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HYPER-CONNECTIVITY INDEX FOR FUZZY GRAPH WITH APPLICATION

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Abstract. Connectivity concept is one of the most important parameters in fuzzy graphs (FGs). The stability of a FG is dependent on the strength of connectedness between each pair of vertices. Depending on "strength of connectedness between each pair of vertices" hyper-connectivity index for fuzzy graph (FHCI) is introduced and studied this index for various FGs like partial fuzzy subgraph, fuzzy subgraph, complete fuzzy graph, saturated cycle, isomorphic fuzzy graphs, etc. A relation of FHCI is established between fuzzy graph and partial fuzzy subgraph. Also, a relation between FHCI and connectivity index for fuzzy graph (FCI) is provided. In the end of the article, a decision-making problem is presented and solved it by using FHCI. Also, a comparison is provided among related indices on the result of application and shown that our method gives better result.

Keyword: Fuzzy graph; Connectivity index; Hyper-connectivity index.

AMS Subject Classification: 05C40, 05C62.

1. Introduction

1.1. **Research background.** In [36], Zadeh first introduced fuzzy set theory in 1965. In 1975, Inspired by this, Rosenfeld [30] provided the concept of fuzzy graph (FG) and also provided the fuzzy relation (FR), fuzzy bridge, fuzzy block and fuzzy distances of a FG. In that time, Yeh et al. [35] also established FG separately and gave an application of FG in clustering analysis. In [27, 29], Poulik et al. introduced empirical and Pragmatic results on bipolar fuzzy graphs. Bipolar fuzzy graph, bipolar fuzzy soft hypergraph are studied by Akram et al. in [1, 31]. Recently, in [14, 15, 16, 17, 18, 19], Mahapatra et al. studied radio fuzzy graph, generalized neutrosophic planer graph, link prediction in neutrosophic graph, edge coloring of a fuzzy graph, coloring of a fuzzy directed graph, etc. In [2, 24, 25, 32, 33], one can see for more details and generalization for FG.

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Connectivity concept is one of the most important parameter in fuzzy graphs (FGs). In 2014, Jicy and Mathew introduced some connectivity parameters for weighted graph in [11]. In [21, 22], Mathew and Sunitha studied node connectivity, arc connectivity and cycle connectivity of a FG. In [23], Mordeson studied various type of connectivity concepts in fuzzy graphs. Different types of algorithm to determine the connectivity of a pair of vertices in a FG is provided in [3, 4, 34]. The stability of a FG is dependent on the strength of connectedness between each pair of vertices. Depending on "strength of connectedness between each pair of vertices", in 2019, Binu et al. [5] introduced connectivity index (FCI) and average connectivity index (AFCI) of a FG. Motivated from this article, Recently, Binu et al. [6] introduced Wiener index of a FG and Islam and Pal also studied Wiener index for FG in [7]. Islam and Pal also introduced Hyper-Wiener index [10], First Zagreb index [8] and F-index [9] for fuzzy graph in 2021. In [13], Mahapatra, Samanta and Pal provided the RSM index in fuzzy graph. Poulik and Ghorai studied Cretain index and Wiener absolute index for bipolar fuzzy graph in [26, 28]. In [12] Kalathian et al. also introduced so many index for fuzzy graphs. Motivated from those articles, fuzzy hyper-connectivity index (FHCI) is introduced and studied those indices for various FGs like partial fuzzy subgraph, fuzzy subgraph, saturated cycle, isomorphic fuzzy graphs, etc. A relation among fuzzy graphs, partial fuzzy subgraphs, fuzzy subgraphs are established with respect to FHCI. Also a relation between fuzzy connectivity index (FCI) and FHCI is provided. In the end of the article, a decision making problem is presented and solved by using FHCI.

1.2. **Significance of the article.** Connectivity concept is one of the most important parameters in FGs. The stability of a FG is dependent on the strength of connectedness between each pair of vertices. Depending on "strength of connectedness between each pair of vertices" FHCI and AFHCI are introduced and studied in this article. Also, depending on AFHCI, vertices of a fuzzy graph are classified in the class HCEN, HCRN and HCNN. A necessary and sufficient condition is provided to determine a vertex is either HCEN or HCRN or HCNN. Value of FHCI is calculated and some bounds are presented for various FGs like partial fuzzy subgraph, fuzzy subgraph, saturated cycle, isomorphic fuzzy graphs, etc. A relation among fuzzy graphs, partial fuzzy subgraphs, fuzzy subgraphs are established with respect to FHCI. Also, a relation between fuzzy connectivity index (FCI) and FHCI is provided. In the end of the article, a decision-making problem is presented and solved by using FHCI.

1.3. **Structure of the study.** Structure of the article is as follows: In sec. 2, some basic definitions and useful results are provided which are essential to the development of our content. In sec. 3, FHCI is introduced and discussed some results on it. In sec. 4, a decision making problem is presented and solved by FHCI.

2. Preliminaries

Some basic definitions and useful results are provided in this section, most of them one can be found in [25].

Definition 2.1. *Let $X(\neq \phi)$ be a given finite set. Now a FG is $G = (\theta, \rho)$, where θ is fuzzy subset(FSS) of X and ρ is FSS of $X \times X$ with $\rho(x, y) \leq \wedge\{\theta(x), \theta(y)\}$, where $\wedge$ indicates the minimum.*

It is noted that ρ is called a FR on θ. It is assumed that ρ is reflexive as well as symmetric. We write $G^* = (\theta^*, \rho^*)$, where $\theta^* = \{x \in X : \theta(x) \neq 0\}$ and $\rho^* = \{(x, y) \in$

$X \times X : \rho(x, y) \neq 0\}$. Here θ^* and ρ^* is known as the vertex set and edge set of the FG, respectively. The fuzzy graph G is trivial if G^* is trivial.

Definition 2.2. *Now $H = (\phi, \omega)$ is called partial fuzzy subgraph(PFSG) of G if for all $x \in \phi^*, \phi(x) \leq \theta(x)$ and for all $(x, y) \in \omega^*, \omega(x, y) \leq \rho(x, y)$. If for all $x \in \phi^*, \phi(x) = \theta(x)$ and for all $(x, y) \in \omega^*, \omega(x, y) = \rho(x, y)$, then H is called FSG of G.*

A FSG, $H = (\phi, \omega)$ spans the FG $G = (\theta, \rho)$ if $\phi = \theta$. It is noted that G_{xy} is a FSG of G with $\omega(x, y) = 0$ and G_x is FSG of G where $\phi(x) = 0$.

