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DISTANCE EIGENVALUES, FORWARDING INDICES, AND DISTANCE-BASED TOPOLOGICAL INDICES OF COMPLEMENT OF TWO CIRCULANT NETWORKS

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Abstract. Let n, a, h, and m be positive integers such that $2 \leq a \leq \frac{n}{2}$ and $m \geq 2$. In this research, we compute the exact value of the distance spectral radius, vertex-forwarding index, and some distance-based topological indices of the connected complement of circulant networks $C_n(1, a)$ and $C_{n=m^h}(1, m, m^2, \ldots, m^{h-1})$. For $a \neq \frac{n}{2}$, the circulant network $C_n(1, a)$ is called a double loop network while the circulant network $C_{n=m^h}(1, m, m^2, \ldots, m^{h-1})$ is called the multiplicative circulant network on m^h vertices.

Keywords: Circulant networks, double loop network, multiplicative circulant graph, graph complement.

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1. Introduction

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For two vertices v_i and v_j in $V(\Gamma)$, the distance between them denoted by $d_\Gamma(v_i, v_j)$ is the length of a shortest path between v_i and v_j. The distance matrix of Γ denoted by $D(\Gamma)$ is the matrix whose $ij-$ entry is $d_\Gamma(v_i, v_j)$ if $v_i \neq v_j$, and 0 otherwise. The distance spectral radius of Γ denoted by $\rho(\Gamma)$ refers to the largest eigenvalue of $D(\Gamma)$.

A graph property related to distance between vertices in a graph is the distance-based topological index. A topological index is a real number associated to a graph which characterizes its topology. It is invariant under graph automorphism. A topological index is said to be distance-based if its computation involves distance between vertices in a graph.

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Many of the known distance-based topological indices have applications in Chemistry. For instance, the applications of the Wiener index W of a graph Γ defined by

$$W(\Gamma) = \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} d_\Gamma(v_i, v_j)$$

is presented in [12]. The Wiener index is the oldest distance-based topological index related to molecular branching. It was introduced by Harry Wiener in 1947, when he studied the relationship between the boiling point and the sum of the distances between any two carbon atoms of paraffin [27].

Another graph property that depends on distance between vertices in a graph is the concept of graph forwarding index. In order to discuss the graph forwarding index, we need to talk first about routings in a graph. In what follows are some of the definitions presented by Xu and Xu in [28].

Let Γ be a graph of order n. A routing R of Γ is a set of $n(n-1)$ elementary paths $R(x, y)$ specified for all ordered pairs (x, y) of vertices of Γ. If each of the paths specified by R is shortest, the routing R is said to be minimal, denoted by R_m. If $R(x, y) = R(y, x)$ specified by R, that is to say the path $R(y, x)$ is the reverse of the path $R(x, y)$ for all x and y, then the routing is symmetric. Finally, the set of all possible routings in Γ is denoted by $\mathcal{R}(\Gamma)$, and the subset of $\mathcal{R}(\Gamma)$ that contains all the minimal routings in Γ is denoted by $\mathcal{R}_m(\Gamma)$.

Now, let $R \in \mathcal{R}(\Gamma)$ and $x \in V(\Gamma)$. The load of a vertex x in R of Γ denoted by $\xi_x(\Gamma, R)$ is the number of paths specified by R passing through x and admitting x as an inner vertex. The vertex-forwarding index of Γ with respect to R, denoted by $\xi(\Gamma, R)$ is the maximum number of paths of R going through any vertex x in Γ. Hence

$$\xi(\Gamma, R) = \max\{\xi_x(\Gamma, R) : x \in V(\Gamma)\}.$$

As an illustrative example, let us consider the graph Γ shown in Figure 1.

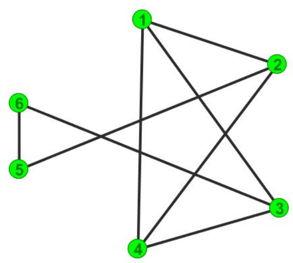

FIGURE 1. The graph Γ.

The sets

$$R_1 = \Big\{(1,2),(1,3),(1,4),(1,2,5),(1,3,6),(2,1),(2,1,3),(2,4),(2,5),(2,5,6),(3,1),(3,1,2),\\ (3,4),(3,6,5),(3,6),(4,1),(4,2),(4,3),(4,2,5),(4,3,6),(5,2,1),(5,2),(5,6,3),(5,2,4),\\ (5,6),(6,3,1),(6,5,2),(6,3),(6,3,4),(6,5)\Big\} \text{ and}$$

$$R_2 = \Big\{(1,4,2),(1,3),(1,4),(1,3,6,5),(1,3,6),(2,1),(2,1,3),(2,4),(2,5),(2,1,3,6),(3,1),\\ (3,1,2),(3,4),(3,4,1,2,5),(3,6),(4,1),(4,2),(4,3),(4,3,6,5),(4,3,6),(5,2,1),\\ (5,6,3,1,2),(5,6,3),(5,2,4),(5,6),(6,3,1),(6,5,2),(6,3),(6,3,4),(6,3,4,2,5)\Big\}$$

are routings of Γ. Observe that R_1 is a minimal routing while R_2 is not. Moreover, the load of vertex 3 in R_1 of Γ is 4, that is $\xi_3(\Gamma, R_1) = 4$. While the load of vertex 3 in R_2 of Γ is 9, that is $\xi_3(\Gamma, R_2) = 9$. Finally, it can be verified that the load of each vertex in the routing R_1 of Γ is given by: 1: 3, 2: 4, 3: 4, 4: 0, 5: 2, 6: 2. Hence, the forwarding index of Γ with respect to R_1 is 4. On the other hand, the load of each vertex in the routing R_2 of Γ is given by: 1: 7, 2: 7, 3: 9, 4: 3, 5: 2, 6: 2. Hence, the forwarding index of Γ with respect to R_2 is 9.

The vertex-forwarding index of Γ, denoted by $\xi(\Gamma)$ is the minimum forwarding index over all possible routings of Γ. In symbol,

$$\xi(\Gamma) = \min\{\xi(\Gamma, R) : R \in \mathcal{R}(\Gamma)\}.$$

A similar definition for the edge-forwarding index of a graph Γ denoted by $\pi(\Gamma)$ can be made by replacing the word "vertex" by "edge" in the definitions being stated.

The concept of graph forwarding indices is applied in network designs. This application was discussed in the works of Xu and Xu [28], Chung et al. [7], and Heydemann [10].

Recently, the exact value of some distance-based topological indices and vertex-forwarding index of some families of circulant networks were computed (see [4, 1, 2, 15, 14]).

