TWMS J. App. and Eng. Math. V.13, N.2, 2023, pp. 661-669

BHARATH HUB NUMBER OF GRAPHS

V. MATHAD[1], ANAND[1] PUNEETH S.[1*], §

Abstract. The mathematical model of a real world problem is designed as Bharath hub number of graphs. In this paper, we study the graph theoretic properties of this variant. Also, we give results for Bharath hub number of join and corona of two connected graphs, cartesian product and lexicographic product of some standard graphs.

Keywords: Bharath hub number, join, corona, cartesian product, lexicographic product.

AMS Subject Classification: 05C69, 05C76, 05C99.

1. Introduction

Bharath, the land of treasure was in the clutches of many kings across several dynasties. As the desire for the treasure among them got increased, so did the rivalry between them. This was the pathway for the arise of conflicts with the neighboring territories. Along with negotiating the attack from the neighbors, withstanding the force of other invaders was the need of the hour. So, the strategy to defend one's region was designed in such a way that by deploying the troops using the minimum military resource, a path was established between any two undeployed regions through which the challenge of hostile was overcome in an optimal way.

Motivated by this, we represent the undefended regions by 0 whenever they are adjacent to defended regions which are weighted 2 and the regions which can defend themselves on their own are weighted either 1 or 2. Thus there always exist a path between any two undefended regions, safeguarding the motherland in the process.

Let $G = (V, E)$ be a simple graph of order n. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. The vertex in G with degree $n-1$ is called the *universal vertex* of G. A set S of vertices is called a 2- packing if for every pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. The 2-*packing number* $P_2(G)$ of G is the maximum cardinality

[1] Department of Mathematics, University of Mysore, Manasagangotri, Mysuru, India.
e-mail: veena_mathad@rediffmail.com; ORCID: https://orcid.org/0000-0002-6621-9596.
e-mail: anandbuoy@gmail.com; ORCID: https://orcid.org/0000-0002-1041-5264.
e-mail: puneethprajwal@gmail.com; ORCID: https://orcid.org/0000-0002-5532-696X.
[*] Corresponding author.
§ Manuscript received: February 16, 2021; accepted: April 30, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.2

of a 2-packing in G [1].

A hub set H of G is a set of vertices with the property that for any pair of vertices outside H, there is a path between them with all intermediate vertices in H. The *hub number* $h(G)$ is then defined to be the size of the smallest hub set of G[11]. A total hub set S of G is a subset of $V(G)$ such that every pair of vertices (whether adjacent or non-adjacent) of $V-S$ are connected by a path, whose all intermediate vertices are in S. The *total hub number* $h_t(G)$ is then defined to be the minimum cardinality of a total hub set of G [8].

A map $f: V(G) \to \{0,1,2\}$ is said to be *Bharath hub function* if for every pair of vertices $u, v \in V(G)$ with $f(u)=0$ and $f(v)=0$ there exists a path $u, v_1, v_2, ..., v_k, v$, $k \geq 1$ such that $f(v_1)=2=f(v_k)$ and $0 \leq f(v_i) \leq 2$, for $2 \leq i \leq k-1$ and if $f(v_i)=0$ then v_i must be adjacent to at least one $w \in V(G)$ with $f(w)=2$. Several variations of hub parameters of graphs and it's graph operations have been extensively studied in [6, 5, 7, 4, 8, 11]. *Bharath hub function* can also be considered as a variation of hub parameters.

A double star graph denoted by $S_{n,m}$ is a graph constructed from two star graphs $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. where, the vertex set $V(S_{n,m})$ is $V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, \ldots, v_{n-1}, u_0, u_1, \ldots, u_{m-1}\}$ and the edge set $E(S_{n,m}) = \{v_0u_0,\ v_0v_i,\ u_0u_j \mid 1 \leq i \leq (n-1); 1 \leq j \leq (m-1)\}$ [9].

Here are the standard definitions of several graph operations used in this paper. The Join or Sum of two graphs G and H, denoted by $G+H$ is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G) \text{ and } v \in V(H)\}$. The Corona of two graphs G and H, denoted by $G \circ H$ is the graph with $V(G \circ H) = V(G) \cup \left(\bigcup_{x \in V(G)} V(H_x)\right)$, where H_x is a copy of H all of whose vertices are adjacent to x for $x \in V(G)$ and $E(G \circ H) = E(G) \cup \left(\bigcup_{x \in V(G)} E(H_x)\right) \cup \{xy : x \in V(G), y \in V(H_x)\}$. The Cartesian product of two graphs G and H, denoted by $G \Box H$, is a graph such that $V(G \Box H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) of $G \Box H$ are adjacent iff either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$ [2]. The lexicographic product of graphs G and H is the graph $G[H]$ with the vertex set $V(G) \times V(H)$ and the edge set $E(G[H]) = \{(a,x)(b,y) \mid ab \in E(G), \text{or } a=b \text{ and } xy \in E(H)\}$ [10].

