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ON THE SOLUTIONS OF CERTAIN q-SHIFT DELAY DIFFERENTIAL EQUATIONS OVER NON-ARCHIMEDEAN FIELD

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ABSTRACT. Inspired by the results of Malmquist [13], in this paper, we have investigated on the solutions of q-shift delay differential equations over non-Archimedean field. We also proved Clunie-type result corresponding to certain q-shift delay differential equations over non-Archimedean field.

Keyword: Non-Archimedean field; meromorphic solution; q-shift delay differential equation.

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1. INTRODUCTION AND DEFINITIONS.

The well known Nevanlinna's [6] value distribution theory is a major part of complex analysis and it plays an important role in the study of differential equations over $\mathbb{C}$. In this article, we are motivated to investigate the solutions of some type of non-linear differential equations over non-Archimedean field. To this end, we recall some basic terminologies related to value distribution theory over non-Archimedean field.

We consider $\mathbb{F}$ be a non-Archimedean algebraically closed field which is complete with respect to a non trivial non-Archimedean absolute value such that characteristic of $\mathbb{F}$ is zero. Let the collection of all power series with radius of convergence $\geq r$ is denoted by $A_{[r}(\mathbb{F})$. We use the notations $\mathscr{A}(\mathbb{F})\big(=A_{[\infty}(\mathbb{F})\big)$ and $\mathscr{M}(\mathbb{F})$ to mean the collections of all entire functions and meromorphic functions respectively on $\mathbb{F}$. Let us denote $\tilde{\mathbb{F}}=\mathbb{F}\cup\{\infty\}$. The disjoint union of any two set U and V is written as $U\sqcup V$.

Let $f(z)$ be a non-constant entire function on $\mathbb{F}$. For a constant $\rho\in\mathbb{R}$ such that $0<\rho\leq r$, the counting function of $f(z)$ is defined as follows

$$N(r,a;f)=\frac{1}{ln\ p}\int_{\rho}^{r}\frac{n(t,a;f)}{t}\,dt,$$

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where $n(t, a; f)$ denotes the number of solution (counting multiplicity) of $f(z) = a$ in the disk $D_t = \{z \in \mathbb{F} : |z| \leq t\}$.

Let us consider a non-constant function $f(z) \in \mathscr{A}(\mathbb{F})$, then $f(z)$ has a power series expansion i.e., we can write $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{F}$. For every $r > 0$, the function $\mu(r, f) : \mathscr{A}(\mathbb{F}) \to \mathbb{R}^+$ is defined as

$$\mu(r, f) = |f(z)| = max\{|a_n|r^n : 0 \leq n < \infty\},$$

for all z such that $|z| = r$ and $f(z) \neq 0$.

Now we consider $f(z) \in \mathscr{M}(\mathbb{F})$. Thus $f(z) = \frac{g(z)}{h(z)}$ such that $g(z), h(z) \in \mathscr{A}(\mathbb{F})$ and having no common zeros. We define $\mu(r, f) = \frac{\mu(r,g)}{\mu(r,h)}$. Let z_0 be a-point (for $a \in \tilde{\mathbb{F}}$) of $f(z)$ (i.e., solution of $f(z) - a = 0$), then the multiplicity of z_0 is denoted by $\mu_f^a(z_0)$. Notice that $\mu_f^a = \mu_{g-ah}^0$, $\mu_f^\infty = \mu_h^0$, $N(r, a; f) = N(r, 0; g - ah)$ and $N(r, \infty; f) = N(r, 0; h)$.

The compensation (or proximity) function of $f(z)$ is defined as follows:

$$m(r, \infty; f) = log^+\mu(r, f) = max\{0, log\ \mu(r, f)\} \text{ and } m(r, a; f) = log^+\frac{1}{\mu(r, f - a)}.$$

The Nevanlinna's characteristic function is defined as:

$$T(r, f) = m(r, \infty; f) + N(r, \infty; f).$$

For the sake of convenience we use $m(r, f)$ and $N(r, f)$ instead of $m(r, \infty; f)$ and $N(r, \infty; f)$. For any non-constant meromorphic function $f(z)$ we define $S(r, f) = o(T(r, f))$.

Definition 1.1. [9] *For $a \in \tilde{\mathbb{F}}$, the defect of $f(z)$ for the value a is denoted by $\delta(a, f)$ and defined by $\delta(a; f) = 1 - \limsup\limits_{r\to\infty} \dfrac{N(r, a; f)}{T(r, f)}$. Also $\Theta(a; f)$ is defined as $\Theta(a; f) = 1 - \limsup\limits_{r\to\infty} \dfrac{\overline{N}(r, a; f)}{T(r, f)}$.*

For some basic results related to Nevanlinna's theory over $\mathbb{F}$ readers can look a glance at [9].

In 1913, Malmquist [13] first investigated on the existence of solutions of a particular type of differential equation over $\mathbb{C}$ and obtained the following result.