Let G be a group and S be a subset of $G\backslash\{e\}$. A graph Γ is a Cayley graph of G with connection (or jump) set S, written $\Gamma = Cay(G, S)$ if $V(\Gamma) = G$ and $E(\Gamma) = \{\{g, sg\} : g \in G, s \in S\}$. If $G = \langle \mathbb{Z}_n, +_n\rangle$, then the graph $\Gamma = Cay(G, S)$ is called the circulant network with connection set S. The circulant network on n vertices with connection set S is denoted by $C_n(S)$. Note that for s and s^{-1} in $\mathbb{Z}_n$, we have $\{\{g, s +_n g\} : g \in \mathbb{Z}_n\} = \{\{g, s^{-1} +_n g\} : g \in \mathbb{Z}_n\}$. Hence, for a circulant network, we have $S \subseteq \{1, 2, \ldots, \frac{n}{2}\}$ if n is even and $S \subseteq \{1, 2, \ldots, \frac{n-1}{2}\}$ if n is odd.

Circulant networks can also be defined in terms of their adjacency matrix. In particular, circulant networks are graphs with circulant adjacency matrix. Recall, an $n \times n$ matrix M is said to be circulant if each row in M is rotated one element to the right relative to the preceding row. Figure 2 shows some examples of circulant networks.

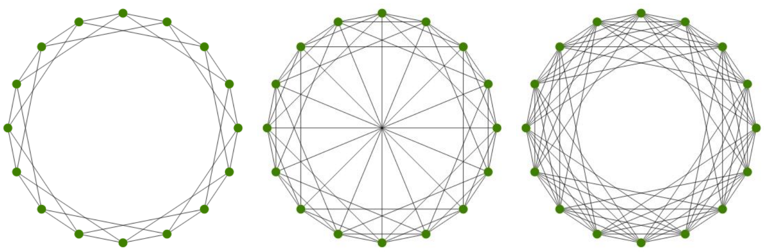

FIGURE 2. The circulant networks $C_{16}(1,3)$, $C_{16}(1,2,4,8)$ and $C_{16}(1,2,3,4,5)$ respectively.

Circulant networks have vast applications in different fields of study. Some of these fields include telecommunication networking [5], VLSI (Very-large-scale integration) design [13], and distributed computing [16]. Other applications of circulant networks are provided in the work of Monakhova [18], and the references therein.

This research note is motivated by our previous work [4] and the works of Ali et al. [1, 2], Liu and Meng [15], as well as Lin et al. [14]. In [4], we were able to determine the distance spectral radius, vertex-forwarding index, and bounds for the edge-forwarding index of the circulant network $C_{m^h}(1, m, m^2, \ldots, m^{h-1})$ where $m \geq 3$ is odd. In [1, 2], Ali et al. computed the Wiener, hyper-Wiener, and Schultz index of circulant network $C_n(1, a)$, where $a = 2, 3, 4$, and 5. While in [15], Liu and Meng computed the forwarding indices of 4−regular circulant networks. Finally, in [14], Lin et al. computed the exact values of the vertex-forwarding index of circulant networks with the following connection sets (i) $S = \{1, a\}$ where $a = \frac{n}{2}$, (ii) $S = \{1, \ldots, a\}$ where $2 \leq a \leq \frac{n}{2}$, and (iii) $S = \{1, a\}$ where $2 \leq a < \frac{n}{2}$. They also obtained an upper and lower bound for the edge-forwarding index of the said networks.

The main objective of this study is to determine the distance matrix of the connected complement of circulant networks (i) $C_n(1, a)$ and (ii) $C_{m^h}(1, m, m^2, \ldots, m^{h-1})$, where $2 \leq a \leq \frac{n}{2}$, $m \geq 2$ and $h \geq 1$. As a consequence, we have determined the exact value of the distance spectral radius and the vertex-forwarding index of the complement of circulant networks (i) and (ii). We also provide an upper and lower bound for the edge-forwarding index of the complement of the circulant networks (i) and (ii). Lastly, we compute for some well-known distance-based topological indices of the complement of circulant networks (i) and (ii).

2. Preliminaries

In this section, we define some important terms and state some useful results that will be used in the presentation of the main results.

2.1. **The Distance-based Topological Indices.** The distance-based topological indices that we will consider in this paper are given in Tables 1-4. Table 1 gives some of the most well-known purely distance-based topological indices. Table 2 gives some of the most well-known distance-degree-based topological indices. Recall, the degree of a vertex v_i refers to the number of edges incident to v_i. Moreover, we say that a graph Γ is vertex-regular if all the vertices in $V(\Gamma)$ have the same degree.

On the other hand, Table 3 gives some transmission-based topological indices of a graph. The transmission of a vertex v_i in Γ denoted by $Tr_\Gamma(v_i)$ or $\sigma_\Gamma(v_i)$ refers to the sum of the distances from v_i to all other vertices in $V(\Gamma)$. In terms of the distance matrix $\mathrm{D}(\Gamma)$, the transmission of vertex v_i is the sum of the entries of the row indexed by v_i in $D(\Gamma)$. Moreover, we say that a graph Γ is transmission-regular if all the vertices in $V(\Gamma)$ have the same transmission.

Finally, Table 4 gives some newly introduced reciprocal transmission-based topological indices. The reciprocal transmission of a vertex v_i in Γ denoted by $rs_\Gamma(v_i)$ refers to the sum of the reciprocal of the distances from v_i to all other vertices in $V(\Gamma)$. In terms of the distance matrix $\mathrm{D}(\Gamma)$, the reciprocal transmission of vertex v_i is the sum of the reciprocal of the entries of the row indexed by v_i in $D(\Gamma)$.

We note that some of the indices in Table 3 and all the indices in Table 4 first formally appeared in [21]. We also note that the Wiener index is also a transmission-based topological index while the Harary index is also a reciprocal transmission-based topological index.

Topological Index	**Mathematical Expression**	**Introduced by and Date Introduced**
Wiener[27]	$W(\Gamma) = \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} d_\Gamma(v_i, v_j)$	Wiener, 1947
Hyper-Wiener[22]	$WW(\Gamma) = \frac{1}{2} \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} \left[d_\Gamma(v_i, v_j) + d_\Gamma(v_i, v_j)^2\right]$	Randic, 1993
Harary([20],[11])	$H(\Gamma) = \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} \frac{1}{d_\Gamma(v_i, v_j)}$	Plavsic et al. & Ivanciuc et al., 1993

Table 1. Some distance-based topological indices.

