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NEW NOTIONS OF TRIPLE IDEAL CONVERGENT ON INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. In this paper, we introduce and study the notions of $\mathcal{M}^{I_3}_{(\mu,v)}(T)$, $\mathcal{M}^{I^0_3}_{(\mu,v)}(T)$, $\mathcal{M}^{I_3}_{(\mu,v)}(T,F)$ and $\mathcal{M}^{I^0_3}_{(\mu,v)}(T,F)$ for triple sequences in intuitionistic fuzzy normed space for the sequence. Furthermore, some topological properties are established on these spaces.

Keywords: Ideal spaces, intuitionistic fuzzy normed spaces, Orlicz function, compact operator, I_3-convergence.
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1. Introduction

In the last decade, the concept of fuzzy set has been as the most active field of research in many branches of mathematics, computer and engineering [1]. Taking into account the work introduced by Zadeh [35], a huge amount of researches have been done on fuzzy set theory and its applications, as well as, fuzzy analogues of the classical theories. Fuzzy set has a wide number of applications in various fields such as population dynamics [2], nonlinear dynamical system [14], chaos control [7], computer programming [9] and much more. In 2006, Saadati and Park [23] defined the concept of intuitionistic fuzzy normed spaces. After that, the study of intuitionistic fuzzy topological spaces [3], intuitionistic fuzzy 2-normed space [22] and intuitionistic fuzzy Zweier ideal convergent sequence spaces [15] are the latest developments in fuzzy topology.

On the other hand, the statistical convergence was derived from the convergence of real sequences by Fast [6] and Schoenberg [26]. After the studies of Salát [24], Fridy [8], and Connor [4] in this area, many studies have been conducted. Kostyrko et al. [18] introduced the concept of ideal convergence by expanding the concept of statistical convergence. After basic properties of I-convergence were given by Kostyrko et al. [19], some studies [21, 25, 27, 10, 11] have been the basis of other studies. Besides, for multiple sequences Tripathy et al. [28] have studied some notions for double sequences and later extended some of these notions for triple sequences (see [29, 30, 31, 32, 33, 34, 5]).

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In this paper, we use the notions presented by [17] and we extend these concepts for triple sequences, i.e. we introduce and study the concepts of $\mathcal{M}^{I_3}_{(\mu,v)}(T)$, $\mathcal{M}^{I^0_3}_{(\mu,v)}(T)$, $\mathcal{M}^{I_3}_{(\mu,v)}(T,F)$ and $\mathcal{M}^{I^0_3}_{(\mu,v)}(T,F)$ for triple sequences in intuitionistic fuzzy normed space. Also, we establish some of their properties. Moreover, notions studied in this paper, can be studied in neutrosophic sers [13] and localized metric spaces [12].

2. Preliminaries

In this section, we recall some well-known notions which are useful for the developing of this paper.

Definition 2.1. ([23]) *The five-tuple $(X, \mu, v, *, \diamond)$ is called an intuitionistic fuzzy normed space (simpy IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, v are fuzzy sets on $X \times (0,\infty)$ satisfying the following conditions for every $r, u \in X$ and $s, t > 0$:*

(1) $\mu(r,t) + v(r,t) \le 1$,
(2) $\mu(r,t) > 0$,
(3) $\mu(r,t) = 1$ *if and only if* $r = 0$,
(4) $\mu(\alpha r,t) = \mu(r, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,
(5) $\mu(r,t) * \mu(u,s) \le \mu(r+u, t+s)$,
(6) $\mu(r,.) : (0,\infty) \to [0,1]$ *is continuous,*
(7) $\lim_{t\to\infty} \mu(r,t) = 1$ *and* $\lim_{t\to 0} \mu(r,t) = 0$,
(8) $v(r,t) < 1$,
(9) $v(r,t) = 0$ *if and only if* $r = 0$,
(10) $v(\alpha r,t) = v(r, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,
(11) $v(r,t) \diamond v(u,s) \ge v(r+u, t+s)$,
(12) $v(x,.) : (0,\infty) \to [0,1]$ *is continuous,*
(13) $\lim_{t\to\infty} v(r,t) = 1$ *and* $\lim_{t\to 0} v(r,t) = 0$,

In this case, (μ, v) is said to be an intuitionistic fuzzy norm.

