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CUBIC PYTHAGOREAN FUZZY LINEAR SPACES

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Abstract. Pythagorean fuzzy sets assist in handling more uncertain and vague data than fuzzy sets and intuitionistic fuzzy sets. The notion of cubic pythagorean fuzzy sets is defined by combining interval valued pythagorean fuzzy sets and pythagorean fuzzy sets In this paper, based on the notion of cubic Pythagorean fuzzy sets we initiate a new theory called cubic pythagorean fuzzy linear spaces. Inspired by the notion of Cubic linear spaces we also present P(resp. R)-union and P(resp. R) intersection of cubic pythagorean fuzzy linear spaces. The concept of internal(resp. external) cubic pythagorean fuzzy linear spaces and its properties are examined.

Keywords: pythagorean fuzzy sets, interval valued pythagorean fuzzy sets, cubic pythagorean fuzzy sets, cubic pythagorean fuzzy linear spaces.

AMS Subject Classification: 08A72, 03E72.

1. Introduction

The concept of intuitionistic fuzzy sets was introduced by Attansov [1] as generalisation of Zadeh's fuzzy sets [17] has wide applications in decision making problems. Zadeh's proposed fuzzy set theory with the membership function efficaciously characterises the uncertain situations and thus assist in making a right choice. Further Attansov's intuitionistic fuzzy sets contain two elements that is membership and non membership degree with the condition that $\mu + \upsilon \leq 1$. The characterization of fuzzy information is more detailed and far reaching than Zadeh's fuzzy sets. The intuitionistic fuzzy sets have been further extended to interval valued intuitionistic fuzzy sets, hesistant fuzzy set, triangular intuitionistic fuzzy sets, trapezoidal intuitionistic fuzzy sets and normal intuitionistic fuzzy sets. The decision makers can characterize the fuzzy information more effectively and make acute decisions without altering the given fuzzy information. In some practical problems whose sum of the membership and non membership degrees provided by the model user (or) decision maker is bigger than 1 but their square sum is less than (or) equal

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to 1. Therefore to deal with such cases Yager [16]recently introduced pythagorean fuzzy sets . Later on the notion of interval valued pythagorean fuzzy sets is given by Peng and Yang [9]. Based on the concept of cubic sets [6], [10], [11], [12] cubic pythagorean fuzzy sets were introduced where the membership degree is interval valued pythagorean fuzzy set and non membership degree is pythagorean fuzzy set. The applications of pythagorean fuzzy sets can be observed in [3], [4], [5], [13].
In the current paper, motivated by the notion of fuzzy linear spaces[7], [14] and cubic pythagorean fuzzy sets [2], interval valued fuzzy linear spaces, cubic linear spaces[15], cubic $\Gamma-$ n normed linear spaces and $N-$ cubic sets applied to linear spaces[8] we introduce the idea of cubic pythagorean fuzzy linear spaces. We introduce the concept of $P-$(resp. R)union and $P-$(resp R)intersection of internal and external cubic pythagorean fuzzy linear spaces and their properties with examples.

2. Basic Definitions

Definition 2.1. [2] *Claiming X to be a fixed set, a pythagorean Fuzzy Set P in X can be defined as*

$$P_y = \{(x, \eta_{p_y}(x), \theta_{p_y}(x))|x \in X\}$$

where $\eta_{p_y} : X \to [0,1]$ represents the degree of membership of an element $x \in X$ and $\theta_{p_y}(x) : X \to [0,1]$ represents the degree of non membership of an element $x \in X$ satisfying the condition that $0 \leq \eta_{p_y}(x) \leq 1$, $0 \leq \theta_{p_y}(x) \leq 1$ and $0 \leq \eta^2_{p_y}(x) + \theta^2_{p_y}(x) \leq 1$ for all $x \in X$.
Now the degree of indeterminacy of x to P is given as

$$D_P(x) = \sqrt{1 - \eta^2_{p_y}(x) - \theta^2_{p_y}(x)}$$

$D_{p_y}(x)$ satisfies the condition that $0 \leq D_{p_y}(x)$ for every $x \in X$.