Theorem A. [13] *Let*

$$f'(z) = R(f) = \frac{A(f)}{B(f)} = \frac{\sum\limits_{s=0}^{n_1} a_s(z) f^s(z)}{\sum\limits_{t=0}^{n_2} b_t(z) f^t(z)},$$

where $A(f), B(f)$ are relatively prime polynomial of f with all coefficients are rational functions. If the equation admits a transcendental meromorphic solution, then $n_2 = 0$ and $n_1 \leq 2$.

In [12], for some general non-linear differential equations, Laine proved Clunie-type [2] theorem. After that in 2004, Korhonen [10] and in 2007, Yang-Ye [17] extended Laine's result. In [5], Hulburd-Korhonen found the difference analogue of Clunie's theorem.

On the other hand, in 2002, Gundersen et al. [4] investigated on the growth of meromorphic solutions of similar types of equations as in *Theorem A*. They considered $f(qz)$ instead of $f'(z)$ and established some results. For more results related to Malmquist-type theorems reader can see Gackstatter-Laine [3], Laine [11], Toda [15], Yosida [18] and He-Xiao [7]. Notice that all the above mentioned results are in the field of complex numbers. So far a very few results can be found in this direction over non-Archimedean field. In this paper one of our aim is to investigate on Clunie-type theorem and solutions of some non-linear delay differential equations over non-Archimedean field.

Let $f(z)$ be a non-constant meromorphic function on $\mathbb{F}$ and $q_i, c_i \in \mathbb{F}\setminus\{0\}$, $|q_i| = 1$ (for $i = 1, 2, \cdots, n$) with $q_0 = 1, c_0 = 0$. Consider the operators $\Omega, \Phi_1, \Phi_2, \Phi_3$ as follows:

$$\Omega = \sum_{\xi\in\Xi} \alpha_\xi \prod_{i=0}^{n} \Big(f(q_i z + c_i)\Big)^{\xi_i}, \tag{1.1}$$

$$\Phi_1 = \sum_{\kappa\in K} \beta_\kappa \prod_{i=0}^{n} \Big(f^{(i)}(z)\Big)^{\kappa_i}, \tag{1.2}$$

$$\Phi_2 = \sum_{\lambda\in\Lambda} \beta_\lambda \prod_{i=0}^{n} \Big(f(q_i z + c_i)\Big)^{\lambda_i}, \tag{1.3}$$

$$\Phi_3 = \sum_{\eta\in H} \beta_\eta \prod_{i=0}^{n} \Big(f^{(i)}(q_i z + c_i)\Big)^{\eta_i}, \tag{1.4}$$

where $\alpha_\xi, \beta_\kappa, \beta_\lambda, \beta_\eta$ are non-constant meromorphic functions on $\mathbb{F}$ and Ξ, K, Λ, H are index sets of non-negative integers with finite cardinality and $\xi = (\xi_0, \xi_1, \xi_2, \cdots, \xi_n)$ similar for κ, λ, η. Define

$$deg(\Omega) = \max_{\xi\in\Xi}\Big\{\sum_{i=0}^{n}\xi_i\Big\},\ \ \gamma(\Omega) = \max_{\xi\in\Xi}\Big\{\sum_{i=1}^{n} i\xi_i\Big\},\ \ \Gamma(\Omega) = \max_{\xi\in\Xi}\Big\{\sum_{i=0}^{n}(i+1)\xi_i\Big\}.$$

Similarly we can define $deg(\Phi_i), \gamma(\Phi_i)$ and $\Gamma(\Phi_i)$ for $i = 1, 2, 3$. For the sake of convenience we call Ω (or Φ_2), Φ_1, Φ_3 as q-shift, differential and delay differential operator respectively.

The non-Archimedian analogue of Clunie-type result for non-linear differential equation was first investigated by Yang-Hu [16] in the following manner.

Theorem B. [16] *Consider Φ_1 as defined in (1.2). Let $f(z) \in \mathscr{M}(\mathbb{F})$ be a solution of*

$$\Phi_1 = R(f) = \frac{A(f)}{B(f)} = \frac{\sum_{s=0}^{n_1} a_s(z) f^s(z)}{\sum_{t=0}^{n_2} b_t(z) f^t(z)},$$

where $A(f), B(f)$ are relatively prime polynomials of $f(z)$ with all coefficients are in $\mathscr{M}(\mathbb{F})$. If $n_2 \geq n_1$ then

$$(i)\ \ m(r, \Phi_1) \leq \sum_{\kappa\in K} m(r, \beta_\kappa) + \sum_{s=0}^{n_1} m(r, a_s) + O\left(m\Big(r, \frac{1}{b_{n_2}}\Big) + \sum_{t=0}^{n_2} m(r, b_t)\right),$$

$$(ii)\ \ N(r, \Phi_1) \leq \sum_{\kappa\in K} N(r, \beta_\kappa) + \sum_{s=0}^{n_1} N(r, a_s) + O\left(\sum_{t=0}^{n_2} N\Big(r, \frac{1}{b_t}\Big)\right).$$

Recently Hu-Luan [8] studied on the solution of non-linear difference equation as follows.