Definition 2.3. *Let $x_0, x_1, \cdots, x_n$ be distinct vertices in G. Then we call $P = x_0x_1 \cdots x_n$ is path in G if $\rho(x_i, x_{i+1}) \neq 0$ for $i = 0, 1, \cdots, n-1$.*

For this case, length of the path is n.

Definition 2.4. *The path, $P = x_1x_2 \cdots x_n$ is called a cycle if $\rho(x_1, x_n) > 0$.*

For this case, length of the cycle is n.

Definition 2.5. *Let $x_0, x_1, \cdots, x_n$ be the vertices of G. Then $G = (x_0, x_1, x_1 \cdots x_n)$ is called a star if $\rho(x_0, x_i) \neq 0$ for $i = 0, 1, \cdots, n$.*

Here x_0 is called the center of the star.

Definition 2.6. *G is called a complete FG (CPFG) if for all $x, y \in \theta^*, \rho(x, y) = \theta(x) \wedge \theta(y)$, where $\wedge$ denotes the minimum.*

Definition 2.7. *Let $P = x_0x_1 \cdots x_n$ be a path in G. Then strength of the path P, $S(P)$ is defined by $S(P) = \wedge\{\rho(x_i, x_{i+1}) : i = 0, 1, \cdots, n-1\}$.*

Definition 2.8. *Let $x, y \in \theta^*$ be any two vertices of G. Then strength of connectedness between x and y is denoted by $CONF_G(x, y)$ and defined by $CONF_G(x, y) = \vee\{S(P) : P \text{ is any } (x, y) \text{ path}\}$, where $\vee$ denotes maximum.*

Definition 2.9. *Let $x, y \in \theta^*$ be any two vertices of G. Then strong strength of connectedness between x and y is denoted by $SCONF_G(x, y)$ and defined by*

$$SCONF_G(x, y) = \frac{1}{2}[CONF_G(x, y) + CONF_G^2(x, y)].$$

Definition 2.10. *For a path P, if $S(P)$ is equal to $CONF_G(x, y)$, then P is (x, y) strongest path.*

Definition 2.11. *G is called connected if for any $x, y \in \theta^*, CONF_G(x, y) > 0$.*

Definition 2.12. *G called tree if there exist a FSG $S = (\theta, \omega)$ which is a tree and spans G, such that for any a, b not in S, there exists a path P between a and b in S such that $\rho(a, b) < S(P)$.*

Also S is a unique maximum spanning tree (MST) of G. Strong edges and δ-edges in a FG are defined in the next definition.

Definition 2.13. *Let (x, y) be an edge of G. Then the edge (x, y) is called α-strong or α-strong edge (α-st) if $\rho(x, y) > CONF_{G_{xy}}(x, y)$. The edge (x, y) is called β-strong or β-strong edge (β-st) if $\rho(x, y) = CONF_{G_{xy}}(x, y)$. The edge (x, y) is said to be δ-edge if it is neither α-st nor α-st. (x, y) is called strong edge if it is not a δ-edge.*

A strong path is a path which does not contain any δ-edge.

Definition 2.14. *A FG G is called α-saturated (α-sat) if each vertex contains at least one α-st edge in G. and β-saturated (β-sat) if each vertex contains at least one β-st edge in G. If it is both α-sat and β-sat then G is called saturated FG (sat-FG).*

The vertices and edges for a saturated cycle are characterized in the next two theorems.

Theorem 2.1. [5] *Let $G = (\theta, \rho)$ be a saturated cycle with n vertex. Then*
(i) n is an even number.
(ii) α-st edge and β-st edge alternatively appear in G.

Theorem 2.2. [5] *Let $G = (\theta, \rho)$ be a saturated cycle. Then each β-st edge has constant strength.*

Isomorphism between two FGs is defined below.

Definition 2.15. [25] *Two FGs $G_1 = (\theta_1, \rho_1)$ and $G_2 = (\theta_2, \rho_2)$ are called isomorphic if there exist a bijective map $h : \theta_1^* \to \theta_2^*$ with any $x, y \in \theta_1^*, \rho_2(h(x), h(y)) = \rho_1(x, y)$.*

3. Hyper-connectivity index for fuzzy graph

In [5] Binu et al. defined the connectivity index for a FG as:

Definition 3.1. [5] *Let $G = (\theta, \rho)$ be a FG. Then connectivity index of G is denoted by $FCI(G)$ and defined by*

$$FCI(G) = \sum_{x,y\in\theta^*} \theta(x)\theta(y)CONF_G(x, y).$$

In this section, FHCI is introduced which is more generalization of FCI and studied for various FG like as complete fuzzy graph, saturated cycles, isomorphic fuzzy graphs, etc. A relation among fuzzy graphs, partial fuzzy subgraph, fuzzy subgraph are established with respect to FHCI. Also a relation between connectivity index and hyper-connectivity index for fuzzy graphs is established and this index is calculated for saturated cycles.

Definition 3.2. *Let $G = (\theta, \rho)$ be a CNFG. The FHCI of the FG, G is defined by:*

$$\begin{aligned} FHCI(G) &= \sum_{a,b\in\theta^*} \theta(a)\theta(b)SCONF_G(a, b) \\ &= \frac{1}{2}\sum_{a,b\in\theta^*} \theta(a)\theta(b)\big[CONF_G(a, b) + CONF_G^2(a, b)\big]. \end{aligned}$$

In the next example, the FHCI of a FG depicted in Fig. 1 is calculated.

