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AN OPTIONAL SERVICE MARKOVIAN QUEUE WITH WORKING DISASTERS AND CUSTOMER'S IMPATIENCE

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Abstract. In this paper, we develop a new class of Markov model with working disasters, second optional service, and reneging of customers. The disasters can occur during regular busy period. Whenever a disaster occurs, server continues to serve the customers with a lower service rate instead of completely stopping the service and after the completion of disaster recovery it switches to the regular busy period. Steady-state solution of the model is obtained by using probability generating function technique and stability condition is derived. Further, some important performance measures are presented. A cost model is developed in order to obtain the optimal service rates during first essential service, second optional service and during disaster period using quadratic fit search method. At the end, we provide some numerical examples to visualize the applicability of the model in practical situations.

Keywords: Queue, first essential service, second optional service, working disasters, reneging.

AMS Subject Classification: 60K25, 90B22.

1. Introduction

Queueing systems with removals of customers before being served are often encountered in many practical situations. One type of such situation occurs in queueing models with reneging and another type appears in systems that are subject to disasters. The crucial difference between the above two situations is that the customers decide to abandon the system according to their own desire in the case of reneging, while they are forced to leave the system in the case of disasters. A disaster is also called as a catastrophe, mass exodus, or queue flushing [4].

There has been a great deal of interest in the development and analysis of queueing models with system disasters. Many authors treated the queueing models with catastrophes under various assumptions. A catastrophe may arise either from outside the system or from another service station. Such cases can be seen in inventory systems, computer network applications, telecommunications, etc. Kumar et al. [6] analyzed an $M/M/1$

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queueing system with catastrophes, server failures and non-zero repair time. Using generating function technique, transient and steady state analysis of an $M/M/1$ queue with balking, catastrophes, server failures and repairs has been carried out by Tarabia [15]. A dynamically changing road traffic model which occasionally suffers a disaster resulting in loss of all the vehicles (diverted to other routes) has been examined by Vinodhini and Vidhya [16]. Some more articles on queueing networks with catastrophes can be found in Kim and Lee [5], Bura [3], Ammar [1], Wang et al. [17]. Recently, an $M/M/1$ retrial queueing system with constant retrial rate and Poisson generated catastrophes has been investigated by Li and Wang [7].

Queues with customer impatience have drawn significant attention in recent years. When considering the customer impatience, the reneging plays an important role. Sudhesh [14] focused on transient solution of single server queue with system disasters and impatient customers. Yechiali [18] investigated a queueing model with system disasters and customer's impatience when system is down and he derived various quality of service measures using generating functions. A single server queue with reneging, catastrophes, server failures and repairs has been discussed by Sampath and Liu [11]. Most recent work on disaster queues with customer's impatience can be seen in Zhang and Gao [19].

Queueing systems with second optional service (SOS) is another extensively researched area in the literature. Queueing situations are commonly encountered in everyday life, wherein the server performs first essential service (FES) to all arriving customers and after completing the FES, SOS will be provided to those customers who demand it. Madan [8] introduced the concept of SOS and illustrated a number of practical applications for this type of two-phase service queue. Recently, Balasubramanian et al. [2] were able to approximate the effects of catastrophe on a retrial queueing system with two kinds of services, multiple vacation policy and customer's impatience.

However, in all the models considered so far in the existing queueing literature with disasters, the underlying assumption was that the disasters are related to the server and causes the instantaneous departure of customers from the system including the one in service. In practice, there are enormous situations where the disasters are biological in nature and may not stop the service of a customer completely. Biological disasters have devastating effects caused by an enormous spread of a certain kind of living organism that may spread a disease, viruses on an epidemic or pandemic level. For example, the corona virus COVID-19 pandemic which has severe health crisis that leads to an unprecedented socio-economic impact. This has great impact on industries, IT sectors, etc. Companies are faced with balancing the health and safety of their employees with the need to keep the lights on. In order to reduce the production loss, instead of completely stopping the process, they are allowing their employees to do work from home or to operate with 50 percent employees. Inspired by this, we have therefore, considered a new class of queueing model with working disasters. The term "working disaster" is different from working vacation, first formulated by Servi and Finn [13]. Working vacations are taken only after completion of service of all the customers present in the system. In our paper, working disaster is taken as "a disaster may occur at any time during busy period and when a disaster occurs instead of completely stopping the service, server continues to provide the service with lower service rate irrespective of the number of customers in the system". The inclusion of SOS makes the model more versatile and closer to many practical scenarios.

Therefore, in this investigation, we analyze an infinite capacity queueing system with SOS, working disasters and customer's impatience. Our model has applications in manufacturing industries, IT sectors, hospitals, etc. The primary objectives of this paper are: (i) To derive the stability condition and steady-state solutions for the proposed queueing model. (ii) To develop a cost model that gives the optimum values of service rates in FES (μ_1), SOS (μ_2), and disaster period (η). (iii) To investigate the effect of various parameters on the performance characteristics of the system and the optimal values of μ_1, μ_2, and η.

The rest of this paper is organized as follows. Section 2 gives a description of the queueing model. In section 3, we derive the stability condition using matrix geometric method and steady state solution using generating functions. Some performance measures and special cases of the model are discussed in Section 4. Section 5 and Section 6 present the cost model and sensitivity analysis, respectively followed by conclusions in Section 8.

2. Model Description

We consider an $M/M/1$ queueing model with SOS, working disasters and customer's reneging. Customers arrive at the system according to a Poisson process with parameters λ_b and λ_d, where λ_b and λ_d correspond to the arrivals during busy and disaster periods, respectively and we assume that $\lambda_d < \lambda_b$. The server provides FES to all arriving customers and only some of them demand the SOS with probability r ($0 \leq r \leq 1$) after the completion of FES. Further, we assume that FES and SOS are provided by the same server, and the server can serve only one customer at a time. It is assumed that the service times during FES and SOS are exponentially distributed with rates μ_1 and μ_2, respectively. During normal busy period, disasters occur randomly with a Poisson arrival rate ξ. The occurrence of disaster resulting in the reduction of service rate. During disaster period, the server serves the customers at an exponential rate η, where $\eta < \mu_1$, and $\eta < \mu_2$. Customers who arrive during this time are only provided with FES. If a disaster occurs during SOS of regular busy period, only that customer continues to get SOS in the disaster period. Recovery time of the disaster is also exponentially distributed with parameter ϕ. After the completion of disaster recovery, server switches its service rate from η to μ_1. The customers in this system are assumed to be impatient only during disaster period. The reneging times are exponentially distributed with parameter α.