Theorem C. [8] *Let $f(z) \in \mathscr{M}(\mathbb{F})$ be a admissible non-constant meromorphic solution of the equation*

$$\Omega = \frac{A(f)}{B(f)},$$

where Ω defined in (1.1) and $A(f) = \sum_{s=0}^{n_1} a_s(z) f^s(z)$, $B(f) = \sum_{t=0}^{n_2} b_t(z) f^t(z)$ with $a_s(z), b_t(z)$ are meromorphic function on $\mathbb{F}$. Then $n_2 = 0$ and $n_1 \leq deg(\Omega)$.

We see that *Theorem B* is on Clunie-type results for non-linear differential equation and *Theorem C* is on solution of non-linear q-shift equation. In this paper, first we investigate Clunie-type result for q-shift delay differential equation. In this perspective, we have the following result.

Theorem 1.1. *Consider the non-linear q-shift delay differential equation*

$$\Omega = \frac{A(f)}{B(f)}\big(\Phi_1 + \Phi_2 + \Phi_3\big), \tag{1.5}$$

where $A(f), B(f)$ are same as in Theorem C and $\Omega, \Phi_1, \Phi_2, \Phi_3$ are defined as in (1.1)-(1.4). Let $\Gamma = \max\big\{\Gamma(\Phi_1), \Gamma(\Phi_2), \Gamma(\Phi_3)\big\}$. If $n_2 \geq n_1 + \Gamma$ then

$$\begin{aligned}(i)\ \ m(r,\Omega) \ \ &\leq \ \ \sum_{\xi\in\Xi} m(r,\alpha_\xi) + \sum_{\tau\in K\sqcup\Lambda\sqcup H} m(r,\beta_\tau) + \sum_{s=0}^{n_1} m(r,a_s) + l_1 m\Big(r, \frac{1}{b_{n_2}}\Big) \\ &\quad + l_1 \sum_{t=0}^{n_2-1} m(r,b_t) + O(1),\end{aligned}$$

where $l_1 = \max\{1, deg(\Omega)\}$;

$$(ii)\ \ N(r,\Omega) \leq \sum_{\xi\in\Xi} N(r,\alpha_\xi) + \sum_{\tau\in K\sqcup\Lambda\sqcup H} N(r,\beta_\tau) + \sum_{s=0}^{n_1} N(r,a_s) + l_2 \sum_{t=0}^{n_2} N\Big(r, \frac{1}{b_t}\Big),$$

where $l_2 = \max\Big\{1, \frac{deg(\Omega)}{n_2}\Big\}$.

Corollary 1.1. *If all the conditions of Theorem 1.1 are satisfied, then*

$$T(r,\Omega) \leq \sum_{\xi\in\Xi} T(r,\alpha_\xi) + \sum_{\tau\in K\sqcup\Lambda\sqcup H} T(r,\beta_\tau) + \sum_{s=0}^{n_1} T(r,a_s) + O\Big(\sum_{t=0}^{n_2} T(r,b_t)\Big) + O(1).$$

Definition 1.2. *Let $f(z) \in \mathscr{M}(\mathbb{F})$ be a non-constant solution of the equation (1.5) then $f(z)$ is said to be admissible solution of (1.5) if $f(z)$ satisfies the following condition*

$$\sum_{\xi\in\Xi} T(r,\alpha_\xi) + \sum_{\tau\in K\sqcup\Lambda\sqcup H} T(r,\beta_\tau) + \sum_{s=0}^{n_1} T(r,a_s) + \sum_{t=0}^{n_2} T(r,b_t) = S(r,f).$$

Notice that if all the coefficient functions of $\Omega, \Phi_i, A(f), B(f)$ (for $i = 1, 2, 3$) are small functions with respect to $f(z)$ then $f(z)$ must be admissible solution of the equation (1.5). Inspired by *Theorem C*, our next result is related to the admissible solution of the equation (1.5).

Theorem 1.2. *Let $f(z)$ be a non-constant admissible meromorphic solution of (1.5). If $(\Phi_1+\Phi_2+\Phi_3)$ is non-constant with $n_2 \geq n_1+\Gamma$, then*

$$n_2 \leq \min\Big\{\sum_{i=1}^{3} deg(\Phi_i)+(1-\Theta(\infty;f))\sum_{i=1}^{3}\gamma(\Phi_i), \sum_{i=1}^{3}\Gamma(\Phi_i)\Big\}$$

and if $(\Phi_1+\Phi_2+\Phi_3)$ is constant, then

$$n_2 = 0, n_1 \leq deg(\Omega).$$

In the next section we discuss on some lemmas which will be useful for proving our main results.

