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# APPLICATIONS OF THE OPERATOR ${}_r\Phi_s$ IN $q$-POLYNOMIALS

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Abstract. We establish ${}_r\Phi_s$ as a general operator for many $q$-operators. A new polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y;q)$ are described as an extension of the bivariate Rogers-Szegö polynomial $h_n(x,y|q)$ and the generalized Al-Salam–Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$. With the use of the operator ${}_r\Phi_s$, we provide an operator proof of the generating function and its extension, Mehler's formula and its extension and Rogers formula and its extension to the polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y;q)$. The generating function and its extension, Mehler's formula and its extension and Rogers formula and its extension for $h_n(x,y|q)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ are deduced by giving special values to parameters of a new polynomial $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$.

Keywords: the $q$-operators, the bivariate Rogers-Szegö polynomials, the generalized Al-Salam–Carlitz $q$-polynomials, generating function, Mehler's formula, Rogers formula.

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## 1. Introduction

In this paper, the notations that was used in [9] is followed, and we assume that $|q|<1$. We're going to mention to a few notations for the $q$-series that we depend on during this paper.

Let $a\in\mathbb{C}$. The $q$-shifted factorial is given as follows [9]:

$$(a;q)_0=1,\quad (a;q)_m=\prod_{k=0}^{m-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),$$

and for the multiple $q$-shifted factorials by:

$$(a_1,a_2,\ldots,a_r;q)_m=(a_1;q)_m(a_2;q)_m\cdots(a_r;q)_m,$$

where $m\in\mathbb{Z}$ or $\infty$.

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The basic hypergeometric series ${}_r\phi_s$ is defined as [9]:

$$ {}_r\phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,x\right)=\sum_{m=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_m}{(q,b_1,\ldots,b_s;q)_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r}x^m. $$

The $q$-binomial coefficient is presented as follows [9]:

$$ \begin{bmatrix} n \\ k \end{bmatrix}=\begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0\leqslant k\leqslant n; \\ 0, & \text{otherwise}, \end{cases} $$

where $n,k\in\mathbb{N}$. The Cauchy identity is given by:

$$ \sum_{m=0}^{\infty}\frac{(a;q)_m}{(q;q)_m}\,x^m=\frac{(ax;q)_\infty}{(x;q)_\infty},\quad |x|<1. \tag{1} $$

Euler has given special case of Cauchy identity (1) as [9]:

$$ \sum_{m=0}^{\infty}\frac{x^m}{(q;q)_m}=\frac{1}{(x;q)_\infty},\quad |x|<1. \tag{2} $$

**Definition 1.1.** [3, 4, 12]. *The $D_q$ operator or the q-derivative is:*

$$ D_q\{f(a)\}=\frac{f(a)-f(aq)}{a}. $$

**Theorem 1.1.** [3, 12]. *For $m\geqslant 0$, we have*

$$ D_q^m\{f(a)g(a)\}=\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}q^{k(k-m)}D_q^k\{f(a)\}\,D_q^{m-k}\left\{g(aq^k)\right\}. \tag{3} $$

The following theorem is easy to demonstrate:

**Theorem 1.2.** [3]. *We have*

$$ D_q^k\{a^n\}=\frac{(q;q)_n}{(q;q)_{n-k}}\,a^{n-k}. \tag{4} $$

$$ D_q^k\left\{\frac{1}{(at;q)_\infty}\right\}=\frac{t^k}{(at;q)_\infty},\quad |at|<1. \tag{5} $$

Hahn polynomials which were first studied by Hahn [10] and then by Al-Salam and Carlitz [1] are defined as follows:

$$ \phi_n^{(a)}(x)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(a;q)_k\,x^k. $$

**Theorem 1.3.** [1].
*The generating function for $\phi_n^{(a)}(x)$ is:*

$$ \sum_{n=0}^{\infty}\phi_n^{(a)}(x)\frac{w^n}{(q;q)_n}=\frac{(axw;q)_\infty}{(w,xw;q)_\infty},\quad \max\{|w|,|xw|\}<1. \tag{6} $$

*Mehler's formula for $\phi_n^{(a)}(x)$ is:*

$$ \sum_{n=0}^{\infty}\phi_n^{(a)}(x)\phi_n^{(b)}(y)\frac{w^n}{(q;q)_n}=\frac{(axw,byw;q)_\infty}{(w,xw,yw;q)_\infty}\,{}_3\phi_2\left(\begin{array}{c} a,b,w \\ axw,byw \end{array};q,xyw\right), \tag{7} $$

*where* $\max\{|w|,|xw|,|yw|,|xyw|\}<1$.