Definition 2.2. [9] *Taking X as a fixed set an interval valued pythagorean fuzzy set on X is defined as*

$$\widetilde{P_y} = \{(x, \widetilde{\eta_{p_y}}(x), \widetilde{\theta_{p_y}}(x)|x \in X)\}$$

where $\widetilde{\eta_{p_y}}(x) = [\eta^l_{p_y}(x), \eta^u_{p_y}(x)] \subset [0,1]$ and $\widetilde{\theta_{p_y}}(x) = [\theta^l_{p_y}(x), \theta^u_{p_y}(x)] \subset [0,1]$ with $\eta^l_{p_y}(x) = inf\ \eta_{p_y}(x)$ and $\eta^u_{p_y}(x) = sup\ \eta_{p_y}(x)$ like wise $\theta^l_{p_y}(x) = inf\ \theta_{p_y}(x)$ and $\theta^u_{p_y}(x) = sup\ \theta_{p_y}(x)$.

Definition 2.3. [2] *Consider X to be a non empty set. A cubic pythagorean fuzzy set of X is a structure of the form*

$$C_{P_y} = \{x, \widetilde{P_y}(x), P_y(x)|x \in X\},$$

in which $\widetilde{P_y}$ is an interval valued pythagorean fuzzy set in X and P is a pythagorean fuzzy set in X.

Definition 2.4. *For any* $C_{P_{y_i}} = \{x, \widetilde{P_{y_i}}(x), P_{y_i}(x)|x \in X\}$ *where* $i \in \Lambda$, *where* Λ *is an index set. Now we define*

$$(i)\bigcup_{i\in\Lambda}{}_{\mathrm{p}} C_{P_{y_i}} = \left\{\langle x, \bigcup_{i\in\Lambda}\widetilde{P_{y_i}}(x), \bigcup_{i\in\Lambda}P_{y_i}(x)|x \in X\rangle\right\}(\mathrm{P}-union)$$

$$(ii)\bigcap_{i\in\Lambda}{}_{\mathrm{p}} C_{P_{y_i}} = \left\{\langle x, \bigcap_{i\in\Lambda}\widetilde{P_{y_i}}(x), \bigcap_{i\in\Lambda}P_{y_i}(x)|x \in X\rangle\right\}(\mathrm{P}-intersection)$$

$$(iii)\bigcup_{i\in\Lambda}{}_{\mathrm{r}} C_{P_{y_i}} = \left\{\langle x, \bigcup_{i\in\Lambda}\widetilde{P_{y_i}}(x), \bigcap_{i\in\Lambda}P_{y_i}(x)|x \in X\rangle\right\}(\mathrm{R}-union)$$

$$(iv)\bigcap_{i\in\Lambda}{}_{\mathrm{r}} C_{P_{y_i}} = \left\{\langle x, \bigcap_{i\in\Lambda}\widetilde{P_{y_i}}(x), \bigcup_{i\in\Lambda}P_{y_i}(x)|x \in X\rangle\right\}(\mathrm{R}-intersection)$$

3. Results

3.1. cubic pythagorean fuzzy linear spaces.

Definition 3.1. *For a non empty linear space L over a field F a pythagorean fuzzy set* $P_y = \{X, \eta_{p_y}(x), \theta_{p_y}(x)|x \in X\}$ *is said to be a pythagorean fuzzy linear space* $Ł^{P_y} = \{L, \eta_{p_y}, \theta_{p_y}\}$ *where* $\eta_{p_y} : L \to [0, 1]$ *and* $\theta_{p_y} : L \to [0, 1]$ *and also satisfies following conditions:*

$$Ł^{P_y}(\alpha l_1 * \beta l_2) \geq Ł^{P_y}(l_1) \cap Ł^{P_y}(l_2)$$

for any l_1, l_2 *in* L *and* $\alpha, \beta \in F$.

Definition 3.2. *An interval valued pythagorean fuzzy set* $\widetilde{P_y} = \langle\widetilde{\eta_{P_y}}, \widetilde{\theta_{P_y}}\rangle$ *on* X *is said to be an interval valued pythagorean fuzzy linear space denoted by* $Ł^{\widetilde{P_y}} = \{L, \widetilde{\eta_{P_y}}, \widetilde{\theta_{P_y}}\}$ *over a field F if the following conditions are satisfied*

$$Ł^{\widetilde{P_y}}(\alpha l_1 * \beta l_2) \geq min\{Ł^{\widetilde{P_y}}(l_1), Ł^{\widetilde{P_y}}(l_2)\}$$

for any l_1, l_2 *in* L *and* $\alpha, \beta \in F$.