2. Lemmas

Lemma 2.1. [1] *Let $f(z)$ be a non-constant meromorphic function on $\mathbb{F}$ and $q, c \in \mathbb{F}$, $|q| = 1$, $c \neq 0$. Then*
(i) $m\Big(r, \frac{f(qz+c)}{f(z)}\Big) = O(1)$;
(ii) $N\big(r, 0; f(qz+c)\big) = N(r, 0; f(z)) + O(1)$;
(iii) $N\big(r, f(qz+c)\big) = N(r, f(z)) + O(1)$;
(iv) $T\big(r, f(qz+c)\big) = T(r, f(z)) + O(1)$.

Lemma 2.2. *Let $f(z)$ be a non-constant meromorphic function on $\mathbb{F}$ and $q, c \in \mathbb{F}$, $|q| = 1$, $c \neq 0$. Then*

$$m\Big(r, \frac{f^{(j)}(qz+c)}{f(z)}\Big) = O(1).$$

Proof. Using the lemma of logarithmic derivative and *Lemma 2.1* we get

$$m\Big(r, \frac{f^{(j)}(qz+c)}{f(z)}\Big) \leq m\Big(r, \frac{f^{(j)}(qz+c)}{f(qz+c)}\Big) + m\Big(r, \frac{f(qz+c)}{f(z)}\Big) = O(1).$$

□

The next lemma is the non-Archimedian analogue of Mokhon'ko [14] lemma which was proved by Yang-Hu [16].

Lemma 2.3. [16] *Let $R(f) = \frac{A(f)}{B(f)}$, where $A(f) = \sum_{s=0}^{n_1} a_s(z) f^s(z)$, $B(f) = \sum_{t=0}^{n_2} b_t(z) f^t(z)$ with $a_s(z)$, $b_t(z)$ are any meromorphic function on $\mathbb{F}$. Then*

$$T(r, R) = \max\{n_1, n_2\} T(r, f) + O\Bigg(\sum_{s=0}^{n_1} T(r, a_s) + \sum_{t=0}^{n_2} T(r, b_t)\Bigg).$$

Lemma 2.4. *Consider the operator Φ_3 as defined in (1.4). Then*

$$T(r, \Phi_3) \leq \sum_{\eta \in H} T\big(r, \beta_\eta\big) + deg(\Phi_3) T(r, f) + \gamma(\Phi_3)\overline{N}(r, f) + O(1).$$

Proof. Using *Lemma 2.2* we get

$$\begin{aligned} m(r,\Phi_3) &\leq \max_{\eta\in H} m\Big(r,\beta_\eta \prod_{i=0}^{n}\Big(f^{(i)}(q_i z+c_i)\Big)^{\eta_i}\Big) \qquad (2.1)\\ &\leq \max_{\eta\in H} m(r,\beta_\eta)+\max_{\eta\in H}\left[m\Big(r,f^{\sum_{i=0}^{n}\eta_i}\Big)+\sum_{i=0}^{n}\eta_i\, m\left(r,\frac{f^{(i)}(q_i z+c_i)}{f(z)}\right)\right]\\ &\leq \sum_{\eta\in H} m(r,\beta_\eta)+m(r,f)\max_{\eta\in H}\Big(\sum_{i=0}^{n}\eta_i\Big)+O(1)\\ &\leq \sum_{\eta\in H} m(r,\beta_\eta)+deg(\Phi_3)m(r,f)+O(1). \end{aligned}$$

Now with the help of *Lemma 2.1* we deduce

$$\begin{aligned} N(r,\Phi_3) &\leq \sum_{\eta\in H} N(r,\beta_\eta)+\max_{\eta\in H}\Big[\sum_{i=0}^{n}\eta_i N\big(r,f^{(i)}\big)\Big] \qquad (2.2)\\ &\leq \sum_{\eta\in H} N(r,\beta_\eta)+\max_{\eta\in H}\Big[\sum_{i=0}^{n}\eta_i\big(N(r,f)+i\overline{N}(r,f)\big)\Big]\\ &\leq \sum_{\eta\in H} N(r,\beta_\eta)+deg(\Phi_3)N(r,f)+\gamma(\Phi_3)\overline{N}(r,f)+O(1). \end{aligned}$$

Combining (2.1) and (2.2) we get

$$T(r,\Phi_3)\leq \sum_{\eta\in H} T\big(r,\beta_\eta\big)+deg(\Phi_3)T(r,f)+\gamma(\Phi_3)\overline{N}(r,f)+O(1).$$

□

Remark 2.1. *From Lemma 2.4 it is easy to verify that*

$$T(r,\Phi_3)\leq \sum_{\eta\in H} T\big(r,\beta_\eta\big)+\Big[deg(\Phi_3)+(1-\Theta(\infty;f))\gamma(\Phi_3)\Big]T(r,f)+O(1).$$

Similar inequalities also hold for Φ_1 *and* Φ_2.