The Cauchy polynomials are defined by [8] as:

$$P_n(x,y) = \begin{cases} (x-y)(x-qy)(x-q^2y)\cdots(x-q^{n-1}y), & \text{if } n>0; \\ 1, & \text{if } n=0. \end{cases} \tag{8}$$

The bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ were introduced in 2003 by Chen et al [2] as:

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,y),$$

where $P_k(x,y)$ is defined as in (8).

**Theorem 1.4.**

*The generating function for the polynomials* $h_n(x,y|q)$ [2]*:*

$$\sum_{n=0}^{\infty} h_n(x,y|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}, \quad \max\{|t|,|xt|\}<1. \tag{9}$$

*Mehler's formula for* $h_n(x,y|q)$ *is* [6]:

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n} = \frac{(yt,xvt;q)_\infty}{(t,xt,xut;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} y,xt,v/u \\ yt,xvt \end{matrix};q,ut\right), \tag{10}$$

*where* $\max\{|t|,|xt|,|ut|,|xut|\}<1.$

*The Rogers formula for* $h_n(x,y|q)$ *is* [6]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\ \frac{t^n}{(q;q)_n}\ \frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} y,xs \\ ys \end{matrix};q,t\right), \tag{11}$$

*where* $\max\{|t|,|s|,|xt|,|xs|\}<1.$

Saad and Sukhi [14], provided a new formula for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ in 2010, as follows:

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (y;q)_k\ x^{n-k}.$$

Notice that

$$\phi_n^{(y/x)}(x) = h_n(x,y|q). \tag{12}$$

$$\phi_n^{(y)}(1/x) = x^{-n}h_n(x,y|q). \tag{13}$$

By using equations (12) and (13), we can find polynomials identities for $h_n(x,y|q)$ from the polynomials identities for $\phi_n^{(a)}(x)$. For example setting $a=y$, $x\to 1/x$ and $w\to xt$ in the generating function for $\phi_n^{(a)}(x)$ (6), we get the generating function for $h_n(x,y|q)$ (9) and setting $a=y$, $x\to 1/x$, $w\to xt$, $b=v/u$, $y=u$ in the Mehler's formula for $\phi_n^{(a)}(x)$ (7), we get Mehler's formula for $h_n(x,y|q)$ (10).

The generalized Al-Salam–Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y)$ was introduced in 2020 by Srivastava and Arjika [16] as:

$$\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}\ \frac{(a_1,a_2,\ldots,a_{s+1};q)_k}{(b_1,b_2,\ldots,b_s;q)_k}\ x^k y^{n-k}.$$

**Theorem 1.5.** [16].
*The generating function for* $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)$ *is:*

$$\sum_{n=0}^{\infty} \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)\frac{t^n}{(q;q)_n} = \frac{1}{(yt;q)_\infty}\ {}_{s+1}\phi_s\left(\begin{array}{c} a_1,\cdots,a_{s+1} \\ b_1,\cdots,b_s \end{array};q,xt\right), \tag{14}$$

*where* $\max\{|xt|,|yt|\}<1$.
*The Rogers formula for* $\phi_{n+m}^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)$ *is:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{n+m}^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}$$
$$= \frac{1}{(yw,yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_{s+1};q)_n}{(q,b_1,\cdots,b_s;q)_n}(xt)^n\ {}_2\phi_0\left(\begin{array}{c} q^{-n},yt \\ - \end{array};q,wq^n/t\right), \tag{15}$$

*where* $\max\{|yw|,|yt|\}<1$.

Our paper is organized as follows: We define a generalized $q$-operator $_r\Phi_s$ in section 2, and then acquire some of its identities that will be used in the next sections. In section 3, we introduce a new polynomial $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$ and find its generating function and its extension, then deduce the generating function and its extension for $h_n(x,y|q)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$. In section 4, we derive Mehler's formula and its extension for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$, then we derive the Mehler formula and its extension for $h_n(x,y|q)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$. The Rogers formula and its extension for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$ are obtained in section 5, then we derive the Rogers formula and its extension for $h_n(x,y|q)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

## 2. The Operator $_r\Phi_s$ and some of its Operator Identities

In this section, the generalized $q$-operator $_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_q\right)$ is introduced, then find some of its operator identities.