Definition 3.3. *For a linear space Ł over field F a cubic pythagorean fuzzy set* $C_{P_y} = \langle\widetilde{P_y}, P_y\rangle$ *is said to be a cubic pythagorean fuzzy linear space of Ł if the following conditions are satisfied:*

$$(i)\ \widetilde{P_y}(\alpha l_1 * \beta l_2) \geq min\{\widetilde{P_y}(l_1), \widetilde{P_y}(l_2)\}$$
$$(ii)\ P_y(\alpha l_1 * \beta l_2) \leq max\{P_y(l_1), P_y(l_2)\}$$

for any $l_1, l_2 \in Ł$ *and* $\alpha, \beta \in F$.

Example 3.1. *Let us take a numerical example for cubic pythagorean Fuzzy set. Taking X to be a non empty universal set consider the values tabulated below.*

Table 1. Values of interval valued pythagorean fuzzy sets and pythagorean Fuzzy sets

X	$\widetilde{P_y}$	P_y
x_1	$([0.6, 0.8], [0.5, 0.6])$	$[0.09, 0.21]$
x_2	$([0.5, 0.7], [0.4, 0.5])$	$[0.5, 0.6]$
x_3	$([0.6, 0.8], [0.6, 0.8])$	$[0.5, 0.7]$

From the above table we observe that $\widetilde{P_y}$ is an interval valued pythagorean fuzzy linear space and P_y is a pythagorean fuzzy linear space of X over the field $GF(2)$ with the binary operation $l_2 * l_3 = l_1$ and $\alpha = \beta = 1$. With the above condition we observe that $[0.6, 0.8] \geq [0.6, 0.8]$ and $[0.5, 0.6] \geq [0.5, 0.7]$ which is sensical.Thus the above example satisfied the condition required for it to be an interval valued pythagorean Fuzzy linear space. And hence the above example indeed satisfied the conditions required for the cubic pythagorean fuzzy set to be a cubic pythagorean fuzzy linear space.

Definition 3.4. *Let $\widetilde{P}_{y_1} = (\widetilde{\eta_{P_{y_1}}}, \widetilde{\theta_{P_{y_1}}})$ and $\widetilde{P_{y_2}} = (\widetilde{\eta_{P_{y_2}}}, \widetilde{\theta_{P_{y_2}}})$ be two interval valued pythagorean fuzzy linear spaces. Then the operations union and intersection can be explained as*

$$\begin{aligned} \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(x) &= min\{\widetilde{P_{y1}}(x), \widetilde{P_{y2}}(x)\} \\ \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(x) &= max\{\widetilde{P_{y1}}(x), \widetilde{P_{y2}}(x)\}, \quad x \in X \end{aligned}$$

In the similar way we can define union and intersection of pythagorean fuzzy linear spaces.

Theorem 3.1. *Let $C_{P_{y_1}} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ and $C_{P_{y_2}} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ be two cubic pythagorean fuzzy linear spaces. Then their R-intersection $(C_{P_{y_1}} \cap C_{P_{y_2}}) = (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}, P_{y_1} \cup P_{y_2})$ is again a cubic pythagorean fuzzy linear space.*

Proof. $\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(x) = min\{\widetilde{P_{y_1}}(x), \widetilde{P_{y_2}}(x)\}$
We have,

$$\begin{aligned} \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(\alpha l_1 * \beta l_2) &= min\Big\{\widetilde{P_{y_1}}(\alpha l_1 * \beta l_2), \widetilde{P_{y_2}}(\alpha l_1 * \beta l_2)\Big\} \\ &\geq min\Big\{min\{\widetilde{P_{y_1}}(l_1), \widetilde{P_{y_1}}(l_2)\}, min\{\widetilde{P_{y_2}}(l_1), \widetilde{P_{y_2}}(l_2)\}\Big\} \\ &= min\Big\{min\{\widetilde{P_{y_1}}(l_1), \widetilde{P_{y_2}}(l_1)\}, min\{\widetilde{P_{y_1}}(l_2), \widetilde{P_{y_2}}(l_2)\}\Big\} \\ &= min\Big\{\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1), \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2)\Big\} \end{aligned}$$

$$\Rightarrow \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(\alpha l_1 * \beta l_2) \geq min\Big\{\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1), \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2)\Big\} \tag{1}$$

Hence $\bigcap \widetilde{P_{y_i}}$ is an interval valued pythagorean fuzzy linear space.

