $\gamma(G_i - u_i - N [v_i]) < \gamma(G_i)$ and $v_j \in D_j$.
The proof is similar to the Case - 4.

(6) $\gamma(G_i - N [u_i] - N [v_i]) < \gamma(G_i)$ and $u_j, v_j \in D_j$.
Consider $\gamma(G_2 - N [u_2] - N [v_2]) < \gamma(G_2)$ and $u_1, v_1 \in D_1$. Let D_3 be a γ - set of $G_2 - N [u_2] - N [v_2]$. Let $D_4 = D_1 \cup D_3 \cup \{ u_{12} \} \cup \{ v_{12} \} - \{ u_1, u_2, v_1, v_2 \}$ is a γ - set for H, implies $| D_4 | < | D |$, a contradiction to our assumption that $\gamma(H) \geq \gamma(G_1) + \gamma(G_2)$.

So, if the hypothesis of the theorem is satisfied, it is not possible that $\gamma(H) \geq \gamma(G_1) + \gamma(G_2)$. Hence we conclude that $\gamma(H) < \gamma(G_1) + \gamma(G_2)$. □

In Theorem 3.3 - 3.5, we provide results relating Euler, Hamiltonian graphs and trees with parallel composition.

Theorem 3.3. *Let G_1 and G_2 be Euler graphs. Let $(u_i, v_i) \in V (G_i)$, i = 1, 2. Let H be the parallel composition graph. Then*

(1) *H is not an Euler graph, if u_i adjacent to v_i.*
(2) *H is an Euler graph, if u_i not adjacent to v_i.*

Proof. Let G_1 and G_2 be Euler graphs. Let H be a parallel composition graph constructed by using G_1 and G_2, where $(u_1, v_1) \in V (G_1)$ and $(u_2, v_2) \in V (G_2)$. The degree of the vertices except for u_{12} and v_{12} of H are even since they were even in G_1 and G_2.

(1) u_1 adjacent to v_1 and u_2 adjacent to v_2.
deg (u_{12}) = deg (u_1) + deg (u_2) - 1 = even + even - 1 = odd.
deg (v_{12}) = deg (v_1) + deg (v_2) - 1 = even + even - 1 = odd.
Implies, H is not an Euler graph.
Consider the other possible cases,

(2) u_1 adjacent to v_1 and u_2 not adjacent to v_2, or u_1 not adjacent to v_1 and u_2 adjacent to v_2, or u_1 not adjacent to v_1 and u_2 not adjacent to v_2,

deg(u_{12}) = deg (u_1) + deg (u_2) = even + even = even.
deg (v_{12}) = deg (v_1) + deg (v_2) = even + even = even.
From the above discussion, we conclude that H is Euler. □

Theorem 3.4. *If G_1 and G_2 are any two graphs such that there is a Hamiltonian path between (u_1, v_1) and (u_2, v_2), then the parallel composition graph H is also Hamiltonian.*

Proof. Let G_1 and G_2 be any two graphs and let P_1 and P_2 be a Hamiltonian path between (u_1, v_1) and (u_2, v_2). While tracing a Hamiltonian path in H, start with a vertex u_{12} through P_1 and reach to a vertex v_{12}, then trace the same from v_{12} to u_{12}, this will form a Hamiltonian circuit in H, H is Hamiltonian. □

Theorem 3.5. *Let G_1 and G_2 be any two trees. The parallel composition graph H is also a tree if and only if u_i is adjacent to v_i, i= 1, 2.*

Proof. Let G_1 and G_2 be any two trees. Let H be the parallel composition tree, where (u_1, v_1) $\in$ V(G_1) and (u_2, v_2) $\in$ V(G_2). Suppose that if the condition of the theorem is not true, then we consider the following cases.
Case 1 u_1 not adjacent to v_1 or u_2 adjacent to v_2.
Since G_1 and G_2 are trees, then there is a unique path P_i between u_i and v_i. Trace a path in H, start with a vertex u_{12} pass through P_1 to reach a vertex v_{12} and through P_2 from v_{12} to u_{12} (since u_2 adjacent to v_2), this generates a circuit in H, implies H is not a tree, a contradiction as H is a tree.
Case 2 u_1 adjacent to v_1 or u_2 not adjacent to v_2.
Proof is similar to Case 1.
Case 3 u_1 not adjacent to v_1 or u_2 not adjacent to v_2.
Let P_i be a unique path between u_i and v_i. Trace a path in H, start with a vertex u_{12} passes through P_1 to reach v_{12} and passes through P_2 from v_{12} to u_{12}, implies this generates a circuit in H, implies H is not a tree.
In all cases, we get a contradiction, which implies u_i is adjacent to v_i.

Conversely assume that u_i is adjacent to v_i. Suppose that, if H is not a tree, then there is atleast one circuit in H between a pair of vertices (x, y).
If there is a circuit C containing an edge (u_{12} v_{12}) (since there is a new edge (u_{12} v_{12}) in H), then C - { u_{ij}, v_{ij} } $\cup$ { u_i, v_j }, i, j = 1, 2, i $\neq$ j is a circuit in G_1 or G_2, a contradiction as G_1 and G_2 are trees.
If there exists a circuit not containing u_{12}, v_{12}, then the same will exists in either G_1 or G_2, a contradiction as G_1 and G_2 are trees.

In all cases, we get a contradiction, which implies H is a tree. □

Definition 3.2. *A graph G is said to be domination subdivision stable (DSS), if the domination number of G does not change by subdividing any edge of G* [10]*. In Theorem 3.6, we prove that DSS graphs are parallel critical graphs.*

Theorem 3.6. *If G_1 or G_2 is DSS, then G_1 and G_2 are parallel critical graphs.*

Proof. Let G_1 be a DSS graph. Then for all (u_i, v_i) $\in$ V(G_i), u_i adjacent to v_i either there is some u_i, $v_i \in D_i$ or there is some $u_i \in D_i$.
pn(u_i, D_i) = v_i or v_i is 2 - dominated.
Case 1 There is some u_i, $v_i \in D_i$
G_1 and G_2 are parallel critical graphs by Condition 1 of Theorem 3.2.
Case 2 There is some $u_i \in D_i$ and pn(u_i, D_i) = v_i.
G_1 and G_2 are parallel critical graphs by Condition 3 of Theorem 3.2.
Case 3 There is some $u_i \in D_i$ and v_i is 2 - dominated.
G_1 and G_2 are parallel critical graphs by Condition 4 or 5 of Theorem 3.2.

From cases 1, 2 and 3, we can conclude that G_1 and G_2 are parallel critical graphs. □

4. Conclusion

Binary operations in graph theory are always a tough one because we are trying to apply this operation on more than one graph. We are familiar with the parallel graph, but relating the parallel graph with domination parameters is new to us. In this paper, we have attempted to find the domination number of parallel composition graph H (where H obtain from G_1 and G_2) using the domination number of G_1 and G_2. Also, we have characterized the parallel critical graph using domination number and a binary operation.

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