**Definition 2.1.** *We define the generalized $q$-operator $_r\Phi_s$ as follows*:

$${}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_q\right) = \sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(cD_q)^n, \tag{16}$$

*where* $W_n = \dfrac{(a_1,\cdots,a_r;q)_n}{(b_1,\ldots,b_s;q)_n}$.

Some special values can be given to the generalized $q$-operator $_r\Phi s$ to obtain several previously defined operators, for details see [3, 7, 11, 16, 13, 15, 17].

We obtain the following operator identities by using the $q$-Leibniz formula (3):

**Theorem 2.1.** *We have*

$${}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_q\right)\left\{\frac{1}{(at,aw;q)_\infty}\right\}$$
$$= \frac{1}{(at,aw;q)_\infty}\sum_{n=0}^{\infty}\frac{(cw)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{W_{n+k}}{(q;q)_k}\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(aw;q)_k\,(ct)^k, \tag{17}$$

*where* $\max\{|at|,|aw|\}<1$.

*Proof.*

$$
\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_q\right)\left\{\frac{1}{(at,aw;q)_\infty}\right\} \\
&= \sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n D_q^n\left\{\frac{1}{(at,aw;q)_\infty}\right\} \qquad \text{(by using (16))} \\
&= \sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n \\
&\quad\times \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk} D_q^k\left\{\frac{1}{(at;q)_\infty}\right\} D_q^{n-k}\left\{\frac{1}{(awq^k;q)_\infty}\right\} \qquad \text{(by using (3))} \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{W_n}{(q;q)_{n-k}(q;q)_k}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n\, q^{k^2-nk}\frac{t^k}{(at;q)_\infty}\frac{(wq^k)^{n-k}}{(awq^k;q)_\infty} \\
&= \frac{1}{(at,aw;q)_\infty}\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{W_n}{(q;q)_{n-k}(q;q)_k}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n (aw;q)_k\, t^k\, w^{n-k} \\
&= \frac{1}{(at,aw;q)_\infty}\sum_{n=0}^{\infty}\frac{(cw)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{W_{n+k}}{(q;q)_k}\left[(-1)^{n+k} q^{\binom{n+k}{2}}\right]^{1+s-r}(aw;q)_k\,(ct)^k \ .
\end{aligned}
$$

□

Setting $w = 0$ in equation (17), we get the following corollary:

**Corollary 2.1.** *We have*

$$
\begin{aligned}
{}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_q\right)\left\{\frac{1}{(at;q)_\infty}\right\} &= \frac{1}{(at;q)_\infty}\sum_{k=0}^{\infty}\frac{W_k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(ct)^k \\
&= \frac{1}{(at;q)_\infty}\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ct\right), \qquad (18)
\end{aligned}
$$

*where* $|at| < 1$.

**Theorem 2.2.** *We have*

$$
\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,yD_q\right)\left\{\frac{x^k}{(xw,xt;q)_\infty}\right\} \\
&= \frac{1}{(xw,xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\ \sum_{i=0}^{k}W_{n+i}\left[(-1)^{n+i} q^{\binom{n+i}{2}}\right]^{1+s-r}\begin{bmatrix} k \\ i \end{bmatrix}(xw,xt;q)_i\, y^i\, x^{k-i} \\
&\quad\times \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}(xtq^i;q)_j\left(\frac{w}{t}\right)^j. \qquad (19)
\end{aligned}
$$