$$P_{y_1} \cup P_{y_2}(x) = max\{P_{y_1}(x), P_{y_2}(x)\}, \quad x \in X$$

$$\begin{aligned} P_{y_1} \cup P_{y_2}(\alpha l_1 * \beta l_2) &= max\Big\{P_{y_1}(\eta l_1 * \theta l_2), P_{y_2}(\alpha x_1 * \beta x_2)\Big\} \\ &\leq max\Big\{max\{P_{y_1}(l_1), P_{y_1}(l_2)\}, max\{P_{y_2}(l_1), P_{y_2}(l_2)\}\Big\} \\ &= max\Big\{max\{P_{y_1}(l_1), P_{y_2}(l_1)\}, \{P_{y_1}(l_2), P_{y_2}(l_2)\}\Big\} \\ &= max\{P_{y_1} \cup P_{y_2}(l_1), P_{y_1} \cup P_{y_2}(l_2)\} \end{aligned}$$

$$\Rightarrow P_{y_1} \cup P_{y_2}(\alpha l_1 * \beta l_2) \leq max\{P_{y_1} \cup P_{y_2}(l_1), P_{y_1} \cup P_{y_2}(l_2)\} \tag{2}$$

□

Thus $\bigcup P_{y_i}$ is a pythagorean fuzzy linear space.
Hence from Eq.(1) and Eq.(2) R− intersection of cubic pythagorean fuzzy linear spaces is again a cubic pythagorean fuzzy linear space.

Remark 3.1. *In this remark we try proving with the example that union of interval valued pythagorean fuzzy linear space need not again be an interval valued pythagorean fuzzy linear space.*

Example 3.2. *Let us consider a vector space and binary operation as defined in the example 3.1. Also consider two interval valued pythagorean fuzzy sets as tabulated below*

TABLE 2. Values of interval valued pythagorean fuzzy sets and pythagorean fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.3, 0.5], [0.5, 0.8])$	$[0.39, 0.66]$
l_2	$([0.1, 0.2], [0.43, 0.74])$	$[0.19, 0.44]$
l_3	$([0.6, 0.8], [0.53, 0.85])$	$[0.62, 0.75]$

TABLE 3. Values of interval valued pythagorean fuzzy sets and pythagorean fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.5, 0.7], [0.53, 0.83])$	$[0.4, 0.72]$
l_2	$([0.6, 0.8], [0.4, 0.5])$	$[0.43, 0.77]$
l_3	$([0.25, 0.38], [0.35, 0.4])$	$[0.2, 0.52]$

The above values in turn satisfy the conditions required for them to be interval valued pythagorean fuzzy linear space in Ł. Let us consider the union of the above tabulated values

$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.5, 0.7], [0.53, 0.83]), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2) = ([0.6, 0.8], [0.43, 0.74])$$
$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3) = ([0.6, 0.8], [0.53, 0.85])$$

For $\alpha = \beta = 1$ in def 3.2 we have

$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2 * l_3) \geq \min\left\{ \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3)\right\}$$
$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) \geq \min\left\{([0.6, 0.8], [0.43, 0.74]), ([0.6, 0.8], [0.53, 0.85])\right\}$$
$$=([0.6, 0.8], [0.43, 0.74])$$

$\Rightarrow \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.5, 0.7], [0.53, 0.83]) \geq ([0.6, 0.8], [0.43, 0.74])$
Since $[0.5, 0.7] \geq [0.6, 0.8]$ and $[0.53, 0.83] \geq [0.43, 0.74]$ which is incorrect we conclude that the union of interval valued pythagorean fuzzy linear space again need not be an interval valued pythagorean fuzzy linear space.

Remark 3.2. *Similarly in this remark we provide an example showing that intersection of pythagorean fuzzy linear spaces do not satisfy the second condition of cubic pythagorean fuzzy linear spaces as in definition* 3.3.

Example 3.3. *Consider two pythagorean fuzzy sets as tabulated in table* 2 *and* 3. *These pythagorean fuzzy sets are indeed pythagorean fuzzy linear spaces in Ł . Now let us consider* *their intersection*

$$P_{y_1} \cap P_{y_2}(l_2 * l_3) \leq \max\Big\{ P_{y_1} \cap P_{y_2}(l_2), P_{y_1} \cap P_{y_2}(l_3)\Big\}$$
$$P_{y_1} \cap P_{y_2}(l_1) \leq \max\Big\{([0.19, 0.44], [0.2, 0.52])\Big\}$$
$$= [0.2, 0.52]$$

$\Rightarrow P_{y_1} \cap P_{y_2}(l_1) = [0.39, 0.66] \leq [0.2, 0.52]$ *which is incorrect.*
Hence from the above example it is clear that intersection of pythagorean fuzzy linear spaces do not satisfy the second condition of cubic pythagorean fuzzy linear spaces.