Practical justification of the model. Our model has real time applications in manufacturing industries. The make-to-order (MTO) strategy in manufacturing industry allows customers to order products built to their specifications which includes computer and computer products, automobiles, etc. Once the equipment is operational, the manufacturing production technician continues to monitor the equipment and materials produced to ensure that the machinery is working correctly. If there are problems with equipment, they repair or recalibrate it. After this process, goods are directed for delivery to those consumers who have chosen for transportation facility. At present, manufacturing industry has been hit in many ways due to COVID-19 pandemic. There has been a reduction in the scale of operations, with consequent effect on quality, cost and production volumes. As a result, size of the orders are also reduced, retailers are canceling orders, forced stores closure, etc. In this scenario, ordering products, production of materials and transportation, COVID-19, reduction of order size, canceling of orders correspond to arrivals during busy period, FES and SOS, disaster, arrivals during disaster, reneging, respectively, in queueing terminology.

3. Mathematical Formulation of the Model

At time t, let $L(t)$ be the number of customers in the system, and $J(t)$ be the state of the server, which is defined as

$$J(t)=\begin{cases} 0, & \text{server is on disaster period,} \\ 1, & \text{server is in FES,} \\ 2, & \text{server is in SOS.} \end{cases}$$

The process $\{(L(t), J(t)), t \geq 0\}$ defines a continuous-time Markov process with state space $\chi = \{(n, j) : n \geq 0,\ j = 0, 1, 2\}$.
For mathematical formulation purpose, we define the following steady-state probabilities: $P_{n,0}$=Probability that there are n number of customers in the system and server is on disaster period, where $n \geq 0$; $P_{n,1}$=Probability that there are n number of customers in the system and server is rendering FES, where $n \geq 0$; $P_{n,2}$=Probability that there are n number of customers in the system and server is rendering SOS, where $n \geq 1$.

3.1. **Steady-state equations.** Using Markov theory, the steady state equations for the model are written as

$$\begin{aligned}
(\lambda_d + \phi)P_{0,0} &= (\eta + \alpha)P_{1,0} + \xi P_{0,1}, && (1)\\
(\lambda_d + \phi + \eta + \alpha)P_{n,0} &= \lambda_d P_{n-1,0} + \eta P_{n+1,0} + \alpha P_{n+1,0} + \xi(P_{n,1} + P_{n,2}), \\
&\quad n \geq 1, && (2)\\
(\lambda_b + \xi)P_{0,1} &= \phi P_{0,0} + (1 - r)\mu_1 P_{1,1} + \mu_2 P_{1,2}, && (3)\\
(\lambda_b + \mu_1 + \xi)P_{n,1} &= \lambda_b P_{n-1,1} + (1 - r)\mu_1 P_{n+1,1} + \mu_2 P_{n+1,2} \\
&\quad + \phi P_{n,0},\ n \geq 1, && (4)\\
(\lambda_b + \mu_2 + \xi)P_{1,2} &= r\mu_1 P_{1,1}, && (5)\\
(\lambda_b + \mu_2 + \xi)P_{n,2} &= r\mu_1 P_{n,1} + \lambda_b P_{n-1,2},\ n \geq 2. && (6)
\end{aligned}$$

3.2. **Stability condition.** For deriving the stability condition, let us define the infinitesimal generator $\mathbf{Q}$ of the process, which is given as

$$\mathbf{Q} = \begin{pmatrix} \hat{\mathbf{A}}_0 & \hat{\mathbf{C}}_0 & & & \\ \hat{\mathbf{B}}_0 & \hat{\mathbf{A}}_1 & \hat{\mathbf{C}}_1 & & \\ & \hat{\mathbf{B}}_1 & \hat{\mathbf{A}}_1 & \hat{\mathbf{C}}_1 & \\ & & \hat{\mathbf{B}}_1 & \hat{\mathbf{A}}_1 & \hat{\mathbf{C}}_1 \\ & & & \vdots & \vdots & \vdots \end{pmatrix}$$

where, $\hat{\mathbf{A}}_0 = \begin{pmatrix} -(\lambda_d + \phi) & \phi \\ \xi & -(\lambda_b + \xi) \end{pmatrix}$, $\hat{\mathbf{C}}_0 = \begin{pmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_b & 0 \end{pmatrix}$, $\hat{\mathbf{C}}_1 = \begin{pmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_b & 0 \\ 0 & 0 & \lambda_b \end{pmatrix}$,

$\hat{\mathbf{B}}_0 = \begin{pmatrix} \eta + \alpha & 0 \\ 0 & (1 - r)\mu_1 \\ 0 & \mu_2 \end{pmatrix}$, $\hat{\mathbf{B}}_1 = \begin{pmatrix} \eta + \alpha & 0 & 0 \\ 0 & (1 - r)\mu_1 & 0 \\ 0 & \mu_2 & 0 \end{pmatrix}$,

$\hat{\mathbf{A}}_1 = \begin{pmatrix} -(\lambda_d + \phi + \eta + \alpha) & \phi & 0 \\ \xi & -(\lambda_b + \mu_1 + \xi) & r\mu_1 \\ \xi & 0 & -(\lambda_b + \mu_2 + \xi) \end{pmatrix}$.

Let us define the matrix $\mathbf{\Psi} = \hat{\mathbf{B}}_1 + \hat{\mathbf{A}}_1 + \hat{\mathbf{C}}_1$, given by

$\mathbf{\Psi} = \begin{pmatrix} -\phi & \phi & 0 \\ \xi & -r\mu_1 - \xi & r\mu_1 \\ \xi & \mu_2 & -\mu_2 - \xi \end{pmatrix}$.