Example 2.1. *Let $(X, \|.\|)$ be a normed space. Denote $a * b = ab$ and $a \diamond b = \min(a+b, 1)$ for all $a, b \in [0,1]$ and let μ_0 and $v0$ be fuzzy sets on $X \times (0,\infty)$ defined as follows:*

$$\mu_0(x,t) = \frac{t}{t + \|x\|} \text{ and } v_0(x,t) = \frac{\|x\|}{t + \|x\|}$$

for all $t \in \mathbb{R}^+$. Then, $(X, \mu, v, *, \diamond)$ is an intuitionistic fuzzy normed space.

Definition 2.2. *An ideal I is a non-empty collection of subsets of X which satisfies the conditions (1) and (2) of the following statements:*

(1) *If $A \subset B$ and $B \in I$, then $A \in I$,*
(2) *If $A, B \in I$, then $A \cup B \in I$.*
(3) *I is called as a non-trivial ideal if $X \notin I$ and $I \neq \emptyset$.*
(4) *A non-trivial ideal I on X is said to be admissible if $\{I \supseteq \{x\}\}$.*
(5) *Throughout this paper, I_3 is a non-trivial strongly ideal on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ which $\mathbb{N} \times \mathbb{N} \times \{i\}$, $\mathbb{N} \times \{i\} \times \mathbb{N}$ and $\{i\} \times \mathbb{N} \times \mathbb{N}$ belongs to I_3 for each $i \in \mathbb{N}$. Besides, it is called maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.*

A non-empty family of subsets of $F \subset 2^X$ is a filter on X if satisfies the conditions (1), (2) and (3) of the following statements:

(1) $\emptyset \in F$,
(2) *If* $A, B \in F$, *then* $A \cap B \in F$,
(3) *If* $A \in F$ *and* $A \subset B$, *then* $B \in F$.
(4) $I \subset 2^X$ *is a non-trivial ideal if and only if* $F = F(I) = \{X - A : A \in I\}$ *is a filter on* X.

Definition 2.3. *Let I_3 be a non-trivial ideal of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $(X, \mu, v, *, \diamond)$ be an intuitionistic fuzzy normed space. A triple sequence $x = (x_{kjq})$ of elements of X is said to be I_3-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, v), if for each $\epsilon > 0$ and $t > 0$,*

$$\{(j,k,q) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mu(x_{kjq} - L, t) \leq 1 - \epsilon \text{ or } v(x_{kjq} - L, t) \geq \epsilon\} \in I_3$$

In this case, we wrtie $I^3_{(\mu,v)}\text{-}\lim x = L$.

Definition 2.4. *([16]) Let X and Y be two normed linear spaces and $T : D(T) \to Y$ be a linear operator where $D \subset X$. Then, the operator T is said to be bounded if there exists a positive real n such that $\|Tx\| \leq n\|x\|$, for all $x \in D(T)$. The set of all bounded linear operator $B(X,Y)$ [20] is a normed linear spaces normed by $\|T\| = \sup_{x \in X, \|x\|=1} \|Tx\|$ and $B(X,Y)$ is a Banach space if Y is a Banach space.*

Definition 2.5. *([16]) Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator), if satisfies the following conditions:*

(1) *T is linear.*
(2) *T mas every bounded sequence (x_n) in X onto a sequence $(T(x_n))$ in Y which has a convergent subsequence.*

The set of all compact linear operators $C(X,Y)$ is a closed subspace of $B(X,Y)$.

Definition 2.6. *([17]) An Orliz function is a function $F : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $F(x) \to \infty$ as $x \to \infty$. If the convexity of Orlicz function F is replaced by $F(x+y) \leq F(x) + F(y)$, then this function is called modulus function. Besides, if F is a Orlicz function, then $F(\lambda x) \leq \lambda F(x)$.*

3. I_3-convergent sequences by using compact operator in IFNS

In this section, we introduce the triple ideal sequence spaces on compact operator in intuitionistic fuzzy normed spaces.

$$\mathcal{M}^{I_3}_{(\mu,v)}(T) = \{(x_{qwe}) \in \ell_\infty : \{(q,w,e) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mu(T(x_{qwe}) - L, t) \leq 1 - \varepsilon \text{ or } v(T(x_{qwe}) - L, t) \geq \varepsilon \in I_3\}\}$$

$$\mathcal{M}^{I^0_3}_{(\mu,v)}(T) = \{(x_{qwe}) \in \ell_\infty : \{(q,w,e) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mu(T(x_{qwe}), t) \leq 1 - \varepsilon \text{ or } v(T(x_{qwe}), t) \geq \varepsilon \in I_3\}\}$$