Topological Index	**Mathematical Expression**	**Introduced by and Date Introduced**
Schultz[25]	$S(\Gamma) = \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} [deg_\Gamma(v_i) + deg_\Gamma(v_j)]d_\Gamma(v_i, v_j)$	Schultz, 1989
Gutman[8]	$G(\Gamma) = \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} [deg_\Gamma(v_i)deg_\Gamma(v_j)]d_\Gamma(v_i, v_j)$	Gutman, 1994
Additively weighted Harary[3]	$H_A(\Gamma) = \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} \frac{deg_\Gamma(v_i) + deg_\Gamma(v_j)}{d_\Gamma(v_i, v_j)}$	Alizadeh et al., 2013
Multiplicatively weighted Harary[3]	$H_M(\Gamma) = \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} \frac{deg_\Gamma(v_i) \cdot deg_\Gamma(v_j)}{d_\Gamma(v_i, v_j)}$	Alizadeh et al., 2013

Table 2. Some distance-degree-based topological indices.

Topological Index	**Mathematical Expression**	**Introduced by and Date Introduced**
T. geometric-arithmetic[19]	$T_{GA}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \frac{2\sqrt{\sigma(v_i)\sigma(v_j)}}{\sigma(v_i) + \sigma(v_j)}$	Narayankar & Selvan , 2017
T. sum-connectivity[24]	$T_{SC}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \frac{1}{\sqrt{\sigma(v_i) + \sigma(v_j)}}$	Sharafdini & Reti, 2020
T. arithmetic-geometric[21]	$T_{AG}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \frac{\sigma(v_i) + \sigma(v_j)}{2\sqrt{\sigma(v_i)\sigma(v_j)}}$	Ramane et al. , 2020
T. atom-bond connectivity[21]	$T_{ABC}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \sqrt{\frac{\sigma(v_i) + \sigma(v_j) - 2}{\sigma(v_i)\sigma(v_j)}}$	Ramane et al. , 2020
T. augmented Zagreb[21]	$T_{AZ}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \left[\frac{\sigma(v_i)\sigma(v_j)}{\sigma(v_i) + \sigma(v_j) - 2}\right]^3$	Ramane et al. , 2020

Table 3. Some transmission-based topological indices.

Topological Index	**Mathematical Expression**	**Introduced by and Date Introduced**
R.T. arithmetic-geometric[21]	$RT_{AG}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \frac{rs(v_i) + rs(v_j)}{2\sqrt{rs(v_i)rs(v_j)}}$	Ramane et al. , 2020
R.T. geometric-arithmetic[21]	$RT_{GA}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \frac{2\sqrt{rs(v_i)rs(v_j)}}{rs(v_i) + rs(v_j)}$	Ramane et al. , 2020
R.T. sum-connectivity[21]	$RT_{SC}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \frac{1}{\sqrt{rs(v_i) + rs(v_j)}}$	Ramane et al., 2020
R.T. atom-bond connectivity[21]	$RT_{ABC}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \sqrt{\frac{rs(v_i) + rs(v_j) - 2}{rs(v_i)rs(v_j)}}$	Ramane et al. , 2020
R.T. augmented Zagreb[21]	$RT_{AZ}(\Gamma) = \sum_{v_i,v_j\in E(\Gamma)} \left[\frac{rs(v_i)rs(v_j)}{rs(v_i) + rs(v_j) - 2}\right]^3$	Ramane et al. , 2020

Table 4. Some reciprocal transmission-based topological indices.

2.2. **Some Useful Results.** In this subsection, we state some important results that will be used in the presentation of the main results. We begin with the result involving the vertex-regularity of circulant networks.

Lemma 2.1. *Let $\Gamma = C_n(S)$ such that $|S| = k$. If $v \in V(\Gamma)$, then*

$$deg_\Gamma(v) = \begin{cases} 2k-1 & \text{if } \frac{n}{2} \in S \\ 2k & \text{otherwise.} \end{cases}$$

The next three results show that the distance matrix of the complement of a circulant network is also circulant. For a graph Γ, we denote by $\overline{\Gamma}$ the complement of Γ.

Lemma 2.2 (Lin et al. [14]). *If Γ is a circulant network, then $\boldsymbol{D}(\Gamma)$ is circulant.*

Lemma 2.3 (Meijer [17]). *If Γ is a circulant network, then $\overline{\Gamma}$ is also a circulant network.*

If we combine Lemma 2.2 and Lemma 2.3 we have

Corollary 2.1. *If Γ is a circulant network, then $\boldsymbol{D}(\overline{\Gamma})$ is circulant.*

On the other hand, if we combine Lemma 2.2 and Corollary 2.1 we get

Corollary 2.2. *If Γ is a circulant network, then Γ and $\overline{\Gamma}$ are transmission-regular.*

The connection of the vertex transmission to the distance spectral radius, vertex-forwarding index, and the bounds for the edge-forwarding index of a circulant network is given in the next series of useful results.

Lemma 2.4 (Lin et al. [14]). *Let Γ be a circulant network with distance spectral radius $\rho(\Gamma)$. If $v \in V(\Gamma)$, then*

$$\rho(\Gamma) = \sigma_\Gamma(v).$$

Lemma 2.5 (Lin et al. [14]). *If Γ is a connected circulant network of order n, then*

$$\xi(\Gamma) = \xi_m(\Gamma) = \rho(\Gamma) - (n-1).$$

Lemma 2.6 (Lin et al. [14]). *If Γ is a connected r-regular circulant network of order n, then*

$$\frac{2\rho(\Gamma)}{r} \leq \pi(\Gamma) \leq n + \rho(\Gamma) - (2r-1).$$

Before going to the last two final results in this section, we recall that the **diameter** of a graph Γ denoted by $diam(\Gamma)$ refers to the maximum distance between any pair of vertices in $V(\Gamma)$.

The two final results of this section are the following:

Lemma 2.7 (Gutman et al. [23]). *Let Γ be a graph with n number of vertices and m number of edges. If for any two adjacent vertices u and v in $V(\Gamma)$, there exists a third vertex w in $V(\Gamma)$ that is not adjacent to either u or v [also called Property *] then*

i $\overline{\Gamma}$ *is connected,*

ii the diameter of $\overline{\Gamma}$ *is two, and*

iii the Wiener index of $\overline{\Gamma}$ *satisfies the identity*

$$W(\overline{\Gamma}) = \binom{n}{2} + m.$$

Lemma 2.8 (Gutman et al. [9]). *If Γ is a connected graph with $diam(\Gamma) \geq 4$, then Γ has property *.*

Remark 2.1. *If a graph Γ satisfies the property *, then for any vertex $u, v \in V(\Gamma)$ we have*

$$d_{\overline{\Gamma}}(u,v) = \begin{cases} 2 & \text{if } u \text{ is adjacent to } v \text{ in } \Gamma \\ 0 & \text{if } u = v \\ 1 & \text{otherwise.} \end{cases}$$

3. Distance Matrix of Complement of Two Circulant Networks

In this section, we determine the distance matrix of the complement of circulant networks (i) and (ii). Note that in order to determine the distance matrix of a circulant network, it is enough to determine the distance of the 0-vertex to all the other vertices of the network. We begin by considering the complement of circulant network (i).