There exists a stationary probability vector $\mathbf{\Pi} = (\pi_0, \pi_1, \pi_2)$ of Ψ such that

$$\mathbf{\Pi\Psi} = \mathbf{0},\ \mathbf{\Pi e} = \mathbf{1}, \tag{7}$$

where $\mathbf{e} = [1, 1, 1]^T$. Using Theorem 3.1.1 in [9], the necessary and sufficient condition for the system to be stable is as follows:

$$\mathbf{\Pi}\hat{\mathbf{B}}_1\mathbf{e} > \mathbf{\Pi}\hat{\mathbf{C}}_1\mathbf{e}. \tag{8}$$

Solving Equations (7) and (8), the stability condition of our model is given by

$$\rho = \frac{\lambda_b(\phi+\xi)(r\mu_1+\mu_2+\xi)}{\xi(\eta+\alpha+\lambda_b-\lambda_d)(r\mu_1+\mu_2+\xi)+\phi(\mu_2+\xi)(1-r)\mu_1+r\phi\mu_1\mu_2} < 1. \tag{9}$$

Remark 3.1. *Suppose that $\alpha = 0$, $\eta = 0$, $\lambda_b = \lambda_d = \lambda$, $\xi = 0$, our model can be reduced to the $M/M/1$ queue with SOS, then equation (9) becomes*

$$\frac{\lambda}{\mu_1} + \frac{r\lambda}{\mu_2} < 1,$$

which is in agreement with the stability condition obtained by Sekar et al. [12] *(Section-5, pp. 176).*

3.3. **Probability generating functions.** Now, we derive the steady-state solution of the model using probability generating function technique.
We define the probability generating functions (PGFs) for $0 < z < 1$ as

$$G_0(z) = \sum_{n=0}^{\infty} P_{n,0}z^n,\ G_1(z) = \sum_{n=0}^{\infty} P_{n,1}z^n,\ G_2(z) = \sum_{n=1}^{\infty} P_{n,2}z^n,$$

with $G_0(1) + G_1(1) + G_2(1) = 1$.

Theorem 3.1. *For $\rho < 1$, $\eta < \mu_1$ and $\eta < \mu_2$ the PGFs can be expressed in terms of $P_{0,0}$ as*

$$\begin{aligned}
G_0(z) &= \frac{\xi z\left[\frac{C_2}{A_2} + \left(\frac{A_2-B_2}{A_2}\right)\left(\frac{C_1A_2-C_2A_1}{A_2B_1-A_1B_2}\right)\right] - (\eta+\alpha)(1-z)P_{0,0}}{\lambda_d z(1-z) + \phi z - (\eta+\alpha)(1-z) - \left(\frac{\xi z D_1(A_2-B_2)}{A_2B_1-A_1B_2}\right)},\\
G_2(z) &= \left[\frac{C_1A_2 - C_2A_1}{A_2B_1 - A_1B_2}\right] + \left[\frac{D_1A_2G_0(z)}{A_2B_1 - A_1B_2}\right],\\
G_1(z) &= \frac{C_2}{A_2} - \frac{B_2}{A_2}G_2(z),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \lambda_b z(1-z) + \xi z + \mu_1 z - (1-r)\mu_1,\ B_1 = -\mu_2,\ C_1 = (\mu_1 z - (1-r)\mu_1)P_{0,1},\\
D_1 &= \phi z, A_2 = r\mu_1, B_2 = -(-\lambda_b(1-z) + \mu_2 + \xi), C_2 = r\mu_1 P_{0,1}, P_{0,1} = \frac{\phi}{\lambda_b+\xi}P_{0,0}.
\end{aligned}$$

Proof. Multiplying equations (1) and (2) with z^n and summing over n gives

$$(\lambda_d z(1-z) + \phi z - (\eta+\alpha)(1-z))G_0(z) = \xi z(G_1(z) + G_2(z)) - (\eta+\alpha)(1-z)P_{0,0}. \tag{10}$$

Similarly, multiplying equations (3), (4) and (5), (6) with z^n and summing over n gives, respectively

$$\begin{aligned}
(\lambda_b z(1-z) + \xi z + \mu_1 z - (1-r)\mu_1)G_1(z) - \mu_2 G_2(z) &= (\mu_1 z - (1-r)\mu_1)P_{0,1}\\
&\quad + \phi z G_0(z), \qquad (11)\\
r\mu_1 G_1(z) - (\lambda_b(1-z) + \mu_2 + \xi)G_2(z) &= r\mu_1 P_{0,1}. \qquad (12)
\end{aligned}$$

Solving equations (11) and (12), we get

$$G_2(z) = \left[\frac{C_1A_2 - C_2A_1}{A_2B_1 - A_1B_2}\right] + \left[\frac{D_1A_2G_0(z)}{A_2B_1 - A_1B_2}\right], \tag{13}$$

$$G_1(z) = \frac{C_2}{A_2} - \frac{B_2}{A_2}G_2(z). \tag{14}$$

Now, substituting equations (13) and (14) in (10), we have

$$G_0(z) = \frac{\xi z\left[\frac{C_2}{A_2} + \left(\frac{A_2-B_2}{A_2}\right)\left(\frac{C_1A_2-C_2A_1}{A_2B_1-A_1B_2}\right)\right] - (\eta+\alpha)(1-z)P_{0,0}}{\lambda_d z(1-z) + \phi z - (\eta+\alpha)(1-z) - \left(\frac{\xi z D_1(A_2-B_2)}{A_2B_1-A_1B_2}\right)}. \tag{15}$$

where

$$\begin{aligned} A_1 &= \lambda_b z(1-z) + \xi z + \mu_1 z - (1-r)\mu_1,\ B_1 = -\mu_2,\ C_1 = (\mu_1 z - (1-r)\mu_1)P_{0,1}, \\ D_1 &= \phi z,\ A_2 = r\mu_1,\ B_2 = -(-\lambda_b(1-z) + \mu_2 + \xi),\ C_2 = r\mu_1 P_{0,1}. \end{aligned}$$