*Proof.*

$$
\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,yD_q\right)\left\{\frac{x^k}{(xw,xt;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} y^n D_q^n\left\{\frac{x^k}{(xw,xt;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} y^n \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-n)} D_q^i\left\{x^k\right\} D_q^{n-i}\left\{\frac{1}{(xwq^i,xtq^i;q)_\infty}\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by using (3))}\\
&=\sum_{i=0}^{\infty}\sum_{n=i}^{\infty}\frac{W_n}{(q;q)_{n-i}(q;q)_i}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} y^n q^{i(i-n)} D_q^i\left\{x^k\right\} D_q^{n-i}\left\{\frac{1}{(xwq^i,xtq^i;q)_\infty}\right\}\\
&=\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{W_{n+i}}{(q;q)_n(q;q)_i}\left[(-1)^{n+i} q^{\binom{n+i}{2}}\right]^{1+s-r} y^{n+i} q^{-ni} D_q^i\left\{x^k\right\} D_q^{n}\left\{\frac{1}{(xwq^i,xtq^i;q)_\infty}\right\}\\
&=\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{W_{n+i}}{(q;q)_n(q;q)_i}\left[(-1)^{n+i} q^{\binom{n+i}{2}}\right]^{1+s-r} y^{n+i} q^{-ni} D_q^i\left\{x^k\right\} \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-n)}\\
&\quad\times D_q^j\left\{\frac{1}{(xwq^i;q)_\infty}\right\} D_q^{n-j}\left\{\frac{1}{(xtq^{i+j};q)_\infty}\right\}\\
&=\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{W_{n+i}}{(q;q)_n(q;q)_i}\left[(-1)^{n+i} q^{\binom{n+i}{2}}\right]^{1+s-r} y^{n+i} q^{-ni}\frac{(q;q)_k}{(q;q)_{k-i}}\, x^{k-i}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-n)}\\
&\quad\times\frac{(wq^i)^j}{(xwq^i;q)_\infty}\frac{(tq^{i+j})^{n-j}}{(xtq^{i+j};q)_\infty}\\
&=\frac{1}{(xw,xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{i=0}^{k} W_{n+i}\left[(-1)^{n+i} q^{\binom{n+i}{2}}\right]^{1+s-r}\begin{bmatrix} k \\ i \end{bmatrix}(xw,xt;q)_i\; y^i\; x^{k-i}\\
&\quad\times\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}(xtq^i;q)_j\left(\frac{w}{t}\right)^j .
\end{aligned}
$$

□

If $w=0$ in equation (19), we obtain the following corollary:

**Corollary 2.2.** *We have*

$$
\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,yD_q\right)\left\{\frac{x^k}{(xt;q)_\infty}\right\}\\
&\quad=\frac{1}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{i=0}^{k} W_{n+i}\left[(-1)^{n+i} q^{\binom{n+i}{2}}\right]^{1+s-r}\begin{bmatrix} k \\ i \end{bmatrix}(xt;q)_i\; y^i x^{k-i}, \qquad (20)
\end{aligned}
$$

*where* $|at|<1$.

## 3. The Generating Function for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$

In this section a polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$ are defined. The generating function and its extension for the polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$ is obtained by using the operator ${}_r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{array};q,cD_q\right)$. We provide some special values for the parameters in the generating function as well as its extension for the polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$ to obtain the generating function and its extension for $h_n(x,y|q)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y)$.

**Definition 3.1.** *The polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$ are defined as follows:*

$$h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}W_k\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}x^{n-k}y^k, \tag{21}$$

*where* $W_k=\dfrac{(a_1,\cdots,a_r;q)_k}{(b_1,\cdots,b_s;q)_k}$.

Setting $r=1$, $s=0$, $y=1$ and then $a_1=y$ we get the new form for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ given by Saad and Sukhi [15]. Setting $r=s+1$ and exchanging $x$ and $y$, we get the generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ defined by Srivastava and Arjika [16].

By using (4), it is easy to prove that

$${}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,yD_q\right)\{x^n\}=h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q). \tag{22}$$

The following theorem can be easily demonstrated by using (22) and (18):

**Theorem 3.1.** (The generating function for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$). *We have*

$$\sum_{n=0}^{\infty}h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)\frac{t^n}{(q;q)_n}=\frac{1}{(xt;q)_\infty}\,{}_r\phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,yt\right), \tag{23}$$

*where* $|xt|<1$.

- Setting $r=1$, $s=0$, $y=1$ and then $a_1=y$ in equation (23), we recover the generating function for the bivariate Rogers-Szegö polynomial $h_n(x,y|q)$ (9).
- Setting $r=s+1$ and exchanging $x$ and $y$ in equation (23), we recover the generating function for the generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ (14).