Besides, we define an open ball with center x and radius r with respect to t as follows:

$$B^3_x(r,t)(T) = \{(y_{qwe}) \in \ell_\infty : \{(q,w,e) : \mu(T(x_{qwe}) - T(y_{qwe}), t) \leq 1 - \varepsilon \text{ or } v(T(x_{qwe}) - T(y_{qwe}), t) \geq \varepsilon \in I_3\}\}$$

Now, we will show and prove our main results.

Theorem 3.1. *The sequence spaces $\mathcal{M}^{I_3}_{(\mu,v)}(T)$ and $\mathcal{M}^{I^0_3}_{(\mu,v)}(T)$ are linear spaces.*

Proof. Let $x = (x_{qwe}), y = (y_{qwe}) \in \mathcal{M}^{I_3}_{(\mu,v)}(T)$ and α, β be scalars. Then, for a given $\epsilon > 0$, we have the sets:

$$P_1 = \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(x_{qwe}) - L_1, \frac{t}{2|\alpha|}) \leq 1-\varepsilon \text{ or } v(T(x_{qwe}) - L_1, \frac{t}{2|\alpha|}) \geq \varepsilon\} \in I_3,$$

$$P_1 = \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(y_{qwe}) - L_2, \frac{t}{2|\beta|}) \leq 1-\varepsilon \text{ or } v(T(y_{qwe}) - L_2, \frac{t}{2|\beta|}) \geq \varepsilon\} \in I_3.$$

This implies

$$P_1^c = \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(x_{qwe}) - L_1, \frac{t}{2|\alpha|}) > 1-\varepsilon \text{ or } v(T(x_{qwe}) - L_1, \frac{t}{2|\alpha|}) < \varepsilon\} \in F(I_3);$$

$$P_2^c = \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(y_{qwe}) - L_2, \frac{t}{2|\beta|}) > 1-\varepsilon \text{ or } v(T(y_{qwe}) - L_2, \frac{t}{2|\beta|}) < \varepsilon\} \in F(I_3);$$

Now, define the set $P_3 = P_1 \cup P_2$, thus $P_3 \in I_3$ and P_3^c is a non-empty set in $F(I_3)$. Then, we shall prove that for each $(x_{qwe}), (y_{qwe}) \in \mathcal{M}^{I_e}_{(\mu,v)}(T)$.

$$P_3^c \subset \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu((\alpha T(x_{qwe}) + \beta T(y_{qwe})) - (\alpha L_1 + \beta L_2), t) > 1-\varepsilon \text{ or } v((\alpha T(x_{qwe}) + \beta T(y_{qwe})) - (\alpha L_1 + \beta L_2), t) < \varepsilon\}.$$

Let $(n,m,a) \in P_3^c$, in this case

$$\mu(T(x_{nma} - L_1, \frac{t}{2|\alpha|}) > 1-\varepsilon \text{ or } v(T(x_{nma} - L_1, \frac{t}{2|\alpha|}) < \varepsilon$$

and

$$\mu(T(y_{nma} - L_2, \frac{t}{2|\beta|}) > 1-\varepsilon \text{ or } v(T(y_{nma} - L_2, \frac{t}{2|\beta|}) < \varepsilon$$

Thus, we have

$$\begin{aligned} &\mu((\alpha T(x_{qwe}) + \beta T(y_{nma})) - (\alpha L_1 + \beta L_2), t) \\ &\geq \mu(\alpha T(x_{qwe}) - \alpha L_1, \frac{t}{2}) * \mu(\beta T(x_{nma}) - \beta L_2, \frac{t}{2}) \\ &= \mu(T(x_{nma}) - L_1, \frac{t}{2|\alpha|}) * \mu(T(x_{nma}) - L_2, \frac{t}{2|\beta|}) \\ &> (1-\varepsilon) * (1-\varepsilon) = 1-\varepsilon \end{aligned}$$

and

$$\begin{aligned} &v((\alpha T(x_{nma}) + \beta T(y_{nma})) - (\alpha L_1 + \beta L_2), t) \\ &\leq v(\alpha T(x_{nma}) - \alpha L_1, \frac{t}{2}) \diamond v(\beta T(x_{nma}) - \beta L_2, \frac{t}{2}) \\ &= v(T(x_{nma}) - L_1, \frac{t}{2|\alpha|}) \diamond v(T(x_{nma}) - L_2, \frac{t}{2|\beta|}) \\ &< \varepsilon \diamond \varepsilon = \varepsilon \end{aligned}$$