Lemma 3.1. *From the above theorem and examples following conclusions can be drawn*
(i) Let $C_{P_{y_1}} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ *and* $C_{P_{y_2}} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ *be two cubic pythagorean fuzzy linear spaces. Then their R-union* $(C_{P_{y_1}} \cup C_{P_{y_2}})_R = (\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}, P_{y_1} \cap P_{y_2})$ *need not again be a cubic pythagorean fuzzy linear space.*
(ii) Let $C_{P_{y_1}} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ *and* $C_{P_{y_2}} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ *be two cubic pythagorean fuzzy linear spaces. Then their P-union* $(C_{P_{y_1}} \cup C_{P_{y_2}})_P = (\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}, P_{y_1} \cup P_{y_2})$ *need not again be a cubic Pythagorean fuzzy linear space.*
(iii) Let $C_{P_{y_1}} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ *and* $C_{P_{y_2}} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ *be two cubic pythagorean fuzzy linear spaces. Then their P-intersection* $(C_{P_{y_1}} \cap C_{P_{y_2}})_P = (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}, P_{y_1} \cap P_{y_2})$ *need not again be a cubic pythagorean fuzzy linear space.*

Proof. (i) From Example 3.2 and 3.3 R-union of two cubic pythagorean fuzzy linear spaces need not again be a cubic pythagorean fuzzy linear space.
(ii)Consider the pythagorean fuzzy linear spaces P_{y_1} and P_{y_2} as in table 2 and 3. Now let us consider the union of pythagorean fuzzy linear spaces we have

$$P_{y_1} \cup P_{y_2}(l_1) = [0.4, 0.72], P_{y_1} \cup P_{y_2}(l_2) = [0.43, 0.77], P_{y_1} \cup P_{y_2}(l_3) = [0.62, 0.75]$$

For $\alpha = \beta = 1$ in 3.3 we have

$$P_{y_1} \cup P_{y_2}(l_2 * l_3) \leq \max\Big\{ P_{y_1} \cup P_{y_2}(l_2), P_{y_1} \cup P_{y_2}(l_3)\Big\}$$
$$P_{y_1} \cup P_{y_2}(l_1) \leq \max\Big\{[0.43, 0.77], [0.62, 0.75]\Big\} = [0.62, 0.75]$$

$\Rightarrow P_{y_1} \cup P_{y_2}(l_1) = [0.4, 0.72] \leq [0.62, 0.75]$ which is appropriate and satisfies the second condition of cubic pythagorean fuzzy linear space.Hence P-union is not a cubic pythagorean fuzzy linear space.
(iii)Again consider the interval valued pythagorean fuzzy linear spaces $\widetilde{P_{y_1}}$ and $\widetilde{P_{y_2}}$ as in table 2 and 3. Now let us consider the intersection of pythagorean fuzzy linear spaces we have

$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) = ([0.3, 0.5], [0.5, 0.8]), \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2) = ([0.1, 0.2], [0.4, 0.5]),$$
$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_3) = ([0.25, 0.38], [0.35, 0.4])$$

For $\alpha = \beta = 1$ in 3.3 we have

$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2 * l_3) \geq \min\Big\{ \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2), \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_3)\Big\}$$
$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) \geq \min\Big\{([0.1, 0.2], [0.4, 0.5]), ([0.25, 0.38], [0.35, 0.4])\Big\}$$
$$=([0.1, 0.2], [0.35, 0.4])$$

$\Rightarrow \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) = ([0.3, 0.5], [0.5, 0.8]) \geq ([0.1, 0.2], [0.35, 0.4])$ which is appropriate and satisfies the first condition of cubic pythagorean fuzzy linear space. Hence P-intersection is not a cubic pythagorean fuzzy linear space. □

4. Internal and External cubic pythagorean fuzzy linear spaces

Definition 4.1. *A cubic pythagorean fuzzy set $C_{P_y} = \{\widetilde{P_y}, P_y\}$ in a linear space Ł over a field F is said to be an internal cubic pythagorean fuzzy linear space if*

$$(\widetilde{P_y})^{-}(\alpha l_1 * \beta l_2) \leq P_y(\alpha l_1 * \beta l_2) \leq (\widetilde{P_y})^{+}(\alpha l_1 * \beta l_2)$$

for all $l_1, l_2 \in$ Ł and $\alpha, \beta \in F$.