Using (22) and (20), it is easy to prove the following theorem:

**Theorem 3.2.** (Extension for generating function for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$). *We have*

$$\sum_{n=0}^{\infty}h_{n+k}(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{x^k}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{i=0}^{k}W_{n+i}\left[(-1)^{n+i}q^{\binom{n+i}{2}}\right]^{1+s-r}\begin{bmatrix}k\\i\end{bmatrix}(xt;q)_i\,(y/x)^i,\quad |xt|<1. \tag{24}$$

- Setting $r=1$, $s=0$, $y=1$ and then $a_1=y$ in equation (24), we obtain an extension to generating function for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ as follows:

$$\sum_{n=0}^{\infty}h_{n+k}(x,y|q)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}\frac{(y,xt;q)_i}{(yt;q)_i}\,x^{k-i},$$

where $\max\{|x|,|xt|\}<1$.

- Setting $r = s+1$ and exchanging $x$ and $y$ in equation (24), we obtain an extension to the generating function for the generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y)$ as follows:

$$\sum_{n=0}^{\infty} \phi_{n+k}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{1}{(yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{i=0}^{k}\frac{(a_1,\cdots,a_{s+1};q)_{i+j}}{(b_1,\cdots,b_s;q)_{i+j}}\begin{bmatrix}k\\i\end{bmatrix}(yt;q)_i\, x^i y^{k-i}, \quad |yt|<1.$$

## 4. Mehler's Formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$

In this section, we plan to present an operator approach to Mehler's formula and its extension for the generalized polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$. By giving special values for variables in the Mehler's formula and its extension for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$, the Miller's formula and its extension for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ and the generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ are obtained.

**Theorem 4.1.** (Mehler's formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$). *We have*

$$\sum_{n=0}^{\infty} h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)h_n(\hat{a}_1,\cdots,\hat{a}_r;\hat{b}_1,\cdots,\hat{b}_s;z,c|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{1}{(xzt;q)_\infty}\sum_{m=0}^{\infty}\widehat{W}_m\,\frac{(xct)^m}{(q;q)_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r}\sum_{k=0}^{\infty}\frac{(yzt)^k}{(q;q)_k}$$
$$\times\sum_{i=0}^{m} W_{k+i}\left[(-1)^{k+i}q^{\binom{k+i}{2}}\right]^{1+s-r}\begin{bmatrix}m\\i\end{bmatrix}(xzt;q)_i\,(y/x)^i, \quad |xzt|<1. \tag{25}$$

*Proof.*

$$\sum_{n=0}^{\infty} h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)h_n(\hat{a}_1,\cdots,\hat{a}_r;\hat{b}_1,\cdots,\hat{b}_s;z,c|q)\frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} {}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,yD_q\right)\{x^n\}\,h_n(\hat{a}_1,\cdots,\hat{a}_r;\hat{b}_1,\cdots,\hat{b}_s;z,c|q)\frac{t^n}{(q;q)_n}$$
(by using (22))
$$= {}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,yD_q\right)\left\{\sum_{n=0}^{\infty} h_n(\hat{a}_1,\cdots,\hat{a}_r;\hat{b}_1,\cdots,\hat{b}_s;z,c|q)\frac{(xt)^n}{(q;q)_n}\right\}$$
$$= {}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,yD_q\right)\left\{\frac{1}{(xzt;q)_\infty}\,{}_r\phi_s\left(\begin{matrix}\hat{a}_1,\cdots,\hat{a}_r\\ \hat{b}_1,\cdots,\hat{b}_s\end{matrix};q,xct\right)\right\} \quad \text{(by using (23))}$$
$$= \sum_{m=0}^{\infty}\widehat{W}_m\,\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r}\frac{(ct)^m}{(q;q)_m}\,{}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,yD_q\right)\left\{\frac{x^m}{(xzt;q)_\infty}\right\}$$
$$= \frac{1}{(xzt;q)_\infty}\sum_{m=0}^{\infty}\widehat{W}_m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r}\frac{(xct)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(yzt)^k}{(q;q)_k}$$
$$\times\sum_{i=0}^{m} W_{k+i}\left[(-1)^{k+i}q^{\binom{k+i}{2}}\right]^{1+s-r}\begin{bmatrix}m\\i\end{bmatrix}(xzt;q)_i\,(y/x)^i. \quad \text{(by using (20))}$$

□

Now, we retrieve the Mehler's formula for $h_n(x,y|q)$ (10) by using special values for variables in the Mehler's formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$.