Theorem 3.1. *Let $n = 2a$. For $a \geq 4$ we have*

$$d_{\overline{C_n(1,a)}}(0,v) = \begin{cases} 0 & \text{if } v = 0 \\ 2 & \text{if } v \in \{1, a, n-1\} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We prove the theorem by considering two cases. The first case is when $4 \leq a \leq 7$. If $4 \leq a \leq 7$, we can manually construct the network $\overline{C_n(1,a)}$ and verify that the result holds.

The second case is when $a > 7$. If $a > 7$, it follows from Theorem 3.1 in [14] (the diameter of $C_n(1,a)$ is $\frac{a}{2}$ if a is even, while $\frac{a+1}{2}$ if a is odd) that $diam(C_n(1,a)) \geq 4$. Hence, by Lemma 2.7, Lemma 2.8, and Remark 2.1, the result follows. □

Theorem 3.2. *Let $n \geq 8$ and $2 \leq a < \frac{n}{2}$. If $v \in \overline{V(C_n(1,a))}$ then*

$$d_{\overline{C_n(1,a)}}(0,v) = \begin{cases} 0 & \text{if } v = 0 \\ 2 & \text{if } v \in \{1, a, n-a, n-1\} \\ 1 & \text{otherwise.} \end{cases}$$

With the exception for the circulant network $\overline{C_8(1,3)}$.

Proof. Here we also consider two cases. The first case is when $8 \leq n < 26$. If $8 \leq n < 26$, using a computing software, we verified that the result holds except for the complement of $C_8(1,3)$ since $\overline{C_8(1,3)}$ is disconnected.

For $n \geq 26$, we denote by $\delta(n) = \min\{diam(C_n(a,b)) : 1 \leq a < \frac{n}{2}, a \neq b\}$. Note that $\delta(n) \leq diam(C_n(1,a))$. Using the lower bound for $\delta(n)$ of Boesch and Wang [6] , we have

$$\left\lceil \frac{\sqrt{2n-1}-1}{2} \right\rceil \leq \delta(n) \leq diam(C_n(1,a)).$$

Note that the expression $\left\lceil \frac{\sqrt{2n-1}-1}{2} \right\rceil$ increases as n increases. So it is enough to find the minimum value of n such that $\left\lceil \frac{\sqrt{2n-1}-1}{2} \right\rceil = 4$. The solution of the last stated equation is $n = 26$. Thus, for $n \geq 26$, we have $diam(C_n(1,a)) \geq 4$. Using Lemma 2.7, Lemma 2.8, and Remark 2.1, the result follows. □

Remark 3.1. *If $n = 7$ for the family of complement circulant network that was considered in Theorem 3.2, we have*

$$d_{\overline{C_7(1,2)}}(0,v) = \begin{cases} 0 & \text{if } v = 0 \\ 1 & \text{if } v \in \{3,4\} \\ 2 & \text{if } v \in \{1,6\} \\ 3 & \text{if } v \in \{2,5\}, \end{cases}$$

$$d_{\overline{C_7(1,3)}}(0,v) = \begin{cases} 0 & \text{if } v = 0 \\ 1 & \text{if } v \in \{2,5\} \\ 2 & \text{if } v \in \{3,4\} \\ 3 & \text{if } v \in \{1,6\}. \end{cases}$$

Next, we determine the distance matrix of the complement of multiplicative circulant network on m^h vertices. For simplicity, we denote by Γ_{m^h} the multiplicative circulant network $C_{m^h}(1, m, m^2, \ldots, m^{h-1})$.

Theorem 3.3. *Let $v \in V(\overline{\Gamma_{m^h}})$. For $m \geq 5$ we have*

$$d_{\overline{\Gamma_{m^h}}}(0,v) = \begin{cases} 0 & \text{if } v = 0 \\ 2 & \text{if } v \in \{1, m, m^2, \ldots, m^{h-1}, m^h - m^{h-1}, \ldots, m^h - 1\} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. To prove the theorem, we use one of the results of Tang et al. in [26]. Using Theorem 4 in [26], we get

$$diam(\Gamma_{m^h}) = \begin{cases} \frac{h(m-1)+1}{2} & \text{if } m \text{ is even and } h \text{ is odd} \\ h\left(\frac{m-1}{2}\right) & \text{otherwise.} \end{cases}$$

Now, we consider two cases. The first case is when $5 \leq m \leq 8$. If $5 \leq m \leq 8$, using the diameter formula above reveals that for $h \geq 2$, we have $diam(\Gamma_{m^h}) \geq 4$. By Lemma 2.7, Lemma 2.8, and Remark 2.1, the result follows. For circulant networks $\overline{\Gamma_5}$, $\overline{\Gamma_6}$, $\overline{\Gamma_7}$, and $\overline{\Gamma_8}$, we manually calculated the distance matrix and verified that the result holds. Hence, the result is true for $\overline{\Gamma_{m^h}}$ where $5 \leq m \leq 8$.

The second case is when $m \geq 8$. If $m \geq 8$, using the diameter formula above reveals that for $h \geq 1$, $diam(\Gamma_{m^h}) \geq 4$. By Lemma 2.7, Lemma 2.8, and Remark 2.1, the result follows. □

Remark 3.2. *The complement networks $\overline{\Gamma_{2^h}}$ where $h \geq 4$, $\overline{\Gamma_{3^h}}$ where $h \geq 2$, and $\overline{\Gamma_{4^h}}$ where $h \geq 2$ also satisfies the result in Theorem 3.3.*

For $\overline{\Gamma_{2^3}}$, we have

$$d_{\overline{\Gamma_{2^3}}}(0,v) = \begin{cases} 0 & \text{if } v = 0 \\ 1 & \text{if } v \in \{3,5\} \\ 2 & \text{if } v \in \{2,6\} \\ 3 & \text{if } v \in \{1,7\} \\ 4 & \text{if } v = 4. \end{cases}$$

4. Forwarding Indices and Some Distance-based Topological Indices of $\overline{C_n(1,a)}$

In this section, we state some of the consequences of Theorem 3.1, Theorem 3.2, and Remark 3.1. We begin by considering the distance spectral radius of the complement circulant network $\overline{C_n(1,\frac{n}{2})}$. The result follows from the definition of distance spectral radius and Theorem 3.1.