This implies

$$P_3^c \subset \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu((\alpha T(x_{qwe}) + \beta T(y_{qwe})) - (\alpha L_1 + \beta L_2), t) > 1-\varepsilon \text{ or } v((\alpha T(x_{qwe}) + \beta T(y_{qwe})) - (\alpha L_1 + \beta L_2), t) < \varepsilon\}.$$

Therefore, the sequence space $\mathcal{M}^{I_3}_{(\mu,v)}(T)$ is a linear space. The proof of $\mathcal{M}^{I_3^0}_{(\mu,v)}(T)$ is made similarly.

□

Remark 3.1. *In the following theorems, we will discuss some problems on convergence in triple sequence spaces. For this, first at all, we have to discuss about the topology of this space. Let*

$$\tau^{I_3}_{(\mu,v)}(T) = \{H \subset \mathcal{M}^{I_3}_{(\mu,v)}(T) \text{: for each } x \in H \text{ there exist } t > 0 \text{ and } r \in (0,1) \text{ such that } B_x^3(r,t)(T) \subset A\}$$

Then, $\tau^{I_3}_{(\mu,v)}(T)$ *is a topology on* $\mathcal{M}^{I_3}_{(\mu,v)}(T)$.

Theorem 3.2. *Let* $\mathcal{M}^{I_3}_{(\mu,v)}(T)$ *be an IFNS and* $\tau^{I_3}_{(\mu,v)}(T)$ *be a topology on* $\mathcal{M}^{I_3}_{(\mu,v)}(T)$*. Then, a triple sequence* $(x_{qwe}) \in \mathcal{M}^{I_3}_{(\mu,v)}(T)$, $x_{qw} \to x$ *if and only if* $\mu(T(x_{qwe}) - T(x), t) \to 1$ *and* $v(T(x_{qwe}) - T(x), t) \to 0$ *as* $q, w \to \infty$.

Proof. Fix $t_0 > 0$ and consider $x_{qwe} \to x$. Then, for $r \in (0,1)$, there exist $n_0, m_0, a_0 \in \mathbb{N}$ such that $(x_{qwe}) \in B_x^3(r,t)(T)$ for all $q \geq n_0$, $w \geq m_0$ and $e \geq a_0$. Thus, we have

$$B_x^3(r,t_0)(T) = \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(x_{qwe}) - T(x), t_0) \leq 1 - r \text{ or } v(T(x_{qwe}) - T(x), t_0) \geq r\} \in I_3.$$

such that $(B_x^3)^c(T) \in F(I_3)$. Then, $1 - \mu(T(x_{qwe}) - T(x), t_0) < r$ and $v(T(x_{qwe}) - T(x), t_0) < r$. Therefore, $\mu(T(x_{qwe}) - T(x), t_0) \to 1$ and $v(T(x_{qwe}) - T(x), t_0) \to 0$ as $q, w, e \to \infty$.

Conversely, if for each $t > 0, \mu(T(x_{qwe}) - T(x), t) \to 1$ and $v(T(x_{qwe}) - T(x), t_0) \to 0$ as $q, w, e \to \infty$, then for $r \in (0,1)$, there exist $n_0, m_0, a_0 \in \mathbb{N}$, such that $1 - \mu(T(x_{qwe}) - T(x), t) < r$ and $v(T(x_{qwe}) - T(x), t) < r$, for all $q \geq n_0$, $w \geq m_0$ and $e \geq a_0$. This shows that $\mu(T(x_{qwe}) - T(x), t) > 1 - r$ and $v(T(x_{qwe}) - T(x), t) < r$ for all $q \geq n_0$, $w \geq m_0$ and $e \geq a_0$. Therefore, $(x_{qwe}) \in (B_x^3)^c(r,t)(T)$ for all $q \geq n_0$, $w \geq m_0$, $e \geq a_0$ and then $x_{qwe} \to x$.