*Proof.* With $r=1$, $s=0$, $y=1$, $z=u$, $c=1$, $\hat{a}_1=v$ and then $a_1=y$ in equation (25), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n}\\
&\quad=\frac{1}{(xut;q)_\infty}\sum_{m=0}^{\infty}\frac{(v;q)_m}{(q;q)_m}\,(xt)^m\,\sum_{i=0}^{m}\begin{bmatrix} m\\ i\end{bmatrix}(xut;q)_i\,(\frac{1}{x})^i\,\sum_{k=0}^{\infty}\frac{(y;q)_{k+i}}{(q;q)_k}\,(ut)^k\\
&\quad=\frac{(yut;q)_\infty}{(xut,ut;q)_\infty}\sum_{i=0}^{\infty}\frac{(y,xut;q)_i}{(q,yut;q)_i}\,(\frac{1}{x})^i\sum_{m=0}^{\infty}\frac{(v;q)_{m+i}}{(q;q)_m}\,(xt)^{m+i}\qquad\text{(by using (1))}\\
&\quad=\frac{(yut;q)_\infty}{(xut,ut;q)_\infty}\sum_{i=0}^{\infty}\frac{(y,xut,v;q)_i}{(q,yut;q)_i}\,t^i\sum_{m=0}^{\infty}\frac{(vq^i;q)_m}{(q;q)_m}\,(xt)^m\\
&\quad=\frac{(yut,xvt;q)_\infty}{(xut,ut,xt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} y,xut,v\\ xvt,yut\end{matrix};q,t\right).\quad\text{(by using (1))}
\end{aligned}$$

Replacing $a$, $b$, $c$, $d$, $e$ by $y$, $xut$, $v$, $xvt$, $yut$, respectively, in transformations of ${}_3\phi_2$ series [9, Appendix III, equation (III.9)], we get the desired result. □

For $r=s+1$ and exchanging $x$ and $y$, $z$ and $c$ in equation (25), we get Mehler's formula for the generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ as follows:

**Corollary 4.1.** (Mehler's formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$). *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)\phi_n^{(\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})}(z,c|q)\frac{t^n}{(q;q)_n}\\
&\quad=\frac{1}{(yct;q)_\infty}\sum_{m=0}^{\infty}\frac{(\hat{a}_1,\cdots,\hat{a}_{s+1};q)_m}{(\hat{b}_1,\cdots,\hat{b}_s;q)_m}\,\frac{(yzt)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(xct)^k}{(q;q)_k}\\
&\quad\times\sum_{i=0}^{m}\frac{(a_1,\cdots,a_{s+1};q)_{k+i}}{(b_1,\cdots,b_s;q)_{k+i}}\,\begin{bmatrix} m\\ i\end{bmatrix}(yct;q)_i\,(x/y)^i,\quad |yct|<1.
\end{aligned}$$

**Theorem 4.2.** (Extension of Mehler's formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$). *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)h_{n+k}(\hat{a}_1,\cdots,\hat{a}_r;\hat{b}_1,\cdots,\hat{b}_s;z,c|q)\frac{t^n}{(q;q)_n}\\
&=\frac{1}{(xzt;q)_\infty}\sum_{j=0}^{\infty}\frac{(xct)^j}{(q;q)_j}\sum_{i=0}^{k}\begin{bmatrix} k\\ i\end{bmatrix}\widehat{W}_{i+j}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}(xzt;q)_i\,y^{k-i}\,c^i\\
&\times\sum_{n=0}^{\infty}\frac{\left(yztq^i\right)^n}{(q;q)_n}\sum_{l=0}^{j}\begin{bmatrix} j\\ l\end{bmatrix}W_{n+l}\left[(-1)^{n+l}q^{\binom{n+l}{2}}\right]^{1+s-r}(xztq^i;q)_l(\frac{y}{x})^l,\quad |xzt|<1.\qquad(26)
\end{aligned}$$