Theorem 4.1. *Let $n = 2a$. If $a \geq 4$, then $\rho(\overline{C_n(1,a)}) = n + 2$.*

Another consequence of Theorem 3.1 talks about the reciprocal transmission of a vertex in $\overline{C_n(1,\frac{n}{2})}$.

Theorem 4.2. *Let $n = 2a$ where $a \geq 4$. If $v \in V(\overline{C_n(1,a)})$, then $rs_{\overline{C_n(1,a)}}(v) = \frac{2n-5}{2}$.*

For the vertex-forwarding index and bounds for the edge-forwarding index of the complement circulant network $\overline{C_n(1,\frac{n}{2})}$, they can be computed by combining Lemma 2.5, Lemma 2.6, and Theorem 3.1. The results are presented in the next two corollaries.

Corollary 4.1. *If $n = 2a$ where $a \geq 4$, then $\xi(\overline{C_n(1,a)}) = 3$.*

Corollary 4.2. *If $n = 2a$ where $a \geq 4$, then $\frac{2(n+2)}{n-4} \leq \pi(\overline{C_n(1,a)}) \leq 11$.*

The next series of results give the exact value of some distance-based topological indices of the complement circulant network $\overline{C_n(1,\frac{n}{2})}$. The results follow from the definition of the topological indices combined with Theorem 3.1, Theorem 4.1, Theorem 4.2, and the fact that $\overline{C_n(1,\frac{n}{2})}$ is a vertex-regular network with vertex-regularity $n-4$.

Corollary 4.3. *If $\overline{\Gamma} = \overline{C_n(1,\frac{n}{2})}$, then*

(i) $W(\overline{\Gamma}) = \frac{n(n+2)}{2}$
(ii) $S(\overline{\Gamma}) = n(n-4)(n+2)$
(iii) $G(\overline{\Gamma}) = \frac{n(n+2)(n-4)^2}{2}$
(iv) $WW(\overline{\Gamma}) = \frac{n(n+5)}{2}$
(v) $H(\overline{\Gamma}) = \frac{n(2n-5)}{4}$
(vi) $H_A(\overline{\Gamma}) = \frac{n(n-4)(2n-5)}{2}$
(vii) $H_M(\overline{\Gamma}) = \frac{n(2n-5)(n-4)^2}{4}$
(viii) $T_{AG}(\overline{\Gamma}) = \frac{n(n-4)}{2}$
(ix) $T_{GA}(\overline{\Gamma}) = \frac{n(n-4)}{2}$
(x) $T_{SC}(\overline{\Gamma}) = \frac{n(n-4)}{2\sqrt{2}\sqrt{n+2}}$
(xi) $T_{ABC}(\overline{\Gamma}) = \frac{n(n-4)\sqrt{n+1}}{\sqrt{2}(n+2)}$
(xii) $T_{AZ}(\overline{\Gamma}) = \frac{n(n-4)(n+2)^6}{16(n+1)^3}$
(xiii) $RT_{AG}(\overline{\Gamma}) = \frac{n(n-4)}{2}$
(xiv) $RT_{GA}(\overline{\Gamma}) = \frac{n(n-4)}{2}$
(xv) $RT_{SC}(\overline{\Gamma}) = \frac{n(n-4)}{2\sqrt{2n-5}}$
(xvi) $RT_{ABC}(\overline{\Gamma}) = \frac{n(n-4)\sqrt{2n-7}}{2n-5}$
(xvii) $RT_{AZ}(\overline{\Gamma}) = \frac{n(n-4)(2n-5)^6}{128(2n-7)^3}$.

Now, we consider the distance spectral radius of complement circulant networks $\overline{C_7(1,2)}$ and $\overline{C_7(1,3)}$. The results follow from the definition of distance spectral radius and Remark 3.1.

Theorem 4.3. *If $\overline{\Gamma_1} = \overline{C_7(1,2)}$ and $\overline{\Gamma_2} = \overline{C_7(1,3)}$, then $\rho(\overline{\Gamma_1}) = \rho(\overline{\Gamma_2}) = 12$.*

Another consequence of Remark 3.1 talks about the reciprocal transmission of a vertex in $\overline{C_7(1,2)}$ and $\overline{C_7(1,3)}$.

Theorem 4.4. *Let $\overline{\Gamma_1} = \overline{C_7(1,2)}$ and $\overline{\Gamma_2} = \overline{C_7(1,3)}$. If $v_1 \in \overline{\Gamma_1}$ and $v_2 \in \overline{\Gamma_2}$, then $rs_{\overline{\Gamma_1}}(v_1) = rs_{\overline{\Gamma_2}}(v_2) = \frac{11}{3}$.*

For the vertex-forwarding index and bounds for the edge-forwarding index of complement circulant networks $\overline{C_7(1,2)}$ and $\overline{C_7(1,3)}$, they can be computed by combining Lemma 2.5, Lemma 2.6, and Remark 3.1. The results are presented in the next two corollaries.

Corollary 4.4. *If* $\overline{\Gamma_1} = \overline{C_7(1,2)}$ *and* $\overline{\Gamma_2} = \overline{C_7(1,3)}$, *then* $\xi(\overline{\Gamma_1}) = \xi(\overline{\Gamma_2}) = 6$.

Corollary 4.5. *If* $\overline{\Gamma_1} = \overline{C_7(1,2)}$ *and* $\overline{\Gamma_2} = \overline{C_7(1,3)}$, *then* $12 \leq \pi(\overline{\Gamma_1}) = \pi(\overline{\Gamma_2}) \leq 16$.

The next series of results give the exact value of some distance-based topological indices of complement circulant networks $\overline{C_7(1,2)}$ and $\overline{C_7(1,3)}$. The results follow from the definition of the topological indices combined with Remark 3.1, Theorem 4.3, Theorem 4.4 and the fact that the two networks are vertex-regular network with vertex-regularity 2.