□

Theorem 3.3. *A triple sequence* $x = (x_{qwe}) \in \mathcal{M}^{I_3}_{(\mu,v)}(T)$ *is* I_3*-convergent if and only if for every* $\varepsilon > 0$ *and* $t > 0$ *there exist numbers* $N = N(x,\varepsilon,t)$, $M = M(x,\varepsilon,t)$ *and* $A = A(x,\varepsilon,t)$ *such that*

$$\{(N,M,A) : \mu(T(X_{NMA}) - L, \frac{t}{2}) > 1 - \varepsilon \text{ or } v(T(x_{NMA} - L, \frac{t}{2}) < \varepsilon\} \in F(I_3).$$

Proof. Consider that $I^2_{(\mu,v)}$-$\lim x_{qwe} = L$ and let $t > 0$. For a given $\varepsilon > 0$, take $s > 0$ such that $(1-\varepsilon) * (1-\varepsilon) > 1 - s$ and $\varepsilon \diamond \varepsilon < s$. Then, for each $x \in \mathcal{M}^{I_3}_{(\mu,v)}(T)$,

$$R_2 = \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(x_{qwe}) - L, \frac{t}{2}) \leq 1 - \varepsilon \text{ or } v(T_{qwe} - L, \frac{t}{2}) \geq \varepsilon\} \in I_3,$$

which implies that

$$R_2^c = \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(x_{qwe}) - L, \frac{t}{2}) > 1-\varepsilon \text{ or } v(T(x_{qwe}) - L, \frac{t}{2}) < \varepsilon\} \in F(I_3).$$

Conversely, let's choose $N, M, A \in R_2^c$. Then,

$$\mu(T(x_{NMA}) - L, \frac{t}{2}) > 1-\varepsilon \text{ or } v(T(x_{NMA}) - L, \frac{t}{2}) < \varepsilon$$

Now, we have to prove that there exist number $N = N(x,\varepsilon,t)$, $M = M(x,\varepsilon,t)$ and $A = A(x,\varepsilon,t)$ such that

$$\{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(x_{qwe}) - T(x_{NMA}), t) \le 1-s \text{ or } v(T(x_{qwe}) - T(x_{NMA}), t) \ge s\} \in I_3.$$

For that, we shall define that for each $x \in \mathcal{M}^{I_3}_{(\mu,v)}(T)$

$$S_2 = \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mu(T(x_{qwe} - T(x_{NMA}, t) \le 1-s \text{ or } v(T(x_{qwe}) - T(x_{NMA}), t) \ge s\} \in I_3$$

Thus, we have to prove that $S_2 \subset R_2$. Let's suppose that $S \subset R$, then there exist $n, m, a \in S_2$ such that $n, m, a \notin R_2$. Then, we have

$$\mu(T(x_{nma} - T(x_{NMA}, t) \le 1-s \text{ or } \mu(T(x_{nma}) - L, \frac{t}{2}) > 1-\varepsilon$$

In particular, $\mu(T(x_{NMA}) - L, \frac{t}{2}) > 1-\varepsilon$. Hence, we have that

$$1-s \ge \mu(T(x_{nma} - T(x_{NMA}), t) \ge \mu(T(x_{nma}) - L, \frac{t}{2}) * \mu(T(x_{NMA}) - L, \frac{t}{2}) \ge (1-\varepsilon) * (1-\varepsilon) > 1-s$$

and this is not possible. Otherwise,

$$v(T(x_{nma}) - T(x_{NMA}), t) \ge s \text{ or } v(T(x_{nma}) - L, \frac{t}{2}) < \varepsilon$$

In particular, $v(T(x_{NMA}) - L, \frac{t}{2}) < \varepsilon$. Thus, we have

$$s \le v(T(x_{nma}) - T(x_{NMA}), t) \le v(T(x_{nma}) - L, \frac{t}{2}) \diamond v(T(x_{NMA}) - L, \frac{t}{2}) \le \varepsilon \diamond \varepsilon < s$$

and this is not possible. Therefore, $S_2 \subset R_2$. Thus, $R_2 \in I_3$ which implies that $S \in I_3$. □

4. I_3-convergent sequences by using Orlicz function in IFNS

In this section, we use the notion of compact operator and Orlicz function for defining a new triple ideal sequence space in intuitionistic fuzzy normed spaces.