Also, from equation (13), we have

$$G_2'(z) = \frac{(C_1A_2 - C_2A_1)' + D_1'A_2G_0(z) + D_1A_2G_0'(z) - (A_2B_1 - A_1B_2)'G_2(z)}{(A_2B_1 - A_1B_2)}. \tag{16}$$

By taking $z = 0$ in the above equation, we get

$$P_{0,1} = \left(\frac{\phi}{\lambda_b + \xi}\right)P_{0,0}. \tag{17}$$

Therefore, we have seen that all the stationary probabilities can be derived in terms of $P_{0,0}$, which can be found from the result below. □

Theorem 3.2. *For $\rho < 1$, $0 < \eta < \mu_1$ and $0 < \eta < \mu_2$, the probability $P_{0,0}$ is given by*

$$\begin{aligned} P_{0,0} &= \left[\left(\frac{K_1a_0 + a_1}{K_2}\right) + \left[\left(\frac{-r\mu_1}{r\mu_1 + \mu_2 + \xi}\right)a_0 + \left(\frac{\phi r\mu_1}{\xi(r\mu_1 + \mu_2 + \xi)}\right)\left(\frac{K_1a_0 + a_1}{K_2}\right)\right]\right. \\ &\quad \left.\left(1 + \frac{\mu_2 + \xi}{r\mu_1}\right) + a_0\right]^{-1}, \end{aligned}$$

where

$$\begin{aligned} a_0 &= \frac{\phi}{\lambda_b + \xi},\ a_1 = \eta + \alpha,\ K_1 = \frac{\mu_1\mu_2 + (1-r)\mu_1\xi}{r\mu_1 + \mu_2 + \xi}, \\ K_2 &= \frac{[\xi(\eta + \alpha - \lambda_d) - \phi\lambda_b](r\mu_1 + \mu_2 + \xi) + \phi\mu_1(\mu_2 + \xi(1-r))}{\xi(r\mu_1 + \mu_2 + \xi).} \end{aligned}$$

Proof. Applying L-Hospital's rule in (15), we have

$$\lim_{z\to 1} G_0(z) = \lim_{z\to 1} \frac{\xi z\left[\frac{C_2}{A_2} + \left(\frac{A_2-B_2}{A_2}\right)\left(\frac{C_1A_2-C_2A_1}{A_2B_1-A_1B_2}\right)\right] - (\eta+\alpha)(1-z)P_{0,0}}{\lambda_d z(1-z) + \phi z - (\eta+\alpha)(1-z) - \left(\frac{\xi z D_1(A_2-B_2)}{A_2B_1-A_1B_2}\right)}. \tag{18}$$

which gives

$$G_0(1) = \frac{\left[\frac{\mu_1\mu_2 + (1-r)\mu_1\xi}{r\mu_1+\mu_2+\xi}\right]\left(\frac{\phi}{\lambda_b+\xi}\right)P_{0,0} + (\eta+\alpha)P_{0,0}}{\zeta}, \tag{19}$$

where

$$\zeta = \frac{\xi(\eta+\alpha-\lambda_d)-\phi\lambda_b)(r\mu_1+\mu_2+\xi)+\phi\mu_1(\mu_2+\xi(1-r))}{\xi(r\mu_1+\mu_2+\xi)}.$$

By taking $z=1$ in (13) and (14), this simplifies to

$$G_2(1) = \frac{-r\mu_1}{\mu_2+\xi+r\mu_1}\left(\frac{\phi}{\lambda_b+\xi}\right)P_{0,0} + \frac{\phi r\mu_1}{\xi(\mu_2+\xi+r\mu_1)}G_0(1), \tag{20}$$

$$G_1(1) = \left(\frac{\phi}{\lambda_b+\xi}\right)P_{0,0} + \frac{\mu_2+\xi}{r\mu_1}G_2(1). \tag{21}$$

Substituting equations (19), (20), and (21) in normalization condition $G_0(1)+G_1(1)+G_2(1)=1$, we get

$$P_{0,0} = \left[\left(\frac{K_1a_0+a_1}{\zeta}\right) + \left[\left(\frac{-r\mu_1}{r\mu_1+\mu_2+\xi}\right)a_0 + \left(\frac{\phi r\mu_1}{\xi(r\mu_1+\mu_2+\xi)}\right)\left(\frac{K_1a_0+a_1}{\zeta}\right)\right]\left(1+\frac{\mu_2+\xi}{r\mu_1}\right)+a_0\right]^{-1},$$

where

$$a_0 = \frac{\phi}{\lambda_b+\xi},\ a_1=\eta+\alpha,\ K_1=\frac{\mu_1\mu_2+(1-r)\mu_1\xi}{r\mu_1+\mu_2+\xi}.$$

□

4. Performance Measures and Busy Period Analysis

In this section, we present a number of important measures of system performance for an $M/M/1$ queueing model with SOS, working disasters and reneging.

4.1. Performance measures.

- Expected number of customers in the system when the server is on disaster period is

$$E[L_0] = G_0'(1) = \frac{(H_1-H_2)-(H_3-H_4)G_0(1)}{2(N_1-N_2)},$$

where

$$\begin{aligned}
H_1 &= \xi(r\mu_1P_{0,1}X_1+Z_1Y_2+Z_2Y_1)+\xi(2Z_2Y_2+r\mu_1P_{0,1}X_3+Z_1Y_3),\\
H_2 &= (\eta+\alpha)(-2r\mu_1X_2)P_{0,0},\\
H_3 &= -2\lambda_dr\mu_1\xi(\mu_2+r\mu_1+\xi)+2(-\lambda_d+\phi+\eta+\alpha)r\mu_1X_2+\phi r\mu_1X_3,\\
X_1 &= \xi(r\mu_1+\mu_2+\xi),\\
X_2 &= (-\lambda_b\mu_2-2\lambda_d\xi+\xi\mu_2+\xi^2+\mu_1\mu_2+\mu_1+\xi-r\mu_1\lambda_b),\\
X_3 &= -2\lambda_b\mu_2-4\lambda_b\xi+2\lambda_b^2-2\lambda_b\mu_1,\ Y_1=-r\mu_1P_{0,1}\xi,\\
Y_2 &= r\mu_1P_{0,1}(\lambda_b-\xi),\ Y_3=2r\mu_1\lambda_bP_{0,1},\ Z_1=r\mu_1+\mu_2+\xi,\ Z_2=-\lambda_b.
\end{aligned}$$