Corollary 4.6. *Let* $\overline{\Gamma}$ *denote either* $\overline{C_7(1,2)}$ *or* $\overline{C_7(1,3)}$.. *Then*

(i) $W(\overline{\Gamma}) = 42$
(ii) $S(\overline{\Gamma}) = 168$
(iii) $G(\overline{\Gamma}) = 168$
(iv) $WW(\overline{\Gamma}) = 70$
(v) $H(\overline{\Gamma}) = \frac{77}{6}$
(vi) $H_A(\overline{\Gamma}) = \frac{154}{3}$
(vii) $H_M(\overline{\Gamma}) = \frac{154}{3}$
(viii) $T_{AG}(\overline{\Gamma}) = 7$
(ix) $T_{GA}(\overline{\Gamma}) = 7$
(x) $T_{SC}(\overline{\Gamma}) = \frac{7\sqrt{6}}{12}$
(xi) $T_{ABC}(\overline{\Gamma}) = \frac{7\sqrt{22}}{12}$
(xii) $T_{AZ}(\overline{\Gamma}) = \frac{2\ 612\ 736}{1\ 331}$
(xiii) $RT_{AG}(\overline{\Gamma}) = 7$
(xiv) $RT_{GA}(\overline{\Gamma}) = 7$
(xv) $RT_{SC}(\overline{\Gamma}) = \frac{7\sqrt{66}}{22}$
(xvi) $RT_{ABC}(\overline{\Gamma}) = \frac{28\sqrt{3}}{11}$
(xvii) $RT_{AZ}(\overline{\Gamma}) = \frac{12\ 400\ 927}{110\ 592}$.

Finally, we consider the complement circulant network $\overline{C_n(1,a)}$ where $n \geq 8$ and $2 \leq a < \frac{n}{2}$. We first determine its distance spectral radius. The result follows from the definition of distance spectral radius and Theorem 3.2.

Theorem 4.5. *Let* $\overline{\Gamma} = \overline{C_n(1,a)}$ *where* $2 \leq a < \frac{n}{2}$. *If* $n \geq 8$, *then* $\rho(\overline{\Gamma}) = n+3$.

Another consequence of Theorem 3.2 talks about the reciprocal transmission of a vertex in $\overline{C_n(1,a)}$ where $2 \leq a < \frac{n}{2}$.

Theorem 4.6. *Let* $\overline{\Gamma} = \overline{C_n(1,a)}$ *where* $n \geq 8$ *and* $2 \leq a < \frac{n}{2}$. *If* $v \in V(\overline{\Gamma})$, *then* $rs_{\overline{\Gamma}}(v) = n-3$.

For the vertex-forwarding index and bounds for the edge-forwarding index of the complement circulant network $\overline{C_n(1,a)}$ where $n \geq 8$ and $2 \leq a < \frac{n}{2}$, they can be computed by combining Lemma 2.5, Lemma 2.6, and Theorem 3.2. The results are presented in the next two corollaries.

Corollary 4.7. *If* $\overline{\Gamma} = \overline{C_n(1,a)}$ *where* $n \geq 8$ *and* $2 \leq a < \frac{n}{2}$, *then* $\xi(\overline{\Gamma}) = 4$.

Corollary 4.8. *If* $\overline{\Gamma} = \overline{C_n(1,a)}$ *where* $n \geq 8$ *and* $2 \leq a < \frac{n}{2}$, *then* $\frac{2(n+3)}{n-5} \leq \pi(\overline{\Gamma}) \leq 14$.

The next series of results give the exact value of some distance-based topological indices of complement circulant network $\overline{C_n(1,a)}$ where $n \geq 8$ and $2 \leq a < \frac{n}{2}$. The results follow from the definition of the topological indices combined with Theorem 3.2, Theorem 4.5, Theorem 4.6 and the fact that the network is vertex-regular with vertex-regularity $n-5$.

Corollary 4.9. *If* $\overline{\Gamma} = \overline{C_n(1,a)}$ *where* $n \geq 8$ *and* $2 \leq a < \frac{n}{2}$, *then*

(i) $W(\overline{\Gamma}) = \frac{n(n+3)}{2}$
(ii) $S(\overline{\Gamma}) = n(n-5)(n+3)$
(iii) $G(\overline{\Gamma}) = \frac{n(n+3)(n-5)^2}{2}$
(iv) $WW(\overline{\Gamma}) = \frac{n(2n+14)}{4}$
(v) $H(\overline{\Gamma}) = \frac{n(n-3)}{2}$
(vi) $H_A(\overline{\Gamma}) = n(n-5)(n-3)$

(vii) $H_M(\overline{\Gamma}) = \frac{n(n-3)(n-5)^2}{2}$
(viii) $T_{AG}(\overline{\Gamma}) = \frac{n(n-5)}{2}$
(ix) $T_{GA}(\overline{\Gamma}) = \frac{n(n-5)}{2}$
(x) $T_{SC}(\overline{\Gamma}) = \frac{n(n-5)}{2\sqrt{2}\sqrt{n+3}}$
(xi) $T_{ABC}(\overline{\Gamma}) = \frac{n(n-5)\sqrt{n+2}}{\sqrt{2}(n+3)}$
(xii) $T_{AZ}(\overline{\Gamma}) = \frac{n(n-5)(n+3)^6}{16(n+2)^3}$
(xiii) $RT_{AG}(\overline{\Gamma}) = \frac{n(n-5)}{2}$
(xiv) $RT_{GA}(\overline{\Gamma}) = \frac{n(n-5)}{2}$
(xv) $RT_{SC}(\overline{\Gamma}) = \frac{n(n-5)}{2\sqrt{2}\sqrt{n-3}}$
(xvi) $RT_{ABC}(\overline{\Gamma}) = \frac{n(n-5)\sqrt{n-4}}{\sqrt{2}(n-3)}$
(xvii) $RT_{AZ}(\overline{\Gamma}) = \frac{n(n-5)(n-3)^6}{16(n-4)^3}$.

5. Forwarding Indices and Some Distance-based Topological Indices of $\overline{C_{m^h}(1, m, m^2, \ldots, m^{h-1})}$

In this section, we state some of the consequences of Theorem 3.3 and Remark 3.2. We first consider the distance spectral radius of the complement circulant network $\overline{C_{2^3}(1, 2, 2^2)}$. The result follows from the definition of distance spectral radius and Remark 3.2.

Theorem 5.1. *If $\overline{\Gamma_{2^3}} = \overline{C_{2^3}(1, 2, 2^2)}$, then $\rho(\overline{\Gamma_{2^3}}) = 16$.*

Another consequence of Remark 3.2 talks about the reciprocal transmission of a vertex in $\overline{C_{2^3}(1, 2, 2^2)}$.

Theorem 5.2. *Let $\overline{\Gamma_{2^3}} = \overline{C_{2^3}(1, 2, 2^2)}$. If $v \in V(\overline{\Gamma})$, then $rs_{\overline{\Gamma_{2^3}}}(v) = \frac{47}{12}$.*

For the vertex-forwarding index and bounds for the edge-forwarding index of the complement circulant network $\overline{C_{2^3}(1, 2, 2^2)}$, it can be computed by combining Lemma 2.5, Lemma 2.6, and Remark 3.2. The results are presented in the next two corollaries.