2. Properties of Bharath hub functions

For a graph $G=(V,E)$, let $f: V \to \{0,1,2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V \mid f(v)=i\}$ and $|V_i| = n_i$, for $i = 0,1,2$. Note that there exists a $1-1$ correspondence between the functions $f: V \to \{0,1,2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$.

A function $f = (V_0, V_1, V_2)$ is a *Bharath hub function (BHF)* if $V_2 \succ V_0$, where '$\succ$' means that the set V_2 dominates the set V_0 i.e $V_0 \subset N[V_2]$. The weight of f is $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1$. The *Bharath hub number*, denoted by $h_B(G)$ equals the minimum weight of an BHF of G , and we say that a function $f = (V_0, V_1, V_2)$ is a h_B- *function* if it is an BHF and $f(V) = h_B(G)$.

Proposition 2.1. *For any graph G, $h_t(G) \leq h_B(G) \leq 2h_t(G)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a h_B- function, and let S be a total hub set of G with minimum cardinality. Then, $V_1 \cup V_2$ is a total hub set of G and $(\emptyset, \emptyset, S)$ is a Bharath hub function. Hence, $h_t(G) = |S| \leq |V_1 \cup V_2| = |V_1| + |V_2| \leq |V_1| + 2|V_2| = h_B(G)$. But

$h_B(G) \leq 2|S| = 2h_t(G)$.

□

Proposition 2.2. *Let $f = (V_0, V_1, V_2)$ be any h_B- function. Then*
(a) *$G[V_1]$, the subgraph induced by V_1 has maximum degree 1.*
(b) *No edge of G joins V_1 and V_2.*
(c) *Each vertex of V_0 is adjacent to atmost two vertices of V_1.*
(d) *V_2 is a total hub set of $G[V_0 \cup V_2]$, which is the subgraph induced by $V_0 \cup V_2$.*

Proposition 2.3. *Let G be a graph. For any $h_B(G)$- function, $f = (V_0, V_1, V_2)$,*
(a) $|V_2| \leq h_B(G) - h_t(G)$.
(b) $|V_1| \geq 2h_t(G) - h_B(G)$.

Proof. Since $V_1 \cup V_2$ is a total hub set of G and $V_1 \cap V_2 = \emptyset$, we have $h_t(G) \leq |V_2| + |V_1|$. So, (a) is declared as $h_t(G) \leq 2|V_2| + |V_1| - |V_2| = h_B(G) - |V_2|$ and (b) is obtained as $2h_t(G) \leq 2|V_2| + 2|V_1| = 2|V_2| + |V_1| + |V_1| = h_B(G) + |V_1|$. □

Remark 2.1. *Let $f = (V_0, V_1, V_2)$ be a h_B- function of an isolate-free graph G, such that n_1 is minimum. Then*
(a) *$V_0 \cup V_2$ is a vertex cover.*
(b) *$V_0 \succ V_1$.*
(c) *Each vertex of V_0 is adjacent to atmost one vertex of V_1 i.e V_1 is a 2-packing.*

Definition 2.1. [1] *A Roman dominating function (RDF) on $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on G is called the Roman domination number of G.*

Proposition 2.4. *For any graph G, $\gamma_R(G) \leq h_B(G)$.*

Proof. Consider the Bharath hub function $f : V(G) \to \{0, 1, 2\}$ so that $f(V) = \sum_{u \in V} f(u)$ is minimum. Then for every $u, v \in V(G)$ with $f(u) = 0 = f(v)$ there exists at least one path $u, v_1, v_2, ..., v_k, v$ such that $f(v_1) = 2 = f(v_k)$ and $0 \leq f(v_i) \leq 2$, $2 \leq i \leq k - 1$. If $f(v_i) = 0$, for some i, then there exists $w \in N(v_i)$ such that $f(w) = 2$. This implies that, for every $u \in V(G)$ with $f(u) = 0$, there exists $v \in V(G)$ such that u is adjacent to v and $f(v) = 2$. Therefore, f is a Roman dominating function. □

Proposition 2.5. [1] *For any graph G of order n and maximum degree Δ,*

$$\frac{2n}{\Delta + 1} \leq \gamma_R(G).$$

Corollary 2.1. *For any graph G of order n and maximum degree Δ,*

$$\frac{2n}{\Delta + 1} \leq h_B(G).$$

Now, we illustrate the Bharath hub number by presenting the value of $h_B(G)$ for several classes of graphs.