- Expected number of customers in the system when the server is rendering FES and SOS, respectively, are given as

$$E[L_1] = G_1'(1) = \frac{\mu_2+\xi}{r\mu_1}G_2'(1) - \frac{(\mu_2+\xi-\lambda_b)}{r\mu_1}G_2(1),$$

and

$$E[L_2] = G_2'(1) = \frac{Y_2+\phi r\mu_1G_0(1)+\phi r\mu_1G_0'(1)-X_2G_2(1)}{X_1}.$$

- Expected number of customers in the system is given by

$$E[L] = E[L_0] + E[L_1] + E[L_2].$$

- Expected reneging rate of the customer is

$$E[RC] = \alpha(G_0(1) - P_{0,0}).$$

- Probability that the server is on disaster state is

$$P_d = G_0(1).$$

- Probability that the server is busy with FES and SOS, respectively, are given as

$$P_1 = G_1(1);\ P_2 = G_2(1).$$

- Probability that the server is idle is

$$P_0 = P_{0,0} + P_{0,1}.$$

- Throughput is given as

$$T_p = \eta G_0(1) + \mu_1 G_1(1) + \mu_2 G_2(1).$$

- The expected delay time is given by

$$E[D] = \frac{E[L]}{T_p}.$$

4.2. **Busy period analysis.** Let the expected length of the idle period, the disaster period, the busy period of the server during FES, SOS, and the busy cycle be denoted by $E[I]$, $E[D]$, $E[B_f]$, $E[B_s]$, and $E[BC]$, respectively. We relate the long run fraction of time in various states to the probabilities of the server in various states to obtain $E[I]$, $E[B_f]$, $E[B_s]$, and $E[BC]$. The expected length of the disaster is given by $E[D] = \frac{1}{\phi}$. On the other hand, the long run fraction of the time server is busy during disaster is $\frac{E[D]}{E[BC]} = P_d$. Therefore, we get

$$E[BC] = \frac{1}{\phi}\left(\frac{\left[\left(\frac{\mu_1\mu_2+(1-r)\mu_1\xi}{r\mu_1+\mu_2+\xi}\right)\left(\frac{\phi}{\lambda_b+\xi}\right)P_{0,0} + (\eta+\alpha)P_{0,0}\right]\xi(r\mu_1+\mu_2+\xi)}{\xi(\eta+\alpha-\lambda_d)-\phi\lambda_b)(r\mu_1+\mu_2+\xi)+\phi\mu_1(\mu_2+\xi(1-r))}\right)^{-1}.$$

The long run fraction of the time server is rendering FES, SOS and the server is idle, respectively are given by

$$\frac{E[B_f]}{E[BC]} = P_1;\ \frac{E[B_s]}{E[BC]} = P_2;\ \frac{E[I]}{E[BC]} = P_0.$$

which leads to

$$E[B_f] = E[BC]\left[\left(\frac{\phi}{\lambda_b+\xi}\right)P_{0,0} + \frac{\mu_2+\xi}{r\mu_1}G_2(1)\right],$$

$$E[B_s] = E[BC]\left[\frac{-r\mu_1}{\mu_2+\xi+r\mu_1}\left(\frac{\phi}{\lambda_b+\xi}\right)P_{0,0} + \frac{\phi r\mu_1}{\xi(\mu_2+\xi+r\mu_1)}G_0(1)\right],\ E[I] = P_0E[BC].$$

4.3. **Special cases.**

(1) $M/M/1$ queueing system with working disasters and reneging.
Setting $r = 0$, $\mu_2 = 0$, our study provides the results for $M/M/1$ queueing system with working disasters and reneging.

(2) $M/M/1$ queueing system with SOS and working disasters.
By taking $\alpha = 0$, our results can be used for $M/M/1$ queueing system with SOS and working disasters.

(3) $M/M/1$ queueing system with SOS.
Setting $\lambda_b = \lambda_d = \lambda$, $\alpha = 0$, $\xi = 0$, $\phi \to \infty$, $\eta = 0$, our model yields $M/M/1$ queueing system with SOS.

5. Cost Model

This section develops a cost model in order to carry out an economic analysis of the queueing system under consideration. We develop the total expected cost function per unit time, in which three decision variables (μ_1, μ_2, η) are considered. Our objective is to determine the optimum values of (μ_1, μ_2, η), say $(\mu_1^*, \mu_2^*, \eta^*)$, so that the expected cost function is minimized.

Now, we define the following cost elements:
$c_f \equiv$ busy cost per unit time when the server is rendering FES, $c_s \equiv$ cost per unit time when the server is rendering SOS, $c_d \equiv$ per unit time cost of the server when the server is on disaster period, $c_r \equiv$ cost per customer lost per unit time, $c_{\mu_1} \equiv$ fixed cost of providing a service rate μ_1 for customers, $c_{\mu_2} \equiv$ fixed cost of providing a service rate μ_2 for customers, $c_\eta \equiv$ fixed cost of providing a service rate η for customers.

Based on the definitions of each cost element listed above and its corresponding system characteristics, the total expected cost function per unit time is given by

$$F[\mu_1, \mu_2, \eta] = c_f E[L_1] + c_s E[L_2] + c_d E[L_0] + c_r E[RC] + c_{\mu_1}\mu_1 + c_{\mu_2}\mu_2 + c_\eta \eta.$$

The cost minimization problem can be illustrated mathematically as We solve the above stated optimization problem using quadratic fit search method (QFSM).