$$\mathcal{M}^{I_3}_{(\mu,v)}(T,F) = \{(x_{qwe}) \in \ell_\infty : \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : F(\frac{\mu(T(x_{qwe}) - L, t)}{\rho}) \le 1-\varepsilon \text{ or } F(\frac{v(T(x_{qwe}) - L, t)}{\rho}) \ge \varepsilon\} \in I_3\}$$

$$\mathcal{M}^{I_3^0}_{(\mu,v)}(T,F) = \{(x_{qwe}) \in \ell_\infty : \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : F(\frac{\mu(T(x_{qwe}), t)}{\rho}) \le 1-\varepsilon \text{ or } F(\frac{v(T(x_{qwe}), t)}{\rho}) \ge \varepsilon\} \in I_3\}$$

Moreover, we define an open ball with center x and radius r with respect to t as follows:

$$B^3_x(r,t)(T,F) = \{(y_{qw3}) \in l_\infty : (q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : F(\frac{\mu(T(x_{qwe}) - T(y_{qwe}), t)}{\rho}) \leq 1-\varepsilon$$
$$\text{or } F(\frac{v(T(x_{qwe}) - T(y_{qwe}), t)}{\rho}) \geq \varepsilon \in I_3\}.$$

Remark 4.1. *The sequences $\mathcal{M}^{I_3}_{(\mu,v)}(T,F)$ and $\mathcal{M}^{I^0_3}_{(\mu,v)}(T,F)$ are linear spaces.*

Theorem 4.1. *Every open ball $B^3_x(r,t)(T,F)$ is an open set in $\mathcal{M}^{I_3}_{(\mu,v)}(T,F)$.*

Proof. Let $B^3_x(r,t)(T,F)$ be an open ball with center x and radius r with respect to t. This is

$$B^3_x(r,t)(T,F) = \{y = (y_{qwe}) \in l_\infty : \{(q,w,e) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : F(\frac{\mu(T(x_{qwe}) - T(y_{qwe}), t)}{\rho}) \leq$$
$$1-r \text{ or } F(\frac{v(T(x_{qwe}) - T(y_{qwe}), t)}{\rho}) \geq r\} \in I_3\}$$

Let $y \in B^3_x(r,t)(T,F)$, then

$$F(\frac{\mu(T(x_{qwe}) - T(y_{qwe}), t)}{\rho}) > 1-r \textit{ and } F(\frac{v(T(x_{qwe}) - T(y_{qwe}), t)}{\rho}) < r.$$

Since $F(\frac{\mu(T(x_{qwe}) - T(y_{qwe}), t)}{\rho}) > 1-r$, there exists $t_0 \in (0,t)$ such that

$$F(\frac{\mu(T(x_{qwe}) - T(y_{qwe}), t_0)}{\rho}) > 1-r \textit{ and } F(\frac{v(T(x_{qwe}) - T(y_{qwe}), t_0)}{\rho}) < r.$$

Taking $r_0 = F(\frac{\mu(T(x_{qwe}) - T(y_{qwe}), t_0)}{\rho})$, thus we have $r_0 > 1-r$, there exists $s \in (0,1)$ such that $r_0 > 1-s > 1-r$. For $r_0 > 1-s$, we have that $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1-s$ and $(1-r_0) \diamond (1-r_0) \leq s$. Putting $r_3 = \max\{r_1, r_2\}$. Now, we take a ball $B^c_y(1-r_e, t-t_0)(T,F)$. Now, we will prove that $B^3_y(1-r_3, t-t_0)(T,F) \subset B^3_x(r,t)(T,F)$.

Let $z = (z_{qwe}) \in B^3_y(1-r_3, t-t_0)(T,F)$, then $F(\frac{\mu(T(y_{qwe}) - T(z_{qwe}), t-t_0)}{\rho}) > r_3$ and $F(\frac{v(T(y_{qwe}) - T(z_{qwe}), t-t_0)}{\rho}) < 1-r_3$. Hence, we have

$$\begin{aligned} &F(\frac{\mu(T(x_{qwe}) - T(z_{qwe}), t)}{\rho}) \\ &\geq F(\frac{\mu(T(x_{qwe}) - T(y_{qwe}), t_0)}{\rho}) * F(\frac{\mu(T(y_{qwe}) - T(z_{qwe}), t-t_0)}{\rho}) \\ &\geq (r_0 * r_3) \geq (r_0 * r_1) \geq (1-s) \geq (1-r) \end{aligned}$$

and

$$\begin{aligned} &F(\frac{v(T(x_{qwe}) - T(z_{qwe}), t)}{\rho}) \\ &\leq F(\frac{v(T(x_{qwe}) - T(y_{qwe}), t_0)}{\rho}) \diamond F(\frac{v(T(y_{qwe}) - T(z_{qwe}), t-t_0)}{\rho}) \\ &\leq (1-r_0) \diamond (1-r_3) \leq (1-r_0) \diamond (1-r_2) \leq s \leq r \end{aligned}$$