Example 3.1. *In Fig. 1, $G = (\theta, \rho)$ be a FG where $\theta^* = \{u, \cdots, z\}\rho(uv) = 0.1, \rho(ux) = 0.2, \rho(vw) = 0.3, \rho(wx) = 0.4, \rho(wy) = 0.5, \rho(xy) = 0.3, \rho(yz) = 0.2$ and $\theta(a) = 1$ for all $a \in \theta^*$. Clearly, $CONF_G(u, v) = CONF_G(u, w) = CONF_G(u, x) = CONF_G(u, y) = CONF_G(u, z) = CONF_G(v, z) = CONF_G(w, z) = CONF_G(x, z) = CONF_G(y, z) = 0.2, CONF_G(v, w) = CONF_G(v, x) = CONF_G(v, y) = 0.3, CONF_G(x, w) = CONF_G(x, y) = 0.4$ and $CONF_G(w, y) = 0.5$. Therefore, $FHCI(G) = 2.6$.*

In [5] Binu et al. defined AFCI, CRN, CEN and CNN as:

Definition 3.3. [5] *Let $G = (\theta, \rho)$ be a n-vertex FG. Then AFCI of G is denoted by $AFCI(G)$ and defined by*

$$AFCI(G) = \frac{1}{\binom{n}{2}} \sum_{x,y\in\theta^*} \theta(x)\theta(y)CONF_G(x, y).$$

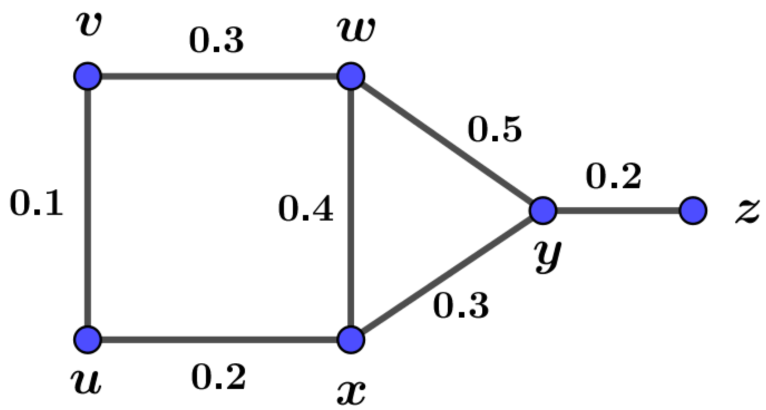

FIGURE 1. A FG G with $FHCI(G) = 2.6$.

Definition 3.4. [5] *Let $G = (\theta, \rho)$ be a FG. Let $x \in \theta^*$. x is called CEN of G if $AFCI(G-x) > AFCI(G)$. x is called CRN of G if $AFCI(G-x) < AFCI(G)$. x is called CNN of G if $AFCI(G-x) = AFCI(G)$.*

In this section, average fuzzy hyper-connectivity index, hyper-connectivity reducing node, hyper-connectivity enhancing node, hyper-connectivity neutral node are introduced.

Definition 3.5. *Let $G = (\theta, \rho)$ be a CNFG. The AFHCI of the FG, G is defined by:*

$$AFHCI(G) = \frac{1}{2 \times \binom{n}{2}} \sum_{a,b \in \theta^*} \theta(a)\theta(b)\big[CONF_G(a,b) + CONF_G^2(a,b)\big].$$

Definition 3.6. [5] *Let $G = (\theta, \rho)$ be a FG. Let $x \in \theta^*$. x is called HCEN of G if $AFHCI(G-x) > AFHCI(G)$. x is called HCRN of G if $AFHCI(G-x) < AFHCI(G)$. x is called HCNN of G if $AFHCI(G-x) = AFHCI(G)$.*

Theorem 3.1. *Let G be a $n-$vertex FG and $x \in \theta^*$. Let us define $r_x = \frac{FHCI(G)}{FHCI(G-x)}$. The necessary and sufficient condition for x be HCNN of G is $r_x = \frac{n}{n-2}$.*

Proof. Suppose x be HCNN of G. then,

$$\begin{aligned} &AFHCI(G-x) = AFHCI(G) \\ \Leftrightarrow &\frac{FHCI(G-x)}{\binom{n-1}{2}} = \frac{FHCI(G)}{\binom{n}{2}} \\ \Leftrightarrow &\frac{FHCI(G-x)}{FHCI(G)} = \frac{\binom{n-1}{2}}{\binom{n}{2}} \\ \Leftrightarrow &r_x = \frac{n}{n-2}. \end{aligned}$$

□

Corollary 3.1. *Let G be a $n-$vertex FG and $x \in \theta^*$. The necessary and sufficient condition for x be HCEN of G is $r_x < \frac{n}{n-2}$.*

Corollary 3.2. *Let G be a $n-$vertex FG and $x \in \theta^*$. The necessary and sufficient condition for x be HCRN of G is $r_x > \frac{n}{n-2}$.*

In the next example, the FHCI of the PFSG is calculated and compared with the original FG in Fig. 1.

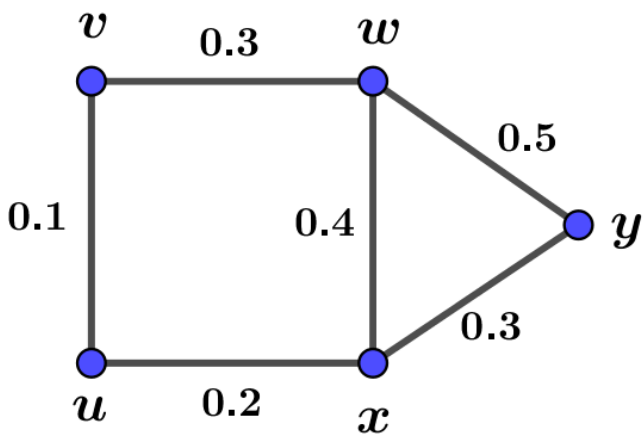

FIGURE 2. A FSG H of G (shown in Figure 1), with $FHCI(G) > FHCI(H)$.

Example 3.2. *Let $H = (\pi, \omega)$ be the PFSG shown in Fig 2 of the FG $G = (\theta, \rho)$ in Fig. 1 with $\pi^* = \{u, \cdots, y\}; \omega(uv) = 0.1, \omega(ux) = 0.2, \omega(vw) = 0.3, \omega(wx) = 0.4, \omega(wy) = 0.5, \omega(xy) = 0.3$ and $\pi(a) = 1$ for all $a \in \pi^*$. Therefore, $FHCI(H) = 2$.*

From the Examples 3.1 and 3.2, $FHCI(H) \leq FHCI(G)$ for PFSG of the FG. In the next proposition, the fact is proved in general.