Proposition 2.6. (a) $h_B(P_p) = p - 1,\ p \geq 3$.
(b) $h_B(C_p) = p - 1,\ p \geq 3$.
(c) $h_B(K_{m,n}) = 4,\ m \geq n$.
(d) $h_B(K_p) = 2,\ p \geq 2$.

(e) $h_B(K_{m_1,m_2,\ldots,m_n}) = 4,\ n \geq 3.$
(f) $h_B(W_n) = 2,\ n \geq 4.$
(g) $h_B(S_{n,m}) = 4,$ *for each* $n, m \geq 2.$

Proposition 2.7. *Let G be a connected graph of order $n \geq 2$. Then the following are equivalent.*

(i) $h_t(G) = 1,$
(ii) $h_B(G) = 2,$
(iii) *G contains a universal vertex.*

Proposition 2.8. *If G is an n-vertex graph, then $h_B(G) \leq n - \Delta(G) + 1$.*

Proof. If v is a vertex of maximum degree, then BHF $(N(v),\ V(G) - N[v]),\ \{v\})$ has weight $n - \Delta(G) + 1$. □

Proposition 2.9. *Let G be a connected graph of order $n \geq 2$. Then $h_B(G) = 3$ if and only if $\Delta(G) = n - 2$.*

Proof. Consider a connected graph G of order $n \geq 2$ with $h_B(G) = 3$. Let $f = (V_0, V_1, V_2)$ be a h_B- *function* of G. If $V_2 = \emptyset$, then $|V_1| = 3$ and thus, n=3. Since G is connected, G is isomorphic to K_3 or P_3, but $h_B(K_3) = h_B(P_3) = 2$, a contradiction. Thus $|V_1| = |V_2| = 1$. Let u and v be vertices with $f(v) = 2$ and $f(u) = 1$. By Proposition 2.2(b), $uv \notin E(G)$, then all vertices from $V(G) \setminus \{u, v\}$ are adjacent to v, implying $\Delta(G) = n - 2$. Now if $\Delta(G) = n - 2$, from Proposition 2.8 , we have $h_B(G) \leq n - (n - 2) + 1 = 3$. On the other hand, Proposition 2.7 implies that $h_B(G) \geq 3$. Therefore $h_B(G) = 3$. □

Proposition 2.10. *Let G be any isolate-free graph of even order n. Then $h_B(G) = n$ if and only if $G = \frac{n}{2}K_2$.*

Proof. If $G = \frac{n}{2}K_2$, then each edge contributes at least two to $h_B(G)$, and hence $h_B(G) = n$. Now suppose that $h_B(G) = n$. If G has two adjacent edges uv and vw, then $f = (V_0, V_1, V_2)$ where $V_0 = \{u, w\}$, $V_1 = V - \{u, v, w\}$ and $V_2 = \{v\}$ defines a *Bharath hub function*. Hence, $h_B(G) \leq |V_1| + 2|V_2| = n - 1$, which is a contradiction. Thus, no two edges of G are adjacent. As G is isolate-free, each component of G is K_2. Also since $h_B(G) = n$ it implies that $G = \frac{n}{2}K_2$. □

3. Bharath hub number of some graph operations

Theorem 3.1. *For any connected graphs G and H,*

$$h_B(G + H) = \begin{cases} 2, & \textit{if } G \textit{ and } H \textit{ are complete or} \\ & \textit{if } G \textit{ is complete and } H \textit{ is non-complete;} \\ min\{h_B(G), h_B(H)\}, & \textit{if } G \textit{ and } H \textit{ both are non-complete.} \end{cases}$$

Proof. Suppose G and H are complete graphs. Then $G + H$ is also complete. By Proposition 2.6(d), $h_B(G + H) = 2$.