QFSM is an optimization technique which can be used when the objective function is highly complex and obtaining its derivative is a difficult task. Given a 3-point pattern, one can fit a quadratic function through corresponding functional values that has a unique minimum, (x^q, y^q, z^q), for the given objective function $F[x, y, z]$. Quadratic fit uses this approximation to improve the current 3-point pattern by replacing one of its points with approximate optimum (x^q, y^q, z^q). The unique optimum (x^q, y^q, z^q) of the quadratic function agreeing with $F[x, y, z]$ at 3-point pattern occurs at
$X^q \cong$

$$\frac{1}{2}\left[\frac{F[x^l, y^l, z^l](s^m - s^h) + F[x^m, y^m, z^m](s^h - s^l) + F[x^h, y^h, z^h](s^l - s^m)}{F[x^l, y^l, z^l](X^m - X^h) + F[x^m, y^m, z^m](X^h - X^l) + F[x^h, y^h, z^h](X^l - X^m)}\right],$$

where $X^l = (x^l, y^l, z^l)^T$, $X^m = (x^m, y^m, z^m)^T$, $X^h = (x^h, y^h, z^h)^T$, $X^q = (x^q, y^q, z^q)^T$, $s^l = (X^l)^2$, $s^m = (X^m)^2$ and $s^h = (X^h)^2$. For the detailed algorithm of QFSM, one may refer [10].

6. Sensitivity Analysis

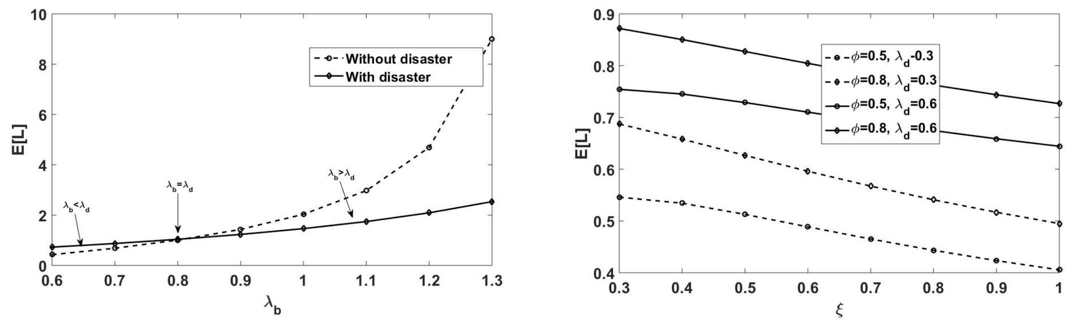
In order to examine the sensitivity of different parameters on the performance measures, we perform some numerical experiments. The model parameters have been chosen arbitrarily yet they bear some close incidence with the practical situations. An example (such as the manufacturing industry mentioned in the practical justification in Section-2) is provided to illustrate the numerical results and cost model.

- The orders for goods (customers) during regular busy period and disaster period (COVID-19) follows a Poisson process with rates $\lambda_b = 0.8$ and $\lambda_d = 0.5$, respectively.
- The service rates of production of goods (FES) and transportation only to those who have chosen it (SOS) during busy period are $\mu_1 = 2.5$ and $\mu_2 = 2.0$, respectively.
- The service rate of production during disaster period is $\eta = 1.5$.
- After the completion of production of the goods, they may go for transportation facility with probability $r = 0.6$ (SOS probability).
- Rate of canceling of orders (reneging) during disaster is $\alpha = 0.5$.
- Disaster occurrence rate is $\xi = 0.7$ and disaster recovery rate is $\phi = 1.0$.

For the cost model,

- Production cost per cycle $c_f = 20$ units.
- Transportation cost per day $c_s = 15$ units.
- Cost per cycle during disaster $c_d = 13$ units.
- Cost of canceling order $c_r = 12$ units per order.
- The fixed costs of production and transportation during busy period are $c_{\mu_1} = 10$ units per service and $c_{\mu_2} = 8$ units per service, respectively.
- Fixed cost during disaster period is $c_\eta = 6$ units per service.

In Figure 1(A), we analyzed the queueing model with disasters by considering all possible arrival rates, viz., $\lambda_b <=> \lambda_d$. Also, in Table 2, for cost analysis, we have considered the case $\lambda_b = \lambda_d$ along with $\lambda_b > \lambda_d$. Here, equal arrival rates during busy and disaster period are referred to as homogeneous arrivals; otherwise, non-homogeneous.



(A) (B)

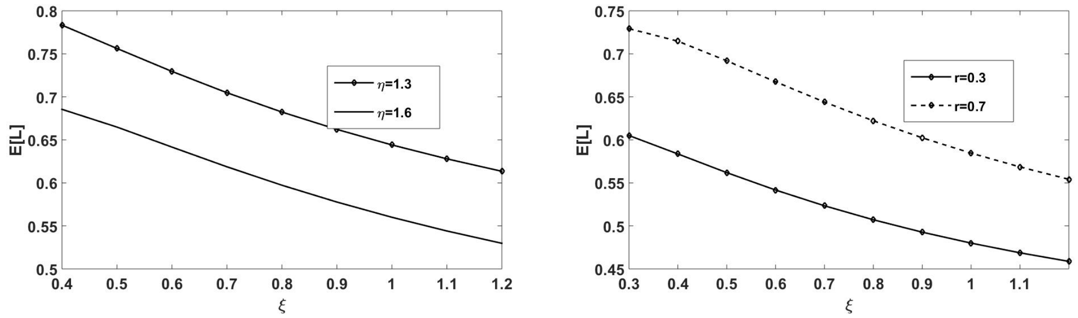
FIGURE 1. Effect of λ_b and ξ on $E[L]$.

Figure 1(A) exhibits the effect of arrival rate during busy period λ_b on system length $E[L]$ with and without disaster. It is clear from the figure that $E[L]$ increases for the increasing values of arrival rates, which is logical. We assumed the values of λ_d as 0.8 and 0.0 for the system with and without disaster, respectively.