Proposition 3.1. *Let $H = (\pi, \omega)$ be a PFSG of the FG, $G = (\theta, \rho)$, then $FHCI(H) \leq FHCI(G)$.*

Proof. Let $a, b \in \pi^*$. As $H = (\pi, \omega)$ be a PFSG of the FG $G = (\theta, \rho)$ then $\pi(u) \leq \theta(u)$ and $CONF_H(a, b) \leq CONF_G(a, b)$ for all $u \in \pi^*$. Hence,

$$\begin{aligned} 2FHCI(H) &= \sum_{a,b \in \pi^*} \pi(a)\pi(b)[CONF_H(a, b) + CONF_H^2(a, b)] \\ &\leq \sum_{a,b \in \theta^*} \theta(a)\theta(b)[CONF_G(a, b) + CONF_G^2(a, b)] \\ &= 2FHCI(G). \end{aligned}$$

Therefore $FHCI(H) \leq FHCI(G)$. □

A fuzzy subgraph is also PFSG, so the above proposition is true for fuzzy subgraph which is stated in the next corollary.

Corollary 3.3. *Let $H = (\pi, \omega)$ be a fuzzy subgraph of the FG, $G = (\theta, \rho)$, then $FHCI(H) \leq FHCI(G)$.*

In the next theorem, a bound for FHCI of a FG is provided.

Theorem 3.2. *Let $G = (\theta, \rho)$ be a FG such that $|\theta^*| = n$ and $G_1 = (\theta_1, \rho_1)$ be CPFG spanned by θ. Then $0 \leq FHCI(G) \leq FHCI(G_1)$.*

Proof. Let $G = (\theta, \rho)$ is a FG. For $|\rho^*| = 0$, implies $FHCI(G) = 0 \leq FHCI(G_1)$. Now for G_1, $|\theta^*| = n$ and $\theta_1 = \theta$. Then $\rho(ab) \leq \rho_1(ab)$. Hence $CONF_G(a, b) \leq CONF_{G_1}(a, b)$ and $CONF_G^2(a, b) \leq CONF_{G_1}^2(a, b)$. Therefore, $0 \leq FHCI(G) \leq FHCI(G_1)$. □

In the next theorem, the value FHCI of a CPFG is provided.

Theorem 3.3. *Let $G = (\theta, \rho)$ be a CPFG such that $\theta^* = \{a_1, a_2, \cdots, a_n\}$ such that $p_1 \leq p_2 \leq \cdots \leq p_n$, where $p_i = \theta(a_i)$ for $1 \leq i \leq n$. Then*

$$FHCI(G) = \frac{1}{2}\sum_{i=1}^{n-1}(p_i^2 + p_i^3)\sum_{j=i+1}^{n} p_j.$$

Proof. Here the vertex a_1 has the least MV p_1. For a CPFG, it is clear that, $CONF_G(a, b) = \rho(ab)$ for all $a, b \in \theta^*$. Thus $\rho(a_1a_i) = p_1$ for $2 \leq i \leq n$, hence $\theta(a_1)\theta(a_i)CONF_G(a_1, a_i) = p_1^2 p_i$ and $\theta(a_1)\theta(a_i)CONF_G^2(a_1, a_i) = p_1^3 p_i$ for $2 \leq i \leq n$. Then

$$\sum_{i=2}^{n}\theta(a_1)\theta(a_i)[CONF_G(a_1, a_i) + CONF_G^2(a_1, a_i)] = \sum_{i=2}^{n} p_1^2 p_i + \sum_{i=2}^{n} p_1^3 p_i.$$

Therefore,

$$\begin{aligned} FHCI(G) &= \frac{1}{2}\Big(\sum_{i=1}^{n-1} p_i^2 \sum_{j=i+1}^{n} p_j + \sum_{i=1}^{n-1} p_i^3 \sum_{j=i+1}^{n} p_j\Big) \\ &= \frac{1}{2}\sum_{i=1}^{n-1}(p_i^2 + p_i^3)\sum_{j=i+1}^{n} p_j. \end{aligned}$$

□

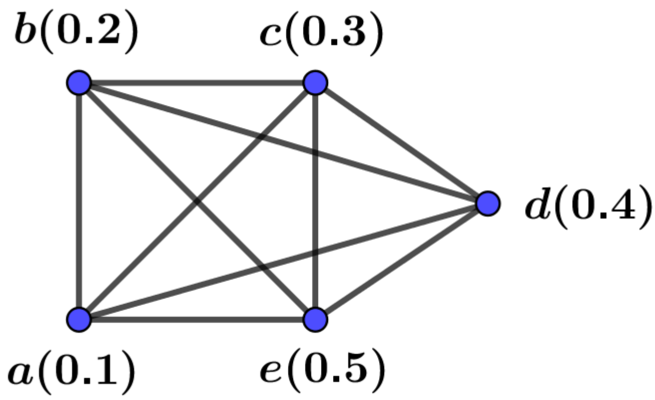

FIGURE 3. A CPFG G_1 with $FHCI(G_1) = 0.14515$.

In the next example, FHCI of the CPFG in Fig. 3 is calculated by using Theorem 3.3.

Example 3.3. *Let $G_1 = (\theta, \rho)$ is the CPFG in Fig. 3 with $\theta^* = \{a, \cdots, e\}$ where* $\theta(a) = 0.1, \theta(b) = 0.2, \theta(c) = 0.3, \theta(d) = 0.4, \theta(e) = 0.5$. *Then* $\rho(ab) = 0.1, \rho(ac) = 0.1, \rho(ad) = 0.1, \rho(ae) = 0.1, \rho(bc) = 0.2, \rho(bd) = 0.2, \rho(be) = 0.2, \rho(cd) = 0.3, \rho(ce) = 0.3, \rho(de) = 0.4$. *Then by using Theorem 3.3,* $2FHCI(G_1) = (0.1^2 + 0.1^3)(0.2 + 0.3 + 0.4 + 0.5) + (0.2^2 + 0.2^3)(0.3 + 0.4 + 0.5) + (0.3^2 + 0.3^3)(0.4 + 0.5) + (0.4^2 + 0.4^3)(0.5) = 0.2903$. *Hence* $FHCI(G_1) = 0.14515$.