Suppose G is complete and H is non-complete. In the join of G and H there exists an edge from each vertex of the complete graph to every other vertex of the non-complete graph. So, by assigning 2 as the weight to one of the fixed vertex of the complete graph and 0 to the rest of the vertices of $G + H$, there exists a path between any two zero weighted vertices of $G + H$ through a vertex weighted 2 on the complete graph. Therefore, $h_B(G + H) = 2$.

Suppose G and H both are non-complete. We initially calculate $h_B(G)$ and $h_B(H)$ separately. As join of G and H forces adjacency between every pair of vertices of G and H the weightage of one of these graphs alone is sufficient to consider for the Bharath hub function. Hence by the definition of Bharath hub number, $h_B(G+H) = \min\{h_B(G), h_B(H)\}$. □

Theorem 3.2. *For any connected graphs G and H, $h_B(G \circ H) \leq 2|V(G)|$ with equality if and only if $G = K_1$ or $H \neq K_1$.*

Proof. Let G and H be connected graphs. We consider the following cases.
Case 1. When $|V(G)| = 1$, $G \circ H$ is simply $G + H$. By Theorem 3.1,
$h_B(G \circ H) = 2 = 2|V(G)|$.
Case 2. Suppose $|V(G)| \geq 2$. We have the following subcases.
Subcase 2.1. Let $G = K_n$ and $H = K_1$. Assigning one of the fixed vertex of K_n say, $v \in V(K_n)$ the weight as 2, we find v adjacent to n number of vertices in $G \circ H$. So by giving weight as zero to those n vertices and 1 to all remaining $(n-1)$ pendant vertices, our BHF is achieved. Suppose if we make the weight of any of the pendant vertices as zero then that vertex will not be adjacent to any vertex weighted 2. Hence, $h_B(G \circ H) = 2 + (n-1) \cdot 1 = 2 + n - 1 = n + 1$. Thus, $h_B(G \circ H) < 2|V(G)|$.
Subcase 2.2. Let G be any graph and $H \neq K_1$. In $G \circ H$, every i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H. By assigning 2 as the weight to each vertex of G and 0 as the weight to the vertices in each copy of H, for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $G \circ H$ such that the vertex succeeding u and the vertex preceding v have weight 2. This is an BHF, as on changing the weight of i^{th} vertex of G to 1, we would not be able to find a vertex of weight 2 which is adjacent to the vertices of i^{th} copy of H which are weighted 0. Thus $h_B(G \circ H) = 2|V(G)|$. □

Theorem 3.3. *Let $3 \leq m \leq n$, $m, n \in \mathbb{Z}$, then $h_B(K_m \square K_n) = 2m$.*

Proof. Consider the vertex set of $K_m \square K_n$, $V(K_m \square K_n) = \{v_{ij} |\ 1 \leq i \leq m, 1 \leq j \leq n\}$ and assign weights to the vertices of $K_m \square K_n$ as given below:

$$f(v_{ij}) = \begin{cases} 2, & i = j; \\ 0, & i \neq j. \end{cases}$$

From this we are able to make all 0 weighted vertices adjacent to at least one vertex with weight 2. Thus, for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $K_m \square K_n$ such that the vertex succeeding u and the vertex preceding v have weight 2. Therefore, $h_B(K_m \square K_n) \leq 2m$. This is minimum because, if $0 \leq f(v_{ii}) \leq 1$ for some i, then no vertex of weight 2 would be adjacent to v_{mi}, which is weighted 0. Therefore, $h_B(K_m \square K_n) = 2m$.
□

Theorem 3.4. *Let G be a non-complete connected graph of order $l \geq 3$. Then*

$$h_B(G \square K_p) = \begin{cases} 2p, & \text{if } \Delta(G) = l-1,\ l > p; \\ 2p + m - 1, & \text{if } G = K_{m,n},\ m \leq n \text{ and } p = 1; \\ 2(p + m - 1), & \text{if } G = K_{m,n},\ 2 \leq m \leq n \text{ and } p \geq 2;\ l > p; \\ 2l, & l \leq p. \end{cases}$$

Proof. Let G and K_p be vertex disjoint graphs with $V(G) = \{1, 2, 3, ..., l\}$ and $V(K_p) = \{1, 2, 3, ..., p\}$. We represent the vertex set of $G \square K_p$ as $\{a_{ij} \mid 1 \leq i \leq l, 1 \leq j \leq p\}$. Consider the following cases.
Case 1. Let $l > p$, $\Delta(G) = l - 1$ and degree of i^{th} vertex in G be $l - 1$ for some i, $1 \leq i \leq l$. Then for fixed i in $G \square K_p$ we assign weight 2 to $a_{ij}'s$, $1 \leq j \leq p$ and 0 as weight to the

rest of the vertices, so that every 0 weighted vertex is adjacent to at least one vertex of weight 2. This forms an BHF, since for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $G\Box K_p$ such that the vertex succeeding u and the vertex preceding v have weight 2. This is minimum because, $l > p$ and if any of the 2 weighted vertex is weighted as 1, then the corresponding $a_{ij}'s$ which are weighted 0 would not be adjacent to any of the 2 weighted vertices.