Corollary 5.1. *If $\overline{\Gamma_{2^3}} = \overline{C_{2^3}(1, 2, 2^2)}$, then $\xi(\overline{\Gamma_{2^3}}) = 9$.*

Corollary 5.2. *If $\overline{\Gamma_{2^3}} = \overline{C_{2^3}(1, 2, 2^2)}$, then $16 \leq \pi(\overline{\Gamma_{2^3}}) \leq 21$.*

The next series of results give the exact value of some distance-based topological indices of complement circulant network $\overline{C_{2^3}(1, 2, 2^2)}$. The results follow from the definition of the topological indices combined with Remark 3.2, Theorem 5.1, Theorem 5.2 and the fact that the network is vertex-regular network with vertex-regularity 2.

Corollary 5.3. *If $\overline{\Gamma_{2^3}} = \overline{C_{2^3}(1, 2, 2^2)}$, then*

(i) $W(\overline{\Gamma_{2^3}}) = 64$
(ii) $S(\overline{\Gamma_{2^3}}) = 256$
(iii) $G(\overline{\Gamma_{2^3}}) = 256$
(iv) $WW(\overline{\Gamma_{2^3}}) = 120$
(v) $H(\overline{\Gamma_{2^3}}) = \frac{47}{3}$
(vi) $H_A(\overline{\Gamma_{2^3}}) = \frac{188}{3}$
(vii) $H_M(\overline{\Gamma_{2^3}}) = \frac{188}{3}$
(viii) $T_{AG}(\overline{\Gamma_{2^3}}) = 8$
(ix) $T_{GA}(\overline{\Gamma_{2^3}}) = 8$
(x) $T_{SC}(\overline{\Gamma_{2^3}}) = \sqrt{2}$
(xi) $T_{ABC}(\overline{\Gamma_{2^3}}) = \frac{\sqrt{30}}{2}$
(xii) $T_{AZ}(\overline{\Gamma_{2^3}}) = \frac{16\ 777\ 216}{3\ 375}$
(xiii) $RT_{AG}(\overline{\Gamma_{2^3}}) = 8$
(xiv) $RT_{GA}(\overline{\Gamma_{2^3}}) = 8$
(xv) $RT_{SC}(\overline{\Gamma_{2^3}}) = \frac{8\sqrt{282}}{47}$
(xvi) $RT_{ABC}(\overline{\Gamma_{2^3}}) = \frac{16\sqrt{210}}{47}$
(xvii) $RT_{AZ}(\overline{\Gamma_{2^3}}) = \frac{10\ 779\ 215\ 329}{74\ 088\ 000}$.

Next, we consider the distance spectral radius of $\overline{C_{2^h}(1, 2, 2^2, \ldots, 2^{h-1})}$ where $h \geq 4$. The result follows from the definition of distance spectral radius and Remark 3.2.

Theorem 5.3. *Let $\overline{\Gamma_{2^h}} = \overline{C_{2^h}(1, 2, 2^2, \ldots, 2^{h-1})}$. If $h \geq 4$, then $\rho(\overline{\Gamma_{2^h}}) = n + 2h - 2$.*

Another consequence of Remark 3.2 talks about the reciprocal transmission of a vertex in $\overline{C_{2^h}(1, 2, 2^2, \ldots, 2^{h-1})}$.

Theorem 5.4. *Let $\overline{\Gamma_{2^h}} = \overline{C_{2^h}(1, 2, 2^2, \ldots, 2^{h-1})}$ and $h \geq 4$. If $v \in V(\overline{\Gamma_{2^h}})$, then $rs_{\overline{\Gamma_{2^h}}}(v) = n - h - \frac{1}{2}$.*

For the vertex-forwarding index and bounds for the edge-forwarding index of complement circulant network $\overline{C_{2^h}(1,2,2^2,\ldots,2^{h-1})}$, it can be computed by combining Lemma 2.5, Lemma 2.6, and Remark 3.2. The results are presented in the next two corollaries.

Corollary 5.4. *Let* $\overline{\Gamma_{2^h}} = \overline{C_{2^h}(1,2,2^2,\ldots,2^{h-1})}$. *If* $h \geq 4$, *then* $\xi(\overline{\Gamma_{2^h}}) = 2h-1$.

Corollary 5.5. *Let* $\overline{\Gamma_{2^h}} = \overline{C_{2^h}(1,2,2^2,\ldots,2^{h-1})}$. *If* $h \geq 4$, *then* $\frac{2(n+2h-2)}{n-2h} \leq \pi(\overline{\Gamma_{2^h}}) \leq 6h-1$.

The next series of results give the exact values of some distance-based topological indices of the complement circulant network $\overline{C_{2^h}(1,2,2^2,\ldots,2^{h-1})}$. The results follow from the definition of the topological indices combined with Theorem 3.3, Theorem 5.3, Theorem 5.4 and the fact that the network is vertex-regular with vertex-regularity $n-2h$.

Corollary 5.6. *Let* $\overline{\Gamma_{2^h}} = \overline{C_{2^h}(1,2,2^2,\ldots,2^{h-1})}$. *If* $h \geq 4$, *then*

(i) $W(\overline{\Gamma_{2^h}}) = \frac{n(n+2h-2)}{2}$
(ii) $S(\overline{\Gamma_{2^h}}) = n(n-2h)(n+2h-2)$
(iii) $G(\overline{\Gamma_{2^h}}) = \frac{n(n+2h-2)(n-2h)^2}{2}$
(iv) $WW(\overline{\Gamma_{2^h}}) = \frac{n(2n+8h-6)}{4}$
(v) $H(\overline{\Gamma_{2^h}}) = \frac{n(n-h-\frac{1}{2})}{2}$
(vi) $H_A(\overline{\Gamma_{2^h}}) = n(n-2h)(n-h-\frac{1}{2})$
(vii) $H_M(\overline{\Gamma_{2^h}}) = \frac{n(n-h-\frac{1}{2})(n-2h)^2}{2}$
(viii) $T_{AG}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)}{2}$
(ix) $T_{GA}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)}{2}$
(x) $T_{SC}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)}{2\sqrt{2}\sqrt{n+2h-2}}$
(xi) $T_{ABC}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)\sqrt{n+2h-3}}{\sqrt{2}(n+2h-2)}$
(xii) $T_{AZ}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)(n+2h-2)^6}{16(n+2h-3)^3}$
(xiii) $RT_{AG}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)}{2}$
(xiv) $RT_{GA}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)}{2}$
(xv) $RT_{SC}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)}{2\sqrt{2n-2h-1}}$
(xvi) $RT_{ABC}(\overline{\Gamma_{2^h}}) = \frac{n(n-2h)\sqrt{2n-2h-3}}{2n-2h-1}$
(xvii) $RT_{AZ}(\overline{\Gamma_{2^h}}) = \frac{n(2h-n)(1+2h-2n)^6}{128(3+2h-2n)^3}$.