In the next example, FCI and FHCI of the FG G_2 depicted in Fig. 4 is calculated and compares FCI and FHCI for the FG G_2.

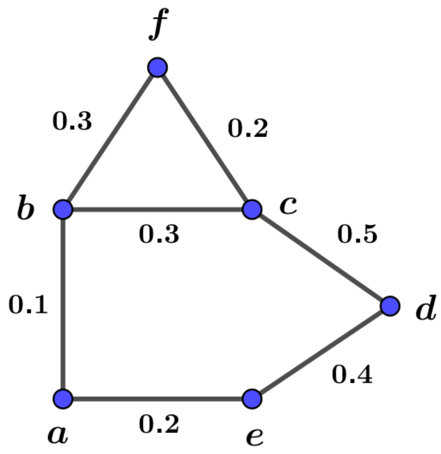

Figure 4. A FG G_2 with $FCI(G_2) > FHCI(G_2)$.

Example 3.4. *Let* $G_2 = (\theta, \rho)$ *be the FG in Fig. 4 with* $\theta^* = \{a, \cdots, f\}$*;* $\rho(ab) = 0.1, \rho(ae) = 0.2, \rho(bc) = 0.3, \rho(cd) = 0.5, \rho(de) = 0.4, \rho(bf) = 0.3, \rho(cf) = 0.2$ *and* $\theta(v) = 1$ *for all* $v \in \theta^*$*. Then,* $FCI(G_2) = 4.4$ *and* $FHCI(G_2) = 2.9$*. So* $FCI(G_2) > FHCI(G_2)$*.*

But for the FG G_2, let $\theta(v) = 1\, \forall\, v \in \theta^*$ and $\rho(e) = 1\, \forall\, e \in \rho^*$. Then $FCI(G_2) = FHCI(G_2)$.

From the Example 3.4, $FCI(G) \geq FHCI(G)$ for a FG. In the next theorem the fact is proved in general.

Theorem 3.4. *Let* $G = (\theta, \rho)$ *be a CNFG. Then* $FCI(G) \geq FHCI(G)$*, equality holds if* $\rho(e) = 1$ *for all* $e \in \rho^*$*.*

Proof. As $0 < \rho(e) \leq 1$ for all $e \in \rho^*$ then $CONF_G(a, b) \geq CONF_G^2(a, b)$, $\forall\, a, b \in \theta^*$. Now

$$\begin{aligned} FCI(G) = &\sum_{a,b \in \theta^*} \theta(a)\theta(b) CONF_G(a, b) \\ =&\frac{1}{2} \sum_{a,b \in \theta^*} \theta(a)\theta(b) \big[CONF_G(a, b) + CONF_G(a, b)\big] \\ \geq&\frac{1}{2} \sum_{a,b \in \theta^*} \theta(a)\theta(b) \big[CONF_G(a, b) + CONF_G^2(a, b)\big] \\ =&FHCI(G). \end{aligned}$$

Equality holds if $\rho(e) = 1$ for all $e \in \rho^*$. □

In the next theorem, the value CI and FHCI of a CNFG is provided.

Theorem 3.5. *Let* $G = (\theta, \rho)$ *be the CNFG with n-vertex and* $\theta(a) = 1\, \forall\, a \in \theta^*$*,* $\rho(e) = x\, \forall\, e \in \rho^*$ *,* $0 < x \leq 1$*. Then* $FCI(G) = \frac{n(n-1)}{2} x$ *and* $FHCI(G) = \frac{n(n-1)}{4}(x + x^2)$*.*

Proof. Clearly $CONF_G(a, b) = x$ and $CONF_G^2(a, b) = x^2$ for all $a, b \in \theta^*$. There are $\binom{n}{2}$ numbers of such pairs (a, b). Hence the results follow. □

HCI of any two isomorphic FGs are discussed below:

Theorem 3.6. *Let two FGs $G_1 = (\theta_1, \rho_1)$ and $G_2 = (\theta_2, \rho_2)$ be isomorphic. Then $FHCI(G_1) = FHCI(G_2)$.*

Proof. Let $G_1 = (\theta_1, \rho_1)$ and $G_2 = (\theta_2, \rho_2)$ be isomorphic FGs. Let ϕ be a bijective map from θ_1^* to θ_2^* such that $\theta_1(a) = \theta_2(\phi(a))$ for $a \in \theta_1^*$ and $\rho_1(ab) = \rho_2(\phi(a)\phi(b))$ for $ab \in \rho_1^*$. Since G_1, G_2 are isomorphic then $CONF_{G_1}(a, b) = CONF_{G_2}(\phi(a), \phi(b))$ for $a, b \in \theta_1^*$. Hence

$$\begin{aligned} FHCI(G_1) =& \frac{1}{2}\sum_{a,b\in\theta_1^*} \theta_1(a)\theta_1(b)\big[CONF_{G_1}(a, b) + CONF^2_{G_1}(a, b)\big] \\ =& \frac{1}{2}\sum_{\phi(a),\phi(b)\in\theta_2^*} \theta_2(\phi(a))\theta_2(\phi(b))\big[CONF_{G_2}(\phi(a), \phi(b)) + CONF^2_{G_2}(\phi(a), \phi(b))\big] \\ =& FHCI(G_2). \end{aligned}$$

□

In the next theorem, the FHCI of a star is calculated.