Case 2. Let $l > p$, and $G = K_{m,n}$. Consider the partition of $V(G) = V_1 \cup V_2$ where $V_1 = \{v_1, v_2, ..., v_m\}$, $V_2 = \{u_1, u_2, ..., u_n\}$ and $V(K_p) = \{1, 2, ..., p\}$. Now, $V(G\Box K_p) = \{v_{ik},\ 1 \leq i \leq m,\ 1 \leq k \leq p\} \cup \{u_{jk},\ 1 \leq j \leq n,\ 1 \leq k \leq p\}$. We have the following subcases.

Subcase 2.1. For $p = 1,\ m \leq n$. As $l > p$ and $m \leq n$, if $f(v_{ix}) = 2$ for some $v_{ix}, 1 \leq i \leq m$ and $f(v_{rx}) = 1$, for every $r \neq i, 1 \leq r \leq m$, then v_{ix} is adjacent to u_{jx} for all $j,\ 1 \leq j \leq n$. Thus $u_{jx} = 0$, for all j. Here, for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $G\Box K_p$ such that the vertex succeeding u and the vertex preceding v have weight 2. Hence, $h_B(G\Box K_p) \leq 2p + (m-1)\cdot 1 = 2p + m - 1$. This is minimum because if $f(v_{kx}) = 0$ for some $v_{kx},\ k \neq i$ then there does not exist v_{ix} adjacent to v_{kx} such that $f(v_{ix}) = 2$. Therefore, $h_B(G\Box K_p) = 2p + m - 1$.

Subcase 2.2. For $p \geq 2,\ 2 \leq m \leq n,\ l > p$. As $m \leq n$, if $f(v_{ik}) = 2$ for some $i, 1 \leq i \leq m$ and for all $k,\ 1 \leq k \leq p$. Since v_{ik} is adjacent to u_{jk} for all $j,\ 1 \leq j \leq n$, we have $f(u_{jk}) = 0,$ for all $j, 1 \leq j \leq n, 1 \leq k \leq p$. Now, fixing one $k,\ 1 \leq k \leq p$, we assign weight 2 to the remaining $(m-1)$ vertices $v_{rk},\ r \neq i$. Also, for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $G\Box K_p$ such that the vertex succeeding u and the vertex preceding v have weight 2. Hence, $h_B(G\Box K_p) \leq 2p + (m-1)\cdot 2 = 2(p+m-1)$. This is minimum because if $f(v_{kx}) = 0$ for some $v_{kx},\ k \neq i$ then there does not exist v_{ix} adjacent to v_{kx} such that $f(v_{ix}) = 2$. Therefore, $h_B(G\Box K_p) = 2(p + m - 1)$.

Case 3. For $l \leq p$, let $V(G) = \{1, 2, ..., l\}$ and $V(K_p) = \{1, 2, ..., p\}$. Now, $V(G\Box K_p) = \{a_{ij},\ 1 \leq i \leq l,\ 1 \leq j \leq p\}$, We assign weights as follows. $f(a_{i1}) = 2, 1 \leq i \leq l$ and $f(a_{ij}) = 0, j \neq i,\ 1 \leq i \leq l$. By this, for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $G\Box K_p$ such that the vertex succeeding u and the vertex preceding v have weight 2. Hence, $h_B(G\Box K_p) \leq 2l$. This is minimum because if $f(v_{kx}) = 0$ for some $v_{kx},\ k \neq i$ then there does not exist v_{ix} adjacent to v_{kx} such that $f(v_{ix}) = 2$. Therefore, $h_B(G\Box K_p) = 2l$. □

Theorem 3.5. *Let $m \leq n,\ m, n \in \mathbb{Z}$, then $h_B(K_m[K_n]) = 2$.*

Proof. The Lexicographic product of two complete graphs is again a complete graph. Therefore by Proposition 2.6(d), $h_B(K_m[K_n]) = 2$. □