Remark 2.2. *We know that* $deg(\Phi_i)\leq\Gamma(\Phi_i)$, *for* $i=1,2,3$. *So (2.1) can be written as*

$$m(r,\Phi_3) \leq \sum_{\eta\in H} m(r,\beta_\eta)+\Gamma(\Phi_3)m(r,f)+O(1). \qquad (2.3)$$

We can also write (2.2) as

$$\begin{aligned} N(r,\Phi_3) &\leq \sum_{\eta\in H} N(r,\beta_\eta)+\max_{\eta\in H}\Big[\sum_{i=0}^{n}\eta_i\big(N(r,f)+i\overline{N}(r,f)\big)\Big] \qquad (2.4)\\ &\leq \sum_{\eta\in H} N(r,\beta_\eta)+\max_{\eta\in H}\Big[\sum_{i=0}^{n}\big(i+1\big)\eta_i\Big]\cdot N(r,f)\\ &\leq \sum_{\eta\in H} N(r,\beta_\eta)+\Gamma(\Phi_3)N(r,f). \end{aligned}$$

Combining (2.3) and (2.4) we obtain

$$T(r,\Phi_3) \leq \sum_{\eta\in H} T(r,\beta_\eta)+\Gamma(\Phi_3)T(r,f)+O(1). \qquad (2.5)$$

It is easy to see that similar inequalities like (2.5) also holds for Φ_1 *and* Φ_2.

3. Proofs of the theorem.

Proof of Theorem 1.1. Take $z \in \mathbb{F}$ such that none of $f(z), \alpha_\xi(z), \beta_\kappa(z), \beta_\lambda(z), \beta_\eta(z), a_s(z)$, $b_t(z)$ are equals to 0 or ∞ for all $\xi \in \Xi, \kappa \in K, \lambda \in \Lambda, \eta \in H, 0 \leq s \leq n_1, 0 \leq t \leq n_2$. Now consider

$$\mathscr{B}(z) = \max_{0 \leq t \leq n_2 - 1} \left\{ 1, \left(\frac{|b_t(z)|}{|b_{n_2}(z)|} \right)^{\frac{1}{n_2 - t}} \right\}.$$

First we consider $|f(z)| > \mathscr{B}(z)$. Then

$$|b_t(z)||f(z)|^t \leq |b_{n_2}(z)| \mathscr{B}^{n_2 - t}(z) |f(z)|^t < |b_{n_2}(z)||f(z)|^{n_2}.$$

Thus

$$|B(f)| = \max_{0 \leq t \leq n_2} |b_t(z)||f(z)|^t = |b_{n_2}(z)||f(z)|^{n_2}. \tag{3.1}$$

Note that $|f(z)| > \mathscr{B}(z) \geq 1$, thus using (3.1) we get

$$\begin{aligned} |\Omega| &\leq \frac{|A(f)|}{|B(f)|} \max\{|\Phi_1|, |\Phi_2|, |\Phi_3|\} \qquad (3.2) \\ &\leq \frac{\max\limits_{0 \leq s \leq n_1} |a_s|}{|b_{n_2}|} \frac{1}{|f|^{(n_2 - n_1)}} \max\{|\Phi_1|, |\Phi_2|, |\Phi_3|\}. \end{aligned}$$

Now

$$\begin{aligned} |\Phi_1| &\leq \max_{\kappa \in K} \left(|\beta_\kappa| \prod_{i=0}^{n} \left| f^{(i)}(z) \right|^{\kappa_i} \right) \\ &\leq |f|^{deg(\Phi_1)} \max_{\kappa \in K} \left(|\beta_\kappa| \prod_{i=0}^{n} \left| \frac{f^{(i)}(z)}{f(z)} \right|^{\kappa_i} \right), \end{aligned}$$

similar inequalities can be deduce for $|\Phi_2|$ and $|\Phi_3|$. Let us assume $\delta = \max\{deg(\Phi_i) : i = 1, 2, 3\}$. So from (3.2) we obtain

$$\begin{aligned} |\Omega| \leq \max_{0 \leq s \leq n_1} \frac{|a_s|}{|b_{n_2}|} \cdot \frac{1}{|f|^{(n_2 - n_1 - \delta)}} \cdot \max \Bigg[&\max_{\kappa \in K} |\beta_\kappa| \prod_{i=0}^{n} \left| \frac{f^{(i)}(z)}{f(z)} \right|^{\kappa_i}, \qquad (3.3) \\ &\max_{\lambda \in \Lambda} |\beta_\lambda| \prod_{i=0}^{n} \left| \frac{f(q_i z + c_i)}{f(z)} \right|^{\lambda_i}, \max_{\eta \in H} |\beta_\eta| \prod_{i=0}^{n} \left| \frac{f^{(i)}(q_i z + c_i)}{f(z)} \right|^{\eta_i} \Bigg]. \end{aligned}$$