Theorem 3.7. *Let $G = (\theta, \rho)$ be a FG where G^* be a star with $\theta^* = \{a_1, a_2, \cdots, a_n\}$ such that $\theta(a_i) = 1$ for $1 \le i \le n$ and $0 < p_1 \le p_2 \le \cdots \le p_{n-1}$, where $p_{i-1} = \rho(a_1 a_i)$ for $2 \le i \le n$. Then, $FHCI(G) = \frac{1}{2}\sum_{i=1}^{n-1}(n - i)(p_i + p_i^2)$.*

Proof. Let a_1 be the center of the star. Then $CONF_G(a_1, a_i) = p_{i-1}$ for $1 \le i \le n$ and $CONF_G(a_i, a_j) = p_{i-1}$ for $i \ne 1, i < j$. So

$$\begin{aligned} 2FHCI(G) &= \sum_{i=1}^{n-1} p_i + (n-2)p_1 + (n-3)p_2 + \cdots + p_{n-2} \\ &+ \sum_{i=1}^{n-1} p_i^2 + (n-2)p_1^2 + (n-3)p_2^2 + \cdots + p_{n-2}^2 \\ &= \sum_{i=1}^{n-1}(n-i)p_i + \sum_{i=1}^{n-1}(n-i)p_i^2 \\ &= \sum_{i=1}^{n-1}(n-i)(p_i + p_i^2). \end{aligned}$$

Therefore

$$FHCI(G) = \frac{1}{2}\sum_{i=1}^{n-1}(n-i)(p_i + p_i^2).$$

□

In the next theorem, the FHCI of a saturated cycle is discussed.

Theorem 3.8. *Let $G = (\theta, \rho)$ be a n-vertex saturated cycle. Let strength of the each α-st edge be σ and each β-st edge be μ, then*

$$FHCI(G) = \frac{n}{4}[(\sigma + \sigma^2) + (n-2)(\mu + \mu^2)].$$

Proof.

$$
\begin{aligned}
2FHCI(G) &= \sum_{a,b\in\theta^*} \theta(a)\theta(b)\big[CONF_G(a,b)CONF_G^2(a,b)\big] \\
&= \frac{n}{2}(\sigma+\mu)+(n-3)\mu+(n-3)\mu+(n-4)\mu+\cdots+\mu \\
&+ \frac{n}{2}(\sigma^2+\mu^2)+(n-3)\mu^2+(n-3)\mu^2+(n-4)\mu^2+\cdots+\mu^2 \\
&= \frac{n}{2}[(\sigma+\sigma^2)+(n-2)(\mu+\mu^2)]
\end{aligned}
$$

Hence

$$FHCI(G) = \frac{n}{4}[(\sigma+\sigma^2)+(n-2)(\mu+\mu^2)].$$

□

Corollary 3.4. *Let $G = (\theta, \rho)$ be a n-vertex saturated cycle. Let strength of the α-st edges are $\sigma_1, \sigma_2, \cdots, \sigma_{\frac{n}{2}}$ and each β-st edges have strength μ. Then*

$$FHCI(G) = \frac{1}{2}\sum_{i=1}^{\frac{n}{2}}(\sigma_i+\sigma_i^2)+\frac{n}{4}(n-2)(\mu+\mu^2).$$

4. Application of fuzzy hyper-connectivity index

Many real life problems can not be represented by crisp graphs. For such circumstances fuzzy graphs are one of the best tools to handle it. In this section, a decision making problem is described and a decision is taken by using fuzzy hyper-connectivity index.

4.1. **Decision-making for selecting a quarter.** Suppose a faculty Dr. X in a university U wants to choose a faculty quarter to stay in the university campus. Let there be only four quarters available for Dr. X. Suppose the quarters are Q_1, Q_2, Q_3, Q_4. Let Dr. X want to choose the best quarter based on the following four parameters:

P_1 = Room size and interior decoration,
P_2 = Distance between quarters and his/her department,
P_3 = Distance between quarters and shopping mall and
P_4 = Distance between quarters and sports complex and gymkhana, etc.

In general, these parameters are different for the quarters. But, all the parameters have their equal importance to evaluate the best quarter. Since the parameter P_1 is linguistic terms, so there is no fixed value for P_1. This term is evaluated by perception of the people residing in the campus and a membership grade is used to represent its value.

For illustration, suppose the value of the membership grade for P_1, distance from quarters to department, shopping mall, sports complex and gymkhana and distance between quarters are listed in Table 1. Let us consider the diameter of the university (D) is around 10 kilometer. Now a fuzzy graph for each parameter is constructed whose vertices are the quarters and each pair of vertices connected by an edge. The vertex and edge membership values are calculated by the following formulas and listed in Table 2,3. The fuzzy graph for each parameter is shown in Figure 5.

$$\sigma_{P_i}(Q_j) = \begin{cases} \text{membership grade of } Q_j \text{ for } P_i & i=1; j=1,2,3,4 \\ \frac{\text{D}-\text{dist}_{P_i}(Q_j)}{\text{D}} & i=2,3,4; j=1,2,3,4. \end{cases}$$

$$\mu_{P_i}(Q_jQ_k) = \min\{\sigma_{P_i}(Q_j)\frac{\text{D}-\text{dist}_{P_i}(Q_jQ_k)}{\text{D}}, \sigma_{P_i}(Q_k)\frac{\text{D}-\text{dist}_{P_i}(Q_jQ_k)}{\text{D}}\} \quad i,j=1,2,3,4.$$

The connectivity matrix of a FG is defined as a symmetric square matrix whose (i,j)th

Table 1. Membership grade for P_1 and distance from quarters to department, shopping mall, sports complex and gymkhana and distance between quarters (distance measured in kilometer).

	P_1	P_2	P_3	P_4		Q_1	Q_2	Q_3	Q_4
Q_1	0.5	4.7	3.9	6.1	Q_1		3.9	4.8	5.6
Q_2	0.4	7.1	4.9	3.5	Q_2	3.9		5.3	5.2
Q_3	0.6	5.8	2.9	5.3	Q_3	4.8	5.3		4.9
Q_4	0.7	3.5	5.7	4.9	Q_4	5.6	5.2	4.9	

Table 2. Vertex membership values.

Quarter	P_1	P_2	P_3	P_4
Q_1	0.5	0.53	0.61	0.39
Q_2	0.4	0.29	0.51	0.65
Q_3	0.6	0.42	0.71	0.47
Q_4	0.7	0.65	0.43	0.51

Table 3. Edge membership values.