Theorem 3.6. *Let $3 \leq m \leq n,\ m, n \in \mathbb{Z}$, then*

$$h_B(P_m[P_n]) = \begin{cases} m, & \text{if } m \equiv 0 \pmod 4; \\ m+1, & \text{if } m \equiv 1, 3 \pmod 4; \\ m+2, & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

Proof. Let $\{a_{ij},\ 1 \leq i \leq m,\ 1 \leq j \leq n\}$, denote the vertices of $P_m[P_n]$. We consider the following cases. Here, in **Fig. 4.1 to 4.4**, the assignment of weights to $V(P_m[P_n])$ has been demonstrated in a matrix form for the corresponding cases.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3(n-1)} & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4(n-1)} & a_{4n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(m-3)1} & a_{(m-3)2} & \cdots & a_{(m-3)(n-1)} & a_{(m-3)n} \\ a_{(m-2)1} & a_{(m-2)2} & \cdots & a_{(m-2)(n-1)} & a_{(m-2)n} \\ a_{(m-1)1} & a_{(m-1)2} & \cdots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \cdots & a_{m(n-1)} & a_{mn} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Fig. 4.1

Case 1. $m \equiv 0 \pmod 4$. For $i \equiv 0, 1 \pmod 4$, we assign $f(a_{ij}) = 0$ and for $i \equiv 2, 3 \pmod 4$, we assign $f(a_{ij}) = 2$ for one fixed j and $f(a_{ij}) = 0$ for the remaining $j's$. Then, we get $\frac{m}{2}$ number of 2 weighted vertices. This forms the Bharath hub function, since for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $P_m[P_n]$ such that the vertex succeeding u and the vertex preceding v have weight 2. Therefore, $h_B(P_m[P_n]) \leq m$.
Case 2. We consider the following subcases.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3(n-1)} & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4(n-1)} & a_{4n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(m-2)1} & a_{(m-2)2} & \cdots & a_{(m-2)(n-1)} & a_{(m-2)n} \\ a_{(m-1)1} & a_{(m-1)2} & \cdots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \cdots & a_{m(n-1)} & a_{mn} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Fig. 4.2

Subcase 2.1. $m \equiv 3 \pmod 4$. Now on assigning weights as in Case 1, we get $\lceil \frac{m}{2} \rceil$ number of 2 weighted vertices. This forms the Bharath hub function, since for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $P_m[P_n]$ such that the vertex succeeding u and the vertex preceding v have weight 2. Therefore $h_B(P_m[P_n]) \leq 2\lceil \frac{m}{2} \rceil = \frac{2(m+1)}{2} = m+1$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3(n-1)} & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4(n-1)} & a_{4n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(m-4)1} & a_{(m-4)2} & \cdots & a_{(m-4)(n-1)} & a_{(m-4)n} \\ a_{(m-3)1} & a_{(m-3)2} & \cdots & a_{(m-3)(n-1)} & a_{(m-3)n} \\ a_{(m-2)1} & a_{(m-2)2} & \cdots & a_{(m-2)(n-1)} & a_{(m-2)n} \\ a_{(m-1)1} & a_{(m-1)2} & \cdots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \cdots & a_{m(n-1)} & a_{mn} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Fig. 4.3

Subcase 2.2. $m \equiv 1 \pmod 4$. This implies, $m - 2 \equiv 3 \pmod 4$. Now, assign weights for first $m - 2$ rows as in Subcase 2.1. Fix j in the next row and assign 2 to $a_{(m-1)j}$ and 0 to the remaining entries in that row. Finally, assign 0 to all the entries in last row.

Thus Bharath hub function is achieved, since for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $P_m[P_n]$ such that the vertex succeeding u and the vertex preceding v have weight 2. Therefore, $h_B(P_m[P_n]) \leq (m-2)+1+2 = m+1$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3(n-1)} & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4(n-1)} & a_{4n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(m-5)1} & a_{(m-5)2} & \cdots & a_{(m-5)(n-1)} & a_{(m-5)n} \\ a_{(m-4)1} & a_{(m-4)2} & \cdots & a_{(m-4)(n-1)} & a_{(m-4)n} \\ a_{(m-3)1} & a_{(m-3)2} & \cdots & a_{(m-3)(n-1)} & a_{(m-3)n} \\ a_{(m-2)1} & a_{(m-2)2} & \cdots & a_{(m-2)(n-1)} & a_{(m-2)n} \\ a_{(m-1)1} & a_{(m-1)2} & \cdots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \cdots & a_{m(n-1)} & a_{mn} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Fig. 4.4