Recall that $|f(z)| > \mathscr{B}(z) \geq 1$ and the given condition $n_2 \geq n_1 + \Gamma \geq n_1 + \delta$. From (3.3)

$$\begin{aligned} |\Omega| \leq \max_{0 \leq s \leq n_1} \frac{|a_s|}{|b_{n_2}|} \cdot \max \Bigg[&\max_{\kappa \in K} |\beta_\kappa| \prod_{i=0}^{n} \left| \frac{f^{(i)}(z)}{f(z)} \right|^{\kappa_i}, \qquad (3.4) \\ &\max_{\lambda \in \Lambda} |\beta_\lambda| \prod_{i=0}^{n} \left| \frac{f(q_i z + c_i)}{f(z)} \right|^{\lambda_i}, \max_{\eta \in H} |\beta_\eta| \prod_{i=0}^{n} \left| \frac{f^{(i)}(q_i z + c_i)}{f(z)} \right|^{\eta_i} \Bigg]. \end{aligned}$$

Let us denote

$$\begin{aligned} A_1 &= \max_{0\leq s\leq n_1}\mu\Big(r,\frac{a_s}{b_{n_2}}\Big)\cdot\max\Bigg[\max_{\kappa\in K}\mu(r,\beta_\kappa)\prod_{i=0}^{n}\bigg(\mu\Big(r,\frac{f^{(i)}(z)}{f(z)}\Big)\bigg)^{\kappa_i},\\ &\max_{\lambda\in\Lambda}\mu(r,\beta_\lambda)\prod_{i=0}^{n}\bigg(\mu\Big(r,\frac{f(q_iz+c_i)}{f(z)}\Big)\bigg)^{\lambda_i},\max_{\eta\in H}\mu(r,\beta_\eta)\prod_{i=0}^{n}\bigg(\mu\Big(r,\frac{f^{(i)}(q_iz+c_i)}{f(z)}\Big)\bigg)^{\eta_i}\Bigg].\end{aligned}$$

Next let us consider $|f(z)|\leq\mathscr{B}(z)$, then from (1.1) we get

$$\begin{aligned} |\Omega| &\leq \mathscr{B}^{deg(\Omega)}(z)\max_{\xi\in\Xi}|\alpha_\xi|\prod_{i=0}^{n}\left|\frac{f(q_iz+c_i)}{f(z)}\right|^{\xi_i}\\ &\leq \max_{0\leq t\leq n_2-1}\left\{1,\left|\frac{b_t}{b_{n_2}}\right|^{\frac{deg(\Omega)}{n_2-t}}\right\}\max_{\xi\in\Xi}|\alpha_\xi|\prod_{i=0}^{n}\left|\frac{f(q_iz+c_i)}{f(z)}\right|^{\xi_i}. \qquad (3.5)\end{aligned}$$

Denote

$$A_2=\max_{0\leq t\leq n_2-1}\left\{1,\Big(\mu\Big(r,\frac{b_t}{b_{n_2}}\Big)\Big)^{\frac{deg(\Omega)}{n_2-t}}\right\}\max_{\xi\in\Xi}\mu(r,\alpha_\xi)\prod_{i=0}^{n}\bigg(\mu\Big(r,\frac{f(q_iz+c_i)}{f(z)}\Big)\bigg)^{\xi_i}.$$

Hence combining (3.4) and (3.5) we get $\mu(r,\Omega)\leq\max\{A_1,A_2\}$. Now with the help of *Lemma 2.1,2.2* and lemma of logarithmic derivative we get

$$\begin{aligned} m(r,\Omega) &\leq \max_{\substack{0\leq s\leq n_1\\ \tau\in K\sqcup\Lambda\sqcup H\\ 0\leq t\leq n_2-1\\ \xi\in\Xi}}\Big[m(r,a_s)+m\Big(r,\frac{1}{b_{n_2}}\Big)+m(r,\beta_\tau),\\ &\quad deg(\Omega)m(r,b_t)+deg(\Omega)m\Big(r,\frac{1}{b_{n_2}}\Big)+m(r,\alpha_\xi)\Big]+O(1)\\ &\leq \sum_{\xi\in\Xi}m(r,\alpha_\xi)+\sum_{\tau\in K\sqcup\Lambda\sqcup H}m(r,\beta_\tau)+\sum_{s=0}^{n_1}m(r,a_s)+l_1m\Big(r,\frac{1}{b_{n_2}}\Big)\\ &\quad +l_1\sum_{t=0}^{n_2-1}m(r,b_t)+O(1),\end{aligned}$$

where $l_1=\max\{1,deg(\Omega)\}$. Therefore (i) of *Theorem 1.1* follows.