*Proof.*

$$\sum_{n=0}^{\infty} h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)h_{n+k}(\hat{a}_1,\cdots,\hat{a}_r;\hat{b}_1,\cdots,\hat{b}_s;z,c|q)\frac{t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty} {}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,yD_q\right)\{x^n\}\, h_{n+k}(\hat{a}_1,\cdots,\hat{a}_r;\hat{b}_1,\cdots,\hat{b}_s;z,c|q)\frac{t^n}{(q;q)_n}$$
$$\text{(by using (22))}$$
$$= {}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,yD_q\right)\left\{\sum_{n=0}^{\infty} h_{n+k}(\hat{a}_1,\cdots,\hat{a}_r;\hat{b}_1,\cdots,\hat{b}_s;z,c|q)\frac{(xt)^n}{(q;q)_n}\right\}$$
$$= {}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,yD_q\right)\left\{\frac{z^k}{(xzt;q)_\infty}\sum_{j=0}^{\infty}\frac{(xct)^j}{(q;q)_j}\sum_{i=0}^{k}\widehat{W}_{i+j}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}\begin{bmatrix}k\\ i\end{bmatrix}\right.$$
$$\left.\times (xzt;q)_i(\frac{c}{z})^i\right\}\quad \text{(by using (24))}$$
$$=\sum_{j=0}^{\infty}\frac{(ct)^j}{(q;q)_j}\sum_{i=0}^{k}\widehat{W}_{i+j}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}\begin{bmatrix}k\\ i\end{bmatrix}z^{k-i}c^i$$
$$\times\ {}_r\phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,yD_q\right)\left\{\frac{x^j}{(xztq^i;q)_\infty}\right\}$$
$$=\frac{1}{(xzt;q)_\infty}\sum_{j=0}^{\infty}\frac{(ct)^j}{(q;q)_j}\sum_{i=0}^{k}\widehat{W}_{i+j}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}\begin{bmatrix}k\\ i\end{bmatrix}z^{k-i}c^i$$
$$\times\sum_{n=0}^{\infty}\frac{\left(yztq^i\right)^n}{(q;q)_n}\sum_{l=0}^{j}W_{n+l}\left[(-1)^{n+l}q^{\binom{n+l}{2}}\right]^{1+s-r}\begin{bmatrix}j\\ l\end{bmatrix}(xztq^i;q)_l\ y^l\ x^{j-l}.\quad\text{(by using (20))}$$

□

Using $r=1$, $s=0$, $y=1$, $z=u$, $c=1$, $\hat{a}_1=v$ and then $a_1=y$ in equation (26), we get an extension of Mehler's formula for bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ as follows:

**Corollary 4.2** (Extension of Mehler's Formula for $h_n(x,y|q)$)**.** *We have*

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_{n+k}(u,v|q)\frac{t^n}{(q;q)_n}=\frac{1}{(xut;q)_\infty}\sum_{j=0}^{\infty}\frac{(xt)^j}{(q;q)_j}\sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}(v;q)_{i+j}u^{k-i}(xut;q)_i$$
$$\times\sum_{m=0}^{\infty}\frac{\left(utq^i\right)^m}{(q;q)_m}\sum_{l=0}^{j}\begin{bmatrix}j\\ l\end{bmatrix}(y;q)_{m+l}(xutq^i;q)_l(\frac{1}{x})^l,$$

*where* $|xut|<1$.

With $r=s+1$ and exchanging $x$ and $y$, $z$ and $c$ in equation (26), we get Mehler's formula for the generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ as follows:

**Corollary 4.3.** (Extension of Mehler's formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$). *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)\phi_{n+k}^{(\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})}(z,c|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{1}{(yct;q)_\infty}\sum_{j=0}^{\infty}\frac{(yzt)^j}{(q;q)_j}\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}\frac{(\hat{a}_1,\cdots,\hat{a}_{s+1};q)_{i+j}}{(\hat{b}_1,\cdots,\hat{b}_s;q)_{i+j}}\ (yct;q)_i\ x^{k-i}\ z^i$$
$$\times\sum_{n=0}^{\infty}\frac{\left(xctq^i\right)^n}{(q;q)_n}\sum_{l=0}^{j}\begin{bmatrix}j\\l\end{bmatrix}\frac{(a_1,\cdots,a_{s+1};q)_{n+l}}{(b_1,\cdots,b_s;q)_{n+l}}\ (yctq^i;q)_l,\quad |yct|<1.$$

## 5. Rogers Formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$

In this section, we intend to present an operator approach to Rogers formula and its extension for the generalized polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$. The Rogers formula and its extension for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ and the generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ are obtained by including special values for variables in the Rogers formula and its expansion for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y;q)$.

**Theorem 5.1.** (Roger's Formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$). *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}=\frac{1}{(xw,xt;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(yw)^n}{(q;q)_n}\sum_{n=0}^{\infty}\frac{W_{n+k}}{(q;q)_k}\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(yt)^k\,(xw;q)_k,\tag{27}$$

*where* $\max\{|xw|,|xt|\}<1$.