Therefore, $z \in B_x(r,t)(T,F)$ and hence, we have that $B_y^e(1-r_3, t-t_0)(T,F) \subset B_x^e(r,t)(T,F)$.

□

Remark 4.2. *$\mathcal{M}^{I_3}_{(\mu,v)}(T,F)$ is an IFNS.*

Define $\tau^{I_3}_{(\mu,v)}(T,F) = \{A \subset \mathcal{M}^{I_3}_{(\mu,v)}(T,F) :$ for each $x \in H$ there exists $t > 0$ and $r \in (0,1)$ such that $B_x^3(r,t)(T,F) \subset H\}$. Then, $\tau^{I_3}_{(\mu,v)}/(T,F)$ is a topology on $\mathcal{M}^{I_3}_{(\mu,v)}(T,F)$.

Remark 4.3. *The topology $\tau^{I_3}_{(\mu,v)}(T,F)$ on $\mathcal{M}^{I_3^0}_{(\mu,v)}(T,F)$ is first countable.*

Theorem 4.2. *$\mathcal{M}^{I_3}_{(\mu,v)}(T,F)$ and $\mathcal{M}^{I_3^0}_{(\mu,v)}(T,F)$ are Hausdorff spaces.*

Proof. Let $u, v \in \mathcal{M}^{I_3}_{(\mu,v)}(T,F)$ such that $u \neq v$. Then, $0 < F(\dfrac{\mu(T(u)-T(v),t)}{\rho}) < 1$ and $0 < F(\dfrac{v(T(u)-T(v),t)}{\rho}) < 1$. Taking $r_1 = F(\dfrac{\mu(T(u)-T(v),t)}{\rho})$, $r_2 = F(\dfrac{v(T(u)-T(v),t)}{\rho})$ and $r = \max\{r_1, 1-r_2\}$. For each $r_0 \in (r,1)$ there exist r_3 and r_4 such that $r_3 * r_4 \geq r_0$ and $(1-r_3) \diamond (1-r_4) \leq 1-r_0$. Putting $r_5 = \max\{r_3, 1-r_4\}$ and consider the open balls $B_u^3(1-r_5, \frac{t}{2})$ and $B_v^3(1-r_5, \frac{t}{2})$. Then, it is clear that $(B_u^3)^c(1-r_5, \frac{t}{2}) \cap (B_v^3)^c(1-r_5, \frac{t}{2}) = \emptyset$, then

$$r_1 = F(\frac{\mu(T(u)-T(v),t)}{\rho}) \geq F(\frac{\mu(T(u)-T(b),\frac{t}{2})}{\rho}) * F(\frac{\mu(T(b)-T(v),\frac{t}{2})}{\rho}) \geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 \geq r_1$$

and

$$r_2 = F(\frac{v(T(u)-T(v),t)}{\rho}) \leq F(\frac{v(T(u)-T(v),\frac{t}{2})}{\rho}) \diamond F(\frac{v(T(b)-T(v),\frac{t}{2})}{\rho}) \leq (1-r_5) \diamond (1-r_5) \leq (1-r_4) \diamond (1-r_4) \leq (1-r_0) \leq r_2$$

and this is a contradiction, Therefore, $\mathcal{M}^{I_3}_{(\mu,v)}(T,F)$ is Hausdorff.

The proof of $\mathcal{M}^{I_3^0}_{(\mu,v)}(T,F)$ is made similarly.

□

5. Conclusion

In this paper we have introduced and studied new concepts on triple sequences spaces by using results presented by [17], these new notions can be extended in higher dimension. On the other hand, applications problem can be obtained such that artificial intelligent, computational simulation and even applied in neutrosophic metric spaces [13] .

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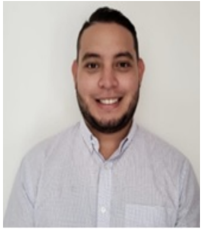

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