Next we will prove (ii). Take a point $z_0\in\mathbb{F}$ such that z_0 is a pole of $f(z)$. We know that

$$\mu^{\infty}_{A(f)}(z_0)\leq n_1\mu^{\infty}_f(z_0)+\sum_{s=0}^{n_1}\mu^{\infty}_{a_s}(z_0), \qquad (3.6)$$

$$\mu^{\infty}_{B(f)}(z_0)\geq n_2\mu^{\infty}_f(z_0)-\sum_{t=0}^{n_2}\mu^{0}_{b_t}(z_0). \qquad (3.7)$$

Now we have the following two cases:

Case 1: Let us assume $\mu^{\infty}_{B(f)}>0$. It can be easily establish the following three inequalities

$$\mu^{\infty}_{\Phi_1}(z_0)\leq\Gamma(\Phi_1)\mu^{\infty}_f(z_0)+\sum_{\kappa\in K}\mu^{\infty}_{\beta_\kappa}(z_0),$$

$$\mu^{\infty}_{\Phi_2}(z_0) \le deg(\Phi_2)\mu^{\infty}_{f}(z_0) + \sum_{\lambda\in\Lambda}\mu^{\infty}_{\beta_\lambda}(z_0) \le \Gamma(\Phi_2)\mu^{\infty}_{f}(z_0) + \sum_{\lambda\in\Lambda}\mu^{\infty}_{\beta_\lambda}(z_0),$$

$$\mu^{\infty}_{\Phi_3}(z_0) \le \Gamma(\Phi_3)\mu^{\infty}_{f}(z_0) + \sum_{\eta\in H}\mu^{\infty}_{\beta_\eta}(z_0).$$

Let us assume $\Gamma = \max\{\Gamma(\Phi_1), \Gamma(\Phi_2), \Gamma(\Phi_3)\}$. Thus

$$\begin{aligned}&\max\left\{\mu^{\infty}_{\Phi_1}(z_0), \mu^{\infty}_{\Phi_2}(z_0), \mu^{\infty}_{\Phi_3}(z_0)\right\}\\ &\le \Gamma\cdot\mu^{\infty}_{f}(z_0) + \sum_{\kappa\in K}\mu^{\infty}_{\beta_\kappa}(z_0) + \sum_{\lambda\in\Lambda}\mu^{\infty}_{\beta_\lambda}(z_0) + \sum_{\eta\in H}\mu^{\infty}_{\beta_\eta}(z_0)\\ &= \Gamma\cdot\mu^{\infty}_{f}(z_0) + \sum_{\tau\in K\sqcup\Lambda\sqcup H}\mu^{\infty}_{\beta_\tau}(z_0).\end{aligned}$$

Using (3.6), (3.7) and the given condition $n_2 \ge n_1 + \Gamma$ we get

$$\begin{aligned}\mu^{\infty}_{\Omega}(z_0) &\le \mu^{\infty}_{A(f)}(z_0) - \mu^{\infty}_{B(f)}(z_0) + \max\left\{\mu^{\infty}_{\Phi_1}(z_0), \mu^{\infty}_{\Phi_2}(z_0), \mu^{\infty}_{\Phi_3}(z_0)\right\} \\ &\le (n_1 + \Gamma - n_2)\mu^{\infty}_{f}(z_0) + \sum_{\tau\in K\sqcup\Lambda\sqcup H}\mu^{\infty}_{\beta_\tau}(z_0) + \sum_{s=0}^{n_1}\mu^{\infty}_{a_s}(z_0) + \sum_{t=0}^{n_2}\mu^{0}_{b_t}(z_0)\\ &\le \sum_{\tau\in K\sqcup\Lambda\sqcup H}\mu^{\infty}_{\beta_\tau}(z_0) + \sum_{s=0}^{n_1}\mu^{\infty}_{a_s}(z_0) + \sum_{t=0}^{n_2}\mu^{0}_{b_t}(z_0).\end{aligned} \tag{3.8}$$

Case 2: Let us assume $\mu^{\infty}_{B(f)} \le 0$. So from (3.7) we get

$$\begin{aligned}&n_2\mu^{\infty}_{f}(z_0) - \sum_{t=0}^{n_2}\mu^{0}_{b_t}(z_0) \le 0\\ &\implies \mu^{\infty}_{f}(z_0) \le \frac{1}{n_2}\sum_{t=0}^{n_2}\mu^{0}_{b_t}(z_0).\end{aligned} \tag{3.9}$$

In the view of (3.9) from (1.1) we deduce

$$\begin{aligned}\mu^{\infty}_{\Omega}(z_0) &\le deg(\Omega)\mu^{\infty}_{f}(z_0) + \sum_{\xi\in\Xi}\mu^{\infty}_{\alpha_\xi}(z_0)\\ &\le \frac{deg(\Omega)}{n_2}\sum_{t=0}^{n_2}\mu^{0}_{b_t}(z_0) + \sum_{\xi\in\Xi}\mu^{\infty}_{\alpha_\xi}(z_0).\end{aligned} \tag{3.10}$$

From Case 1 and Case 2, combining (3.8) and (3.10) we obtain

$$\mu^{\infty}_{\Omega}(z_0) \le \sum_{\tau\in K\sqcup\Lambda\sqcup H}\mu^{\infty}_{\beta_\tau}(z_0) + \sum_{s=0}^{n_1}\mu^{\infty}_{a_s}(z_0) + \max\left\{1, \frac{deg(\Omega)}{n_2}\right\}\sum_{t=0}^{n_2}\mu^{0}_{b_t}(z_0) + \sum_{\xi\in\Xi}\mu^{\infty}_{\alpha_\xi}(z_0).$$