Example 4.1. *Let us consider the values of interval valued pythagorean fuzzy set and pythagorean fuzzy set as in table 2. Now for $\alpha = \beta = 1$ and $l_2 * l_3 = l_1$ in 4.1 we have*

$$(\widetilde{P_y})^{-}(l_2 * l_3) \leq P_y(l_2 * l_3) \leq (\widetilde{P_y})^{+}(l_2 * l_3)$$
$$(\widetilde{P_y})^{-}(l_1) \leq P_y(l_1) \leq (\widetilde{P_y})^{+}(l_1)$$

$\Rightarrow 0.39 \in [0.3, 0.5], 0.66 \in [0.5, 0.9]$. *Hence $C_{P_y} = \{\widetilde{P_y}, P_y\}$ is an internal cubic pythagorean fuzzy linear space.*

Definition 4.2. *A cubic pythagorean fuzzy set $C_{P_y} = \{\widetilde{P_y}, P_y\}$ in a linear space Ł over a field F is said to be an external cubic pythagorean fuzzy linear space if*

$$P_y(\alpha l_1 * \beta l_2) \notin \left(\left(\widetilde{P_y}\right)^{-}(\alpha l_1 * \beta l_2), \left(\widetilde{P_y}\right)^{+}(\alpha l_1 * \beta l_2) \right)$$

for all $l_1, l_2 \in$ Ł and $\alpha, \beta \in F$.

Example 4.2. *Let us consider the values of interval valued pythagorean fuzzy set and pythagorean fuzzy set as in table 1. Now for $\alpha = \beta = 1$ and $l_2 * l_3 = l_1$ in 3.1 we have*

$$P_y(l_2 * l_3) \notin \left(\left(\widetilde{P_y}\right)^{-}(l_2 * l_3), \left(\widetilde{P_y}\right)^{+}(l_2 * l_3) \right)$$
$$P_y(l_1) \notin \left(\left(\widetilde{P_y}\right)^{-}(l_1), \left(\widetilde{P_y}\right)^{+}(l_1) \right)$$

$\Rightarrow 0.09 \notin [06, 0.8]$ *and* $0.21 \notin [0.5, 0.6]$. *Hence $C_{P_y} = \{\widetilde{P_y}, P_y\}$ is an external cubic pythagorean fuzzy linear space.*

Proposition 4.1. *Let $C_{P_{y_1}{}^I} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ and $C_{P_y{}^I} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ be two ICPyFLS. Then their $R-$intersection $C_{P_{y_1}{}^I} \cap C_{P_{y_2}{}^I} = (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}, P_{y_1} \cup P_{y_2})$ is an ICPyFLS.*

Proof. Since $C_{P_{y_1}{}^I}$ and $C_{P_{y_2}{}^I}$ are ICPyFLS in Ł, we have

$$\left(\widetilde{P_{y_1}}\right)^{-}(\alpha l_1 * \beta l_2) \leq P_{y_1}(\alpha l_1 * \beta l_2) \leq \left(\widetilde{P_{y_1}}\right)^{+}(\alpha l_1 * \beta l_2)$$
$$\left(\widetilde{P_{y_2}}\right)^{-}(\alpha l_1 * \beta l_2) \leq P_{y_2}(\alpha l_1 * \beta l_2) \leq \left(\widetilde{P_{y_2}}\right)^{+}(\alpha l_1 * \beta l_2)$$

for all $l_1, l_2 \in$ Ł and $\alpha, \beta \in F$. Now from theorem 3.1 we have

$$(\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}})^{-}(\alpha l_1 * \beta l_2) \leq P_{y_1} \cup P_{y_2}(\alpha l_1 * \beta l_2) \leq (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}})^{+}(\alpha l_1 * \beta l_2)$$

Hence, $C_{P_{y_1}{}^I} \cap C_{P_{y_2}{}^I} = (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}, P_{y_1} \cup P_{y_2})$ is an ICPyFLS. □

Proposition 4.2. *Let $C_{P_{y_1}{}^I} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ and $C_{P_y{}^I} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ be two ECPyFLS. Then their $R-$intersection $C_{P_{y_1}{}^I} \cap C_{P_{y_2}{}^I} = (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}, P_{y_1} \cup P_{y_2})$ is an ECPyFLS.*