P_1	Q_1	Q_2	Q_3	Q_4	P_2	Q_1	Q_2	Q_3	Q_4
Q_1		0.244	0.26	0.22	Q_1		0.1769	0.2184	0.2332
Q_2	0.244		0.188	0.192	Q_2	0.1769		0.1363	0.1392
Q_3	0.26	0.118		0.306	Q_3	0.2184	0.1363		0.2142
Q_4	0.22	0.192	0.306		Q_4	0.2332	0.1392	0.2142	
P_3	Q_1	Q_2	Q_3	Q_4	P_4	Q_1	Q_2	Q_3	Q_4
Q_1		0.3111	0.3172	0.1892	Q_1		0.2379	0.2028	0.1716
Q_2	0.3111		0.2397	0.2064	Q_2	0.2379		0.2209	0.2448
Q_3	0.3172	0.2397		0.2193	Q_3	0.2028	0.2209		0.2397
Q_4	0.1892	0.2064	0.2193		Q_4	0.1716	0.2448	0.2397	

Table 4. Connectivity matrix of the fuzzy graphs shown in Figure 5.

G_{P_1}	Q_1	Q_2	Q_3	Q_4	G_{P_2}	Q_1	Q_2	Q_3	Q_4
Q_1		0.244	0.26	0.26	Q_1		0.1769	0.2184	0.2332
Q_2	0.244		0.244	0.244	Q_2	0.1769		0.1769	0.1769
Q_3	0.26	0.244		0.306	Q_3	0.2184	0.1769		0.2184
Q_4	0.26	0.244	0.306		Q_4	0.2332	0.1769	0.2184	
G_{P_3}	Q_1	Q_2	Q_3	Q_4	G_{P_4}	Q_1	Q_2	Q_3	Q_4
Q_1		0.3111	0.3172	0.2193	Q_1		0.2379	0.2379	0.2379
Q_2	0.3111		0.3111	0.2193	Q_2	0.2379		0.2397	0.2448
Q_3	0.3172	0.3111		0.2193	Q_3	0.2379	0.2397		0.2397
Q_4	0.2193	0.2193	0.2193		Q_4	0.2379	0.2448	0.2397	

entry is $CONF(v_i, v_j)$. The connectivity matrix of the FGs shown in Figure 5 is listed in Table 4. Using the connectivity matrix shown in Table 4, the value of the FHCI for each parameter is 0.299667, 0.163776, 0.332977 and 0.225119 respectively. Now the total

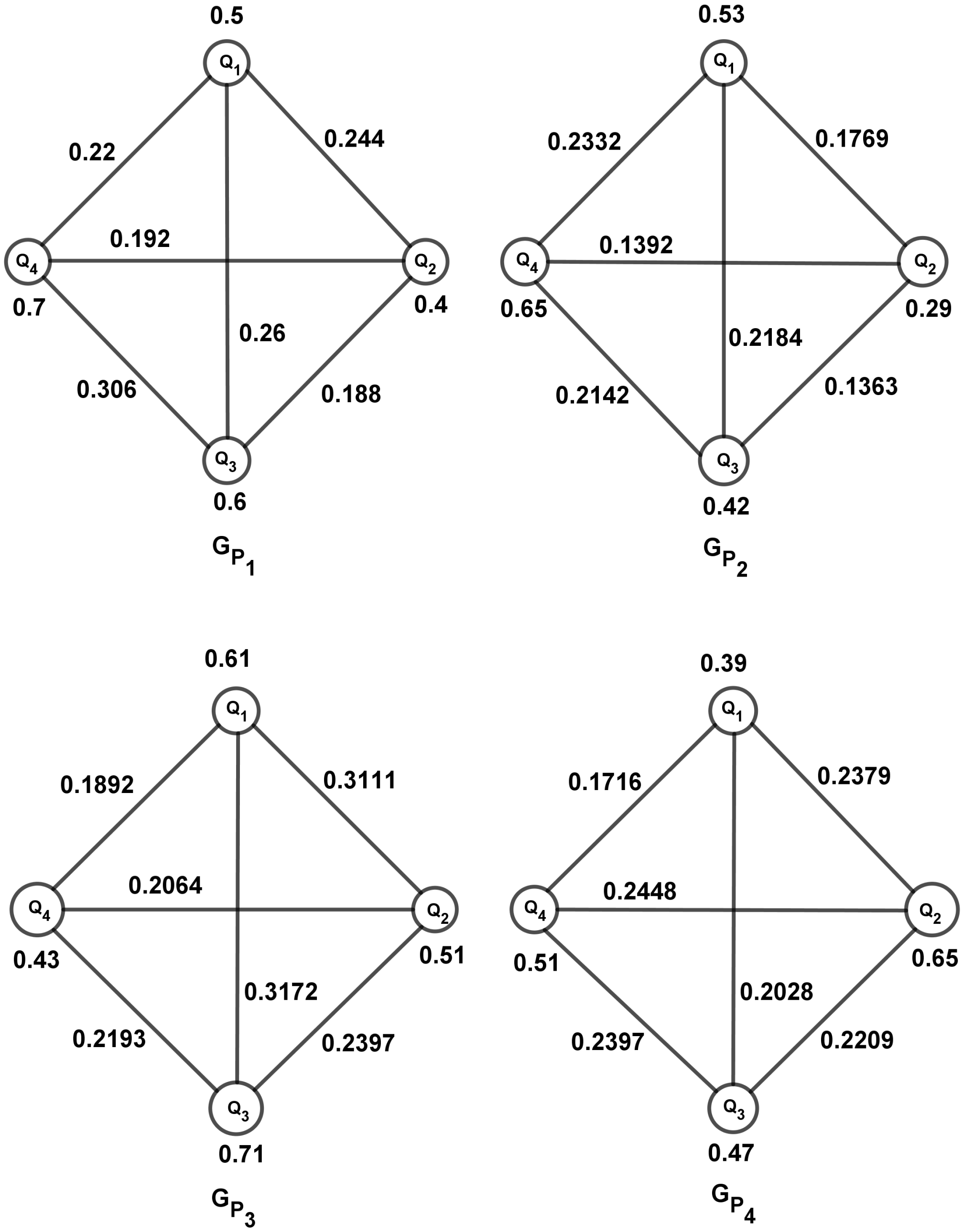

FIGURE 5. Fuzzy graph representation of the decision making problem.

hyper-connectivity index (TFHCI) is defined as the sum of FHCI for each parameter. For the fuzzy graph G, TFHCI is 1.021539.