Thus

$$N(r,\Omega) \le \sum_{\xi\in\Xi}N(r,\alpha_\xi) + \sum_{\tau\in K\sqcup\Lambda\sqcup H}N(r,\beta_\tau) + \sum_{s=0}^{n_1}N(r,a_s) + l_2\sum_{t=0}^{n_2}N\Big(r,\frac{1}{b_t}\Big),$$

where $l_2 = \max\left\{1, \frac{deg(\Omega)}{n_2}\right\}$. This completes the proof of (ii) in *Theorem 1.1.* □

Proof of Theorem 1.2. Let us assume $f(z)$ be non-constant admissible meromorphic solution of (1.5). Now we have following two cases:
Case 1: Let $(\Phi_1+\Phi_2+\Phi_3)$ is non-constant. So from *Corollary 1.1* and *Definition 1.1* we get

$$T(r,\Omega)=S(r,f). \tag{3.11}$$

Notice that (1.5) can be written as

$$\frac{\Omega}{\sum_{i=1}^{3}\Phi_i}=\frac{A(f)}{B(f)}.$$

From the given condition it is obvious that $n_2>n_1$. Using *Lemma 2.3* and the fact $f(z)$ is admissible solution then we deduce

$$\begin{aligned} T\left(r,\frac{\Omega}{\sum_{i=1}^{3}\Phi_i}\right) &= T\left(r,\frac{A(f)}{B(f)}\right) \\ &= n_2T(r,f)+O\left(\sum_{s=0}^{n_1}T(r,a_s)+\sum_{t=0}^{n_2}T(r,b_t)\right) \\ &= n_2T(r,f)+S(r,f). \end{aligned} \tag{3.12}$$

With the help of (3.11) we obtain

$$\begin{aligned} T\left(r,\frac{\Omega}{\sum_{i=1}^{3}\Phi_i}\right) &\leq T(r,\Omega)+T\left(r,\sum_{i=1}^{3}\Phi_i\right)+O(1) \\ &\leq \sum_{i=1}^{3}T(r,\Phi_i)+S(r,f). \end{aligned} \tag{3.13}$$

As $f(z)$ is admissible so from *Remark 2.1, 2.2* we get the following two inequalities

$$\begin{aligned} \sum_{i=1}^{3}T(r,\Phi_i) &\leq \Big[\sum_{i=1}^{3}deg(\Phi_i)+(1-\Theta(\infty;f))\sum_{i=1}^{3}\gamma(\Phi_i)\Big]T(r,f) \\ &\quad+\sum_{\tau\in K\sqcup\Lambda\sqcup H}T(r,\beta_\tau)+S(r,f) \\ &= \Big[\sum_{i=1}^{3}deg(\Phi_i)+(1-\Theta(\infty;f))\sum_{i=1}^{3}\gamma(\Phi_i)\Big]T(r,f)+S(r,f), \end{aligned} \tag{3.14}$$

$$\begin{aligned} \sum_{i=1}^{3}T(r,\Phi_i) &\leq \Big(\sum_{i=1}^{3}\Gamma(\Phi_i)\Big)T(r,f)+\sum_{\tau\in K\sqcup\Lambda\sqcup H}T(r,\beta_\tau)+S(r,f) \\ &= \Big(\sum_{i=1}^{3}\Gamma(\Phi_i)\Big)T(r,f)+S(r,f). \end{aligned} \tag{3.15}$$

Thus from (3.13), (3.14), (3.15) we deduce

$$\begin{aligned} & T\left(r, \frac{\Omega}{\sum_{i=1}^{3}\Phi_i}\right) \\ \leq \quad & \min\Big\{\sum_{i=1}^{3} deg(\Phi_i) + (1-\Theta(\infty; f))\sum_{i=1}^{3}\gamma(\Phi_i), \sum_{i=1}^{3}\Gamma(\Phi_i)\Big\}\cdot T(r,f) \qquad (3.16) \\ & +S(r,f). \end{aligned}$$

Therefore from (3.12) and (3.16) we get

$$n_2 \leq \min\Big\{\sum_{i=1}^{3} deg(\Phi_i) + (1-\Theta(\infty; f))\sum_{i=1}^{3}\gamma(\Phi_i), \sum_{i=1}^{3}\Gamma(\Phi_i)\Big\}.$$

Case 2: Let $(\Phi_1 + \Phi_2 + \Phi_3)$ is constant. Then proceeding as similar method used in the proof of *Theorem 1.3* in [8], we can get $n_2 = 0, n_1 \leq deg(\Omega)$. Therefore this completes the proof of *Theorem 1.2.* □

4. Conclusion

Solutions of delay differential equations have lots of applications in various areas of mathematics as well as in physics. In this article, we consider a class of non-linear q-shift delay-differential equations and investigate the existence of admissible solutions.

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