Proof. Since $C_{{P_{y_1}}^I}$ and $C_{{P_{y_2}}^I}$ are ECPyFLS in Ł, we have

$$P_{y_1}(\alpha l_1 * \beta l_2) \notin \Big((\widetilde{P_{y_1}})^-(\alpha l_1 * \beta l_2), (\widetilde{P_{y_1}})^+(\alpha l_1 * \beta l_2) \Big)$$

$$P_{y_2}(\alpha l_1 * \beta l_2) \notin \Big((\widetilde{P_{y_2}})^-(\alpha l_1 * \beta l_2), (\widetilde{P_{y_2}})^+(\alpha l_1 * \beta l_2) \Big)$$

which implies that

$$P_{y_1} \cup P_{y_2}(\alpha l_1 * \beta l_2) \notin \Big((\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}})^-(\alpha l_1 * \beta l_2), (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}})^+(\alpha l_1 * \beta l_2) \Big)$$

Hence $C_{{P_{y_1}}^I} \cap C_{{P_{y_2}}^I} = (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}, P_{y_1} \cup P_{y_2})$ is an ECPyFLS. □

Proposition 4.3. *Let $C_{{P_{y_1}}^I} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ and $C_{{P_y}^I} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ be two ICPyFLS. Then their P−intersection $C_{{P_{y_1}}^I} \cap C_{{P_{y_2}}^I} = (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}, P_{y_1} \cap P_{y_2})$ need not be an ICPyFLS.*

Proof. Let us consider the values of interval valued pythagorean fuzzy linear spaces and pythagorean fuzzy linear space as in the 2 and also values tabulated below

TABLE 4. Values of interval valued pythagorean fuzzy sets and pythagorean fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.2, 0.4], [0.4, 0.8])$	$[0.29, 0.63]$
l_2	$([0.1, 0.3], [0.33, 0.64])$	$[0.36, 0.7]$
l_3	$([0.55, 0.75], [0.53, 0.85])$	$[0.07, 0.27]$

$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) = ([0.2, 0.4], [0.4, 0.8]), \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2) = ([0.1, 0.2], [0.33, 0.64]),$$
$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_3) = ([0.55, 0.75], [0.53, 0.85])$$

For $\alpha = \beta = 1$ 3.3 we have,

$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2 * l_3) \geq \min \Big\{ \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2), \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_3) \Big\}$$
$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) \geq \min \Big\{ ([0.1, 0.2], [0.33, 0.64]), ([0.55, 0.75], [0.53, 0.85]) \Big\}$$
$$= ([0.1, 0.2], [0.33, 0.64])$$

$\Rightarrow \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) = ([0.2, 0.4], [0.4, 0.8]) \geq ([0.1, 0.2], [0.33, 0.64])$ which is correct.

$$P_{y_1} \cap P_{y_2}(l_1) = [0.29, 0.63], P_{y_1} \cap P_{y_2}(l_2) = [0.26, 0.59], P_{y_1} \cap P_{y_2}(l_3) = [0.07, 0.27]$$

For $\alpha = \beta = 1$ 3.3 we have,

$$P_{y_1} \cap P_{y_2}(l_2 * l_3) \leq \max \Big\{ P_{y_1} \cap P_{y_2}(l_2), P_{y_1} \cap P_{y_2}(l_3) \Big\}$$
$$P_{y_1} \cap P_{y_2}(l_1) \leq \max \Big\{ [0.26, 0.59], [0.07, 0.27] \Big\}$$
$$= [0.26, 0.59]$$

$\Rightarrow P_{y_1} \cap P_{y_2}(l_1) = [0.29, 0.63] \leq [0.26, 0.59]$ which is incorrect.
Therefore, the $P-$intersection of two ICPyLS need not be ICPyLS. □

Proposition 4.4. *Let* $C_{P_{y_1}{}^I} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ *and* $C_{P_y{}^I} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ *be two ECPyFLS. Then their* $P-$*intersection* $C_{P_{y_1}{}^I} \cap C_{P_{y_2}{}^I} = (\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}, P_{y_1} \cap P_{y_2})$ *need not be an ECPyFLS.*

Proof. Let us consider the values of interval valued pythagorean fuzzy linear spaces and pythagorean fuzzy linear space as in the 1 and also values tabulated below

TABLE 5. Values of interval valued pythagorean fuzzy sets and pythagorean fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.32, 0.49], [0.6, 0.72])$	$[0.19, 0.31]$
l_2	$([0.26, 0.42], [0.43, 0.77])$	$[0.13, 0.19]$
l_3	$([0.27, 0.52], [0.2, 0.3])$	$[0.24, 0.37]$