- For the case "system with disaster", when $\lambda_b > \lambda_d$, we observed that system length is smaller when compared to the system length of "without disaster" case. This is reasonable because of the higher service rate during busy period. On the other hand, opposite trend is observed with $\lambda_b < \lambda_d$. Moreover, $E[L]$ coincides for $\lambda_b = \lambda_d$.

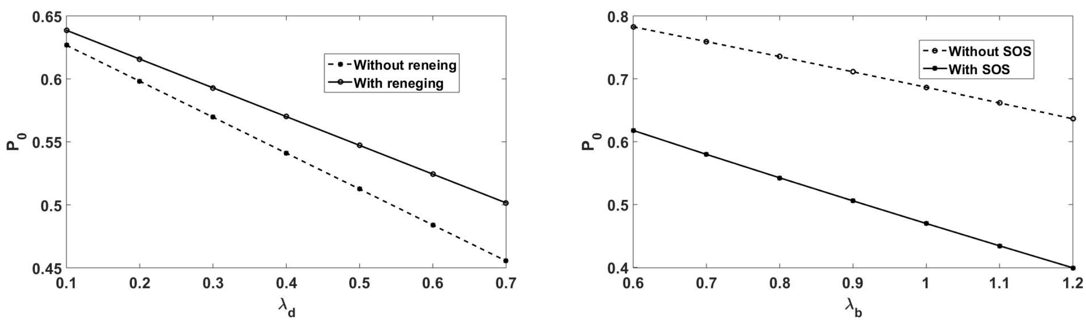
The impact of disaster occurrence rate ξ on $E[L]$ for various disaster recovery rates ϕ, arrival rates during disaster λ_d, service rates during disaster η and SOS probabilities r is shown in Figures 1(B), 2(A) and 2(B), respectively. We observe that

- As disaster occurrence rate increases, the number of customers in the system decreases, as expected.
- From Figure 1(B), for a fixed λ_d, an increment in disaster recovery rate ϕ leads to an increases of $E[L]$. This is apparently because as ϕ is increasing, the customers move out of the disaster period and join the regular busy stream wherein the arrival rates are larger and for the fixed service rate, the queue length is increasing. Further with the increase of λ_d, $E[L]$ again increases, as it should be.
- From Figures 2(A) and 2(B), $E[L]$ decreases with increase of service rate during disaster and increases with increase of SOS probability, which is intuitively true.



(A) (B)

FIGURE 2. Effect of ξ on $E[L]$.



(A) (B)

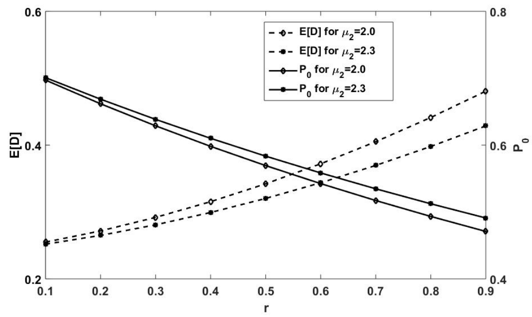
FIGURE 3. Effect of λ_d and λ_b on P_0.

Figure 3(A) depicts the effect of arrival rate during disaster λ_d on idle server probability P_0. It may be noted that, as the value of λ_d grows, P_0 decreases. Meanwhile, for a fixed λ_d, server idleness probability increases in the presence of reneging, which is intuitively true.

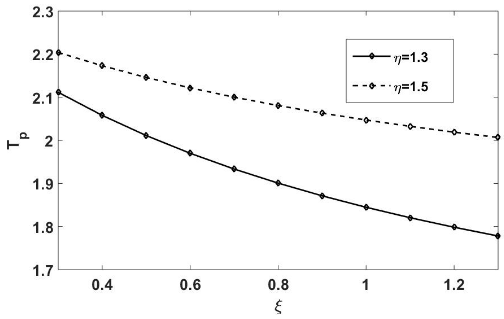
TABLE 1. Effect of r and α on optimum cost.

		μ_1^*	μ_2^*	η^*	$F[\mu_1^*, \mu_2^*, \eta^*]$
	$r = 0.4$	1.7634	1.5283	1.2931	58.6823
$\alpha = 0.3$	$r = 0.6$	1.8946	1.6468	1.3989	62.1194
	$r = 0.8$	2.0191	1.7592	1.4993	65.4660
	$r = 0.4$	1.6775	1.4506	1.2238	57.9645
$\alpha = 0.5$	$r = 0.6$	1.8009	1.5622	1.3234	61.3416
	$r = 0.8$	1.9198	1.6645	1.4192	64.6394

The variation of λ_b on idle server probability P_0 with and without SOS is described in Figure 3(B). We observe that P_0 decreases with the increase of λ_b because higher the number of customers, lesser the chances of idle server. Also, from the figure, it demonstrates that P_0 is smaller when there is SOS as it should be.



(A)



(B)

FIGURE 4. (A) Effect of r on $E[D]$ and P_0. (B) Effect of ξ on T_p.

Figure 4(A) illustrates the impact of SOS probability r on expected delay $E[D]$ and idle server probability P_0 for various values of service rate in SOS μ_2. For a fixed μ_2, $E[D]$ and P_0 increases and decreases respectively with the increase of r. The point of r at which $E[D]$ and P_0 is maximum and minimum respectively is given by the point of intersection of two curves. Furthermore, for a fixed r, as μ_2 increases, opposite trend is observed. This is due to the fact that as service rate increases, the number of customers leaving the system increases which leads to decrease in $E[D]$ and increase of P_0.

Figures 4(B) plot the effect of disaster occurrence rate ξ on throughput T_p for different values of η. It is clear that an increase in ξ causes the frequent service interruption which results in decrease of T_p.

Table 1 shows the optimal service rates $(\mu_1^*, \mu_2^*, \eta^*)$ and optimum cost $F[\mu_1^*, \mu_2^*, \eta^*]$ for different values of r and α. Table 1 reveals that the optimal service rates and minimum cost increases as r increases. This is reasonable because the cost of maintaining the system increases as the number of customers opting for SOS increases. Furthermore, the reverse trend is seen with the increase of reneging rate α as we expect.