Finally, we consider the complement circulant network $\overline{C_{m^h}(1,m,m^2,\ldots,m^{h-1})}$ where $m \geq 3$. We begin by determining its distance spectral radius. The result follows from the definition of distance spectral radius, Theorem 3.3, and Remark 3.2.

Theorem 5.5. *Let* $\overline{\Gamma_{m^h}} = \overline{C_{m^h}(1,m,m^2,\ldots,m^{h-1})}$. *For (i)* $m=3$ *and* $h \geq 2$, *(ii)* $m=4$ *and* $h \geq 2$, *and, (iii)* $m \geq 5$ *and* $h \geq 1$, *we have*

$$\rho(\overline{\Gamma_{m^h}}) = n+2h-1.$$

Another consequence of Theorem 3.3 and Remark 3.2 talks about the reciprocal transmission of a vertex in $\overline{C_{m^h}(1,m,m^2,\ldots,m^{h-1})}$.

Theorem 5.6. *Let* $\overline{\Gamma_{m^h}} = \overline{C_{m^h}(1,m,m^2,\ldots,m^{h-1})}$ *and* $v \in V(\overline{\Gamma_{m^h}})$. *For (i)* $m=3$ *and* $h \geq 2$, *(ii)* $m=4$ *and* $h \geq 2$, *and, (iii)* $m \geq 5$ *and* $h \geq 1$, *we have*

$$rs_{\overline{\Gamma_{m^h}}}(v) = n-h-1.$$

For the vertex-forwarding index and bounds for the edge-forwarding index of the complement circulant network $\overline{C_{m^h}(1,m,m^2,\ldots,m^{h-1})}$, it can be computed by combining Lemma 2.5, Lemma 2.6, Theorem 3.3, and Remark 3.2. The results are presented in the next two corollaries.

Corollary 5.7. *Let* $\overline{\Gamma_{m^h}} = \overline{C_{m^h}(1, m, m^2, \ldots, m^{h-1})}$*. For (i)* $m = 3$ *and* $h \geq 2$*, (ii)* $m = 4$ *and* $h \geq 2$*, and, (iii)* $m \geq 5$ *and* $h \geq 1$*, we have*

$$\xi(\overline{\Gamma_{m^h}}) = 2h.$$

Corollary 5.8. *Let* $\overline{\Gamma_{m^h}} = \overline{C_{m^h}(1, m, m^2, \ldots, m^{h-1})}$*. For (i)* $m = 3$ *and* $h \geq 2$*, (ii)* $m = 4$ *and* $h \geq 2$*, and, (iii)* $m \geq 5$ *and* $h \geq 1$*, we have*

$$\frac{2(n+2h-1)}{n-2h-1} \leq \pi(\overline{\Gamma_{m^h}}) \leq 6h + 2.$$

The next series of results give the exact value of some distance-based topological indices of complement circulant network $\overline{C_{m^h}(1, m, m^2, \ldots, m^{h-1})}$. The results follow from the definition of the topological indices combined with Theorem 3.3, Remark 3.2, Theorem 5.5, Theorem 5.6 and the fact that the network is vertex-regular with vertex-regularity $n - 2h - 1$.

Corollary 5.9. *Let* $\overline{\Gamma_{m^h}} = \overline{C_{m^h}(1, m, m^2, \ldots, m^{h-1})}$*. For (i)* $m = 3$ *and* $h \geq 2$*, (ii)* $m = 4$ *and* $h \geq 2$*, and, (iii)* $m \geq 5$ *and* $h \geq 1$*, we have*

(i) $W(\overline{\Gamma_{m^h}}) = \frac{n(n+2h-1)}{2}$

(ii) $S(\overline{\Gamma_{m^h}}) = n(n-2h-1)(n+2h-1)$

(iii) $G(\overline{\Gamma_{m^h}}) = \frac{n(n+2h-1)(n-2h-1)^2}{2}$

(iv) $WW(\overline{\Gamma_{m^h}}) = \frac{n(n+4h-1)}{2}$

(v) $H(\overline{\Gamma_{m^h}}) = \frac{n(n-h-1)}{2}$

(vi) $H_A(\overline{\Gamma_{m^h}}) = n(n-2h-1)(n-h-1)$

(vii) $H_M(\overline{\Gamma_{m^h}}) = \frac{n(n-h-1)(n-2h-1)^2}{2}$

(viii) $T_{AG}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)}{2}$

(ix) $T_{GA}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)}{2}$

(x) $T_{SC}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)}{2\sqrt{2}\sqrt{n+2h-1}}$

(xi) $T_{ABC}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)\sqrt{\frac{n}{2}+h-1}}{\sqrt{2}(n+2h-1)}$

(xii) $T_{AZ}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)(n+2h-1)^6}{16(n+2h-2)^3}$

(xiii) $RT_{AG}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)}{2}$

(xiv) $RT_{GA}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)}{2}$

(xv) $RT_{SC}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)}{2\sqrt{2}\sqrt{n-h-1}}$

(xvi) $RT_{ABC}(\overline{\Gamma_{m^h}}) = \frac{n(n-2h-1)\sqrt{n-h-2}}{\sqrt{2}(n-h-1)}$

(xvii) $RT_{AZ}(\overline{\Gamma_{m^h}}) = \frac{n(1+2h-n)(1+h-n)^6}{16(2+h-n)^3}$.

6. Conclusion

In this research note, we were able to determine the distance matrix of the connected complement of the circulant networks $C_n(1, a)$ where $2 \leq a \leq \frac{n}{2}$ and $C_{m^h}(1, m, m^2, \ldots, m^{h-1})$ where $m \geq 2$. As a consequence, we were able to compute for the distance spectral radius, vertex-forwarding index, and some distance-based topological indices of the connected complement circulant networks $\overline{C_n(1, a)}$ and $\overline{C_{m^h}(1, m, m^2, \ldots, m^{h-1})}$. As a possible research problem, we recommend the determination of the distance matrix, as well as the distance spectral radius, vertex-forwarding index and some distance-based topological indices of the connected complement of the circulant network $C_n(1, 2, \ldots, a)$, where $3 \leq a \leq \frac{n}{2}$.

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