Now a FSG is constructed from the FG G by deleting the vertex Q_i and calling it G_{Q_i}. Using a similar manner, FHCI and TFHCI are calculated for the FSG G_{Q_i} for $i = 1, 2, 3, 4$ and listed in Table 5. Now score of Q_i is calculated by the formula:

$$S(Q_i) = TFHCI(G) - TFHCI(G_{Q_i}).$$

TABLE 5. Hyper-connectivity index and total hyper-connectivity index.

Fuzzy graph	Hyper-connectivity index for P_1	Hyper-connectivity index for P_2	Hyper-connectivity index for P_3	Hyper-connectivity index for P_4	Total hyper-connectivity index
G_{Q_1}	0.1434282	0.06010431	0.12393716	0.131513257	0.458982927
G_{Q_2}	0.19039356	0.115475264	0.166364009	0.082228768	0.554461601
G_{Q_3}	0.1148996	0.085157968	0.123405664	0.117123595	0.440586827
G_{Q_4}	0.11591792	0.05829544	0.227771509	0.10324132	0.505226189

TABLE 6. Score and normalized score of the quarters.

Quarter	Q_1	Q_2	Q_3	Q_4
Score (FHCI)	0.562556073	0.467077399	0.580952173	0.516312811
Score (FCI)	0.816861	0.729974	0.886138	0.819639
Score (First ZI)	1.680354	1.790257	1.565369	1.648194

The score of the quarters are calculated and listed in Table 6. As score is directly related to connectivity index and if we increase the vertex membership value of a vertex the score is also increased and same occurs when we increase an edge membership value of an edge joined with that vertex. As the score of quarter Q_3 is higher than other quarters, so quarter Q_3 is the best choice for the faculty Dr. X.

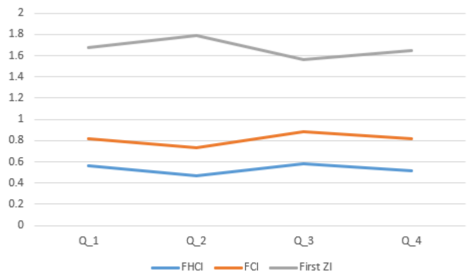

FIGURE 6. Comparison graph.

4.2. **Analysis of hyper-connectivity index for fuzzy graph and its comparison with existing indices on result of application.** The comparisons with other exiting method are useful to evaluate the quality of our proposed method. We summarize it as under:

(i) It is noticed that topological indices for crisp graph is depends on the structure of the graph but does not depend on the behavior of the vertices or the relation between the pair of vertices. As indices which are defined on fuzzy graph are depend on the behavior of the vertices and relation between the each pair of vertices, hence those indices are more suitable for real life application. Hence the indices which are defined in crisp graph is ignored in this comparison. For this comparison, the decision-making problem described in Subsection 4.1 are chosen and compared with the indices: first Zagreb index for fuzzy graph, connectivity index for fuzzy graph.

(ii) Score of each quarter with respect to those indices are listed in Table 6 and comparison graph of the score of those indices are drown in Figure 6.

(iii) Preference of quarter choice using hyper-connectivity index for fuzzy graph is: $Q_3 \succ Q_1 \succ Q_4 \succ Q_2$.

(iv) Preference of quarter choice using connectivity index for fuzzy graph is: $Q_3 \succ Q_4 \succ Q_1 \succ Q_2$.

(v) Preference of quarter choice using first Zagreb index for fuzzy graph is: $Q_2 \succ Q_1 \succ Q_4 \succ Q_3$.

(vi) The decision getting by using connectivity index and hyper-connectivity index for fuzzy graph is same but second and third choice is alternate for those index. But the decision getting by using first Zagreb index for fuzzy graph does not related with other two indices.

(vii) Connectivity index for fuzzy graph is depends proportionally on the strength of connectedness between each pair of vertices but hyper-connectivity index for fuzzy graph is depends proportionally on the strong strength of connectedness between each pair of vertices. Hence, hyper-connectivity index for fuzzy graph gives the better results for connection related real life application than connectivity index for fuzzy graph.

(viii) As first Zagreb index for fuzzy graph depends on degree i.e. depends only the neighboring vertices and does not depends on the other vertices, hence this index is not appropriate for connectivity related decision making problem.

Hence, hyper-connectivity index for fuzzy graph provides the better result on connectivity related real life problem than other indices.

5. Conclusion

FHCI has an important parameter in fuzzy graph theory. In this article, FHCI is introduced and studied various fuzzy graphs like fuzzy partial subgraph, fuzzy subgraph, saturated cycle, isomorphic fuzzy graphs, etc. A relation among fuzzy graphs, partial fuzzy subgraphs, fuzzy subgraphs are established with respect to FHCI. Also a relation between connectivity index and hyper-connectivity index for fuzzy graphs is established and this index is calculated for saturated cycles. Also in the end of this article, a decision making problem is presented and solved by using FHCI.

The limitations and future scopes of the study are:

(i) Good lower bound of the FHCI for n-vertex connected FG is presented here but we cannot provide the good upper bound.

(ii) Good upper bound of the FHCI for path and star are presented here but we cannot provide the good lower bound.

(iii) Which $n-$vertex tree (fuzzy) has the maximum FHCI?

(iv) Which $n-$vertex tree (fuzzy) has the minimum FHCI?

(v) Which $n-$vertex connected FG has the maximum FHCI?

(vi) Which $n-$vertex connected FG has the minimum FHCI?

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Sk Rabiul Islam for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.13, N.2.

M. Pal for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.13, N.2.

Table 7. The list of abbreviation.

Abbreviation	Meaning
FSS	Fuzzy subset
FG	Fuzzy graph
FSG	Fuzzy subgraph
PFSG	Partial fuzzy subgraph
CNFG	Connected fuzzy graph
CPFG	Complete fuzzy graph
CI	Connectivity index
FHCI	fuzzy Hyper-connectivity index
AFHCI	Average fuzzy Hyper-connectivity index
HCRN	Hyper-connectivity reducing node
HCEN	Hyper-connectivity enhancing node
HCNN	Hyper-connectivity neutral node
MV	Membership value