The effect of disaster occurrence rate ξ on $(\mu_1^*, \mu_2^*, \eta^*)$ and $F[\mu_1^*, \mu_2^*, \eta^*]$ for homogeneous and non-homogeneous traffic $(\lambda_b > \lambda_d)$ is explored in Table 2. It is obvious from the table that an increase of ξ results in decrease of optimal service rates and optimum cost. Also, it is noticeable that the optimum cost is greater in the case of homogeneous arrivals. Since, the system length will be larger with homogeneous arrival rates than non-homogeneous

Table 2. Optimum cost for homogeneous and non-homogeneous arrival rates w.r.t ξ

	for homogeneous arrival rates			
ξ	μ_1^*	μ_2^*	η^*	$F[\mu_1^*, \mu_2^*, \eta^*]$
0.4	2.2193	1.8843	1.6110	68.7179
0.6	2.0598	1.7959	1.5321	67.7725
0.8	1.9826	1.7262	1.4698	66.6602
	for non-homogeneous arrival rates			
ξ	μ_1^*	μ_2^*	η^*	$F[\mu_1^*, \mu_2^*, \eta^*]$
0.4	2.0131	1.7537	1.4944	64.5869
0.6	1.8636	1.6187	1.3739	62.4466
0.8	1.7448	1.5115	1.2781	60.2743

Table 3. Effect of λ_b and ξ on optimum cost for various values of cost parameters.

	(λ_b, ξ)	(0.9, 0.6)	(0.9, 0.8)	(1.1, 0.8)
Case 1	$(\mu_1^*, \mu_2^*, \eta^*)$	(2.009,1.749,1.491)	(1.873,1.627,1.381)	(2.128,1.858,1.587)
	$F[\mu_1^*, \mu_2^*, \eta^*]$	66.6175	64.0714	71.5287
Case 2	$(\mu_1^*, \mu_2^*, \eta^*)$	(2.041,1.779,1.517)	(1.902,1.653,1.404)	(2.159,1.806,1.613)
	$F[\mu_1^*, \mu_2^*, \eta^*]$	67.9394	65.3539	72.9055
Case 3	$(\mu_1^*, \mu_2^*, \eta^*)$	(2.024,1.764,1.503)	(1.885,1.638,1.391)	(2.142,1.870,1.598)
	$F[\mu_1^*, \mu_2^*, \eta^*]$	67.1112	64.5138	72.0397
Case 4	$(\mu_1^*, \mu_2^*, \eta^*)$	(2.027,1.766,1.505)	(1.896,1.648,1.399)	(2.153,1.880,1.607)
	$F^[\mu_1^*, \mu_2^*, \eta^*]$	67.2528	64.8347	72.3453
Case 5	$(\mu_1^*, \mu_2^*, \eta^*)$	(2.008,1.749,1.491)	(1.872,1.627,1.381)	(2.128,1.857,1.587)
	$F[\mu_1^*, \mu_2^*, \eta^*]$	67.1800	64.5159	71.9732

case. Now, we discuss the effect of (λ_b, ξ) on $(\mu_1^*, \mu_2^*, \eta^*)$ and $F[\mu_1^*, \mu_2^*, \eta^*]$ based on changes in specific values of the cost parameters in Table 3. The following cost elements are used.
Case 1: $c_f = 20$, $c_s = 15$, $c_d = 13$, $c_r = 12$, $c_{\mu_1} = 10$, $c_{\mu_2} = 8$, $c_\eta = 6$.
Case 2: $c_f = 22$, $c_s = 15$, $c_d = 13$, $c_r = 12$, $c_{\mu_1} = 10$, $c_{\mu_2} = 8$, $c_\eta = 6$.
Case 3: $c_f = 20$, $c_s = 17$, $c_d = 13$, $c_r = 12$, $c_{\mu_1} = 10$, $c_{\mu_2} = 8$, $c_\eta = 6$.
Case 4: $c_f = 20$, $c_s = 15$, $c_d = 15$, $c_r = 12$, $c_{\mu_1} = 10$, $c_{\mu_2} = 8$, $c_\eta = 6$.
Case 5: $c_f = 20$, $c_s = 15$, $c_d = 13$, $c_r = 14$, $c_{\mu_1} = 10$, $c_{\mu_2} = 8$, $c_\eta = 6$.
From Table 3, as would be expected, (i) $(\mu_1^*, \mu_2^*, \eta^*)$ and $F[\mu_1^*, \mu_2^*, \eta^*]$ decreases (increases) with ξ (λ_b) for any case; (ii) $(\mu_1^*, \mu_2^*, \eta^*)$ are almost same for Case 1 and Case 5. Intuitively, c_r rarely affects the optimal values of (μ_1, μ_2, η); and (iii) $(\mu_1^*, \mu_2^*, \eta^*)$ and $F[\mu_1^*, \mu_2^*, \eta^*]$ increases as c_f, c_s, and c_d increases, it means that these values affects the optimal values of (μ_1, μ_2, η) significantly.

7. Conclusion

In this paper, we investigated an $M/M/1$ queueing model with SOS, working disasters and reneging. The innovative feature of our study is that after the occurrence of a disaster, server continues to stay in service with lower service rate without completely stopping the service of a customer. For this model, we derived the stability condition using matrix geometric method and steady-state solution of the system using probability generating function method. Various system characteristics such as expected system length, expected length of the idle period, etc. are performed along with the cost analysis. The effects of various parameters on the system performance measures were explored by numerical experiments. This study shows that (i) Choosing $\lambda_d < \lambda_b$ helps to improve the performance of the system when there is a disaster. (ii) An increase of disaster occurrence rate decreases throughput of the system. (iii)The optimum service rates and minimum cost decreases with the increase of disaster occurrence rate and increase with the increase of SOS probability. The sensitivity analysis and cost function will helpful to decision makers and system designers to make appropriate decisions. This investigation further extended by incorporating bulk arrivals, server breakdowns, etc.

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