*Proof.*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,yD_q\right)\left\{x^{n+m}\right\}\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}\qquad\text{(by using (22))}$$
$$={}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,yD_q\right)\left\{\frac{1}{(xt,xw;q)_\infty}\right\}\qquad\text{(by using (2))}$$
$$=\frac{1}{(xt,xw;q)_\infty}\sum_{k=0}^{\infty}\frac{(yw)^n}{(q;q)_n}\sum_{n=0}^{\infty}\frac{W_{n+k}}{(q;q)_k}\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(yt)^k\,(xw;q)_k.$$

(by using (17))

□

Now, we recover the Rogers formula for $h_n(x,y|q)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ by using unique values for the variables in the Rogers formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$.

With $r=1$, $s=0$, $y=1$ and then $a_1=y$ and $w=s$ in equation (27), we recover Rogers formula for $h_n(x,y|q)$ (11).

For $r=s+1$ , and exchanging $x$ and $y$ in equation (27), we recover Rogers formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ (15).

**Theorem 5.2.** (Extension of Roger's formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$). *Let* $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$ *be defined as in* (21), *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m+k}(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}$$
$$=\frac{1}{(xw,xt;q)_\infty}\sum_{n=0}^{\infty}\frac{W_{n+i}}{(q;q)_n}\ (yt)^n\ \left[(-1)^{n+i}q^{\binom{n+i}{2}}\right]^{1+s-r}$$
$$\times\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}(xw,xt;q)_i\ x^{k-i}\ \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(xtq^i;q)_j\ \left(\frac{w}{t}\right)^j, \tag{28}$$

*where* $\max\{|xw|,|xt|\}<1$.

*Proof.*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m+k}(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,yD_q\right)\left\{x^{n+m+k}\right\}\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}\quad\text{(by using (22)).}$$
$$={}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,yD_q\right)\left\{\frac{x^k}{(xw,xt;q)_\infty}\right\}\quad\text{(by using (2))}$$
$$=\frac{1}{(xw,xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\ \sum_{i=0}^{k}W_{n+i}\left[(-1)^{n+i}q^{\binom{n+i}{2}}\right]^{1+s-r}\begin{bmatrix}k\\i\end{bmatrix}(xw,xt;q)_i\ y^i\ x^{k-i}$$
$$\times\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(xtq^i;q)_j\left(\frac{w}{t}\right)^j.\quad\text{(by using (19))}$$

□

Now, we get an extension of the Rogers formula for $h_n(x,y|q)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ by using specific values for variables in the extension of the Rogers formula for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$.

When $r=1$, $s=0$, $y=1$ and then $a_1=y$ and $w=s$ in equation (28), we get an extension of Roger's formula for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ as follows:

**Corollary 5.1** (Extension of Roger's Formula for $h_n(x,y|q)$). *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m+k}(x,y|q)\ \frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{1}{(xs,xt;q)_\infty}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{i=0}^{k}(y;q)_{n+i}\begin{bmatrix}k\\i\end{bmatrix}\ (xs,xt;q)_i\ x^{k-i}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\ (xtq^i;q)_j\ \left(\frac{s}{t}\right)^j,$$

*where* $\max\{|xs|,|xt|\}<1$.

With $r=s+1$ and exchanging $x$ and $y$ in equation (28), we get an extension of Roger's formula for generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ as follows:

**Corollary 5.2** (Extension of Roger's Formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$)**.** *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{n+m+k}^{(a,b)}(x,y|q)\ \frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}=\frac{1}{(yw,yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_{s+1};q)_n}{(q,b_1,\cdots,b_s;q)_n}\ (xt)^n$$
$$\times\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}\ (yw,yt;q)_i x^i\ y^{k-i}\ \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\ (ytq^i;q)_j\ \left(\frac{w}{t}\right)^j,$$

*where* $\max\{|yw|,|yt|\}<1$.

## 6. Conclusions

(1) Many operators can be obtained by assigning some special values to the generalized $q$-operator ${}_r\Phi s$.
(2) The bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ and the the generalized Al-Salam-Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ are special cases of the polynomials $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y;q)$.
(3) The polynomials identities for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y;q)$ are an extension of the polynomials identities for the bivariate Rogers-Szegö polynomial $h_n(x,y|q)$ and the generalized Al-Salam–Carlitz $q$-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

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**Husam Luti Saad** for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.12, N.2.

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