Case 3. $m \equiv 2 \pmod 4$. This implies, $m-2 \equiv 0 \pmod 4$. Now, assign weights for first $m-2$ rows as in Case 1. Further by assigning 2 as the weight for a fixed j and 0 for the remaining $j's$ in the last two rows, Bharath hub function is attained. Since for any two vertices u, v with $f(u) = 0$ and $f(v) = 0$ there is a uv-path in $P_m[P_n]$ such that the vertex succeeding u and the vertex preceding v have weight 2. Thus, $h_B(P_m[P_n]) \leq m-2+4 = m+2$.

In all the above cases, the corresponding allotment of weights forms the h_B-*function*, since on changing the weight of any 2 weighted vertex, say a_{ij} to 1, there will be no more required path between $a_{(i-1)j}$ and $a_{(i+1)j}$. Hence the result. □

4. Conclusion

In this paper, we have introduced the concept of Bharath hub number of graphs and have initiated the study of corresponding parameters. Further, the scope can be extended to the field of Operation research that deals with the application of advanced analytical methods to help make better decisions. And the resource allocation problem seeks to find an optimal allocation of a fixed amount of resources to activities so as to minimize the cost incurred by the allocation.

References

[1] Cockayne, E. J., Dreyer Jr, P. A. and Hedetniemi, S. M., (2004), Roman domination in graphs, Discrete Mathematics, 278(1-3), pp. 11-22.

[2] Cuaresma Jr, E. C. and Paluga, R. N., (2015), On the Hub Number of Some Graphs, Annals of Studies in Science and Humanities, 1(1) pp. 17-24.

[3] Harary, F., (1969), Graph Theory, Addison Wesley, Reading Mass.

[4] Khalaf, S. I., Mathad, V. and Mahde, S. S., (2018), Hubtic number in graphs, Opuscula Mathematica, 38(6), pp. 841-847.

[5] Khalaf, S. I. and Mathad, V., (2019), On hubtic and restrained hubtic of a graph, TWMS Journal of Applied and Engineering Mathematics, 9(4), pp. 930-935.

[6] Khalaf, S. I. and Mathad, V., (2019), Restrained hub number in graphs, Bulletin of the International Mathematical Virtual Institute, 9, pp. 103-109.

[7] Khalaf, S. I. and Mathad, V., (2020), Hub and global hub numbers of a graph, Proceedings of the Jangjeon Mathematical Society, 23(2), pp. 231-239.

[8] Mathad, V., Sahal, A. M. and Kiran, S., (2014), The Total Hub Number of Graphs, Bulletin of International Mathematical Virtual Institute, 4, pp. 61-67.

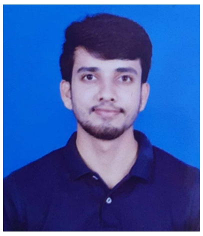
[9] Padmapriya, P. and Mathad, V., (2017), The eccentric-distance sum of some graphs, Electronic Journal of Graph Theory and Applications (EJGTA), 5, pp. 51-62.

[10] Šumenjak,T. K., Pavlič, P. and Tepeh, A., (2012), On the Roman domination in the lexicographic product of graphs, Discrete Applied Mathematics, 160(13-14), pp. 2030-2036.

[11] Walsh, M., (2006), The hub number of a graph, International Journal of Mathematics and Computer Science, 1, pp. 117-124.

[12] Yero, I. G. and Rodríguez-Velázquez,J. A., (2013), Roman domination of Cartesian product graphs and Strong product graphs, Applicable Analysis and Discrete Mathematics, 7, pp. 262-274 .

V. Mathad for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.12, N.4.



Anand got his B.Sc. degree in 2017, and M.Sc. degree in 2019 from University of Mysore. He is presently pursuing his Ph.D degree at the same university and carrying out his research in graph theory. His research interests are hub parameters and energy of graphs.

Puneeth Shivakumar got his B.Sc. degree in 2017, and M.Sc. degree in 2019 from Bangalore University. He is presently pursuing his Ph.D degree at University of Mysore and carrying out his research in graph theory. His research interests are hub parameters and graphical indices.