$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) = ([0.32, 0.49], [0.5, 0.6]), \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2) = ([0.26, 0.42], [0.4, 0.5]),$$
$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_3) = ([0.2, 0.35], [0.2, 0.3])$$

For $\alpha = \beta = 1$ 3.3 we have,

$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2 * l_3) \geq \min\left\{\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_2), \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_3)\right\}$$
$$\widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) \geq \min\left\{([0.26, 0.42], [0.4, 0.5]), ([0.2, 0.35], [0.2, 0.3])\right\}$$
$$=([0.2, 0.35], [0.2, 0.3])$$

$\Rightarrow \widetilde{P_{y_1}} \cap \widetilde{P_{y_2}}(l_1) = ([0.32, 0.49], [0.5, 0.6]) \geq ([0.2, 0.35], [0.2, 0.3])$ which is correct.

$$P_{y_1} \cap P_{y_2}(l_1) = [0.09, 0.21], P_{y_1} \cap P_{y_2}(l_2) = [0.13, 0.19], P_{y_1} \cap P_{y_2}(l_3) = [0.24, 0.37]$$

For $\alpha = \beta = 1$ 3.3 we have,

$$P_{y_1} \cap P_{y_2}(l_2 * l_3) \leq \max\left\{P_{y_1} \cap P_{y_2}(l_2), P_{y_1} \cap P_{y_2}(l_3)\right\}$$
$$P_{y_1} \cap P_{y_2}(l_1) \leq \max\left\{[0.13, 0.19], [0.24, 0.37]\right\}$$
$$= [0.24, 0.37]$$

$\Rightarrow P_{y_1} \cap P_{y_2}(l_1) = [0.09, 0.21] \leq [0.24, 0.37]$ which is incorrect.
Therefore, the $P-$intersection of two ECPyLS need not be ECPyLS. □

Proposition 4.5. *Let* $C_{P_{y_1}{}^I} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ *and* $C_{P_y{}^I} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ *be two ICPyFLS. Then their* $P-$*union* $C_{P_{y_1}{}^I} \cup C_{P_{y_2}{}^I} = (\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}, P_{y_1} \cup P_{y_2})$ *need not be an ICPyFLS.*

Proof. Let us consider the values of interval valued pythagorean fuzzy linear spaces and pythagorean fuzzy linear space as in the table 2 and also values tabulated below

TABLE 6. Values of interval valued pythagorean fuzzy sets and pythagorean Fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.4, 0.6], [0.45, 0.75])$	$[0.42, 0.7]$
l_2	$([0.5, 0.7], [0.53, 0.84])$	$[0.25, 0.54]$
l_3	$([0.35, 0.53], [0.52, 0.79])$	$[0.45, 0.79]$

$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.4, 0.6], [0.5, 0.8]), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2) = ([0.5, 0.7], [0.53, 0.84]),$
$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3) = ([0.6, 0.8], [0.53, 0.85])$

For $\alpha = \beta = 1$ 3.3 we have,

$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2 * l_3) \geq \min\left\{\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3)\right\}$$
$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) \geq \min\left\{([0.5, 0.7], [0.53, 0.84]), ([0.6, 0.8], [0.53, 0.85])\right\}$$
$$=([0.6, 0.8], [0.53, 0.85])$$

$\Rightarrow \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.4, 0.6], [0.5, 0.8]) \geq ([0.6, 0.8], [0.53, 0.85])$ which is incorrect.

$$P_{y_1} \cup P_{y_2}(l_1) = [0.42, 0.7], P_{y_1} \cup P_{y_2}(l_2) = [0.26, 0.59], P_{y_1} \cup P_{y_2}(l_3) = [0.45, 0.79]$$

For $\alpha = \beta = 1$ 3.3 we have,

$$P_{y_1} \cup P_{y_2}(l_2 * l_1) \leq \max\left\{P_{y_1} \cup P_{y_2}(l_2), P_{y_1} \cup P_{y_2}(l_3)\right\}$$
$$P_{y_1} \cup P_{y_2}(l_1) \leq \max\left\{[0.26, 0.59], [0.45, 0.79]\right\}$$
$$= [0.45, 0.79]$$

$\Rightarrow P_{y_1} \cup P_{y_2}(l_1) = [0.42, 0.7] \leq [0.45, 0.79]$ which is correct.
Therefore, the $P-$union of two ICPyLS need not be ICPyLS. □

Proposition 4.6. *Let $C_{P_{y_1}{}^I} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ and $C_{P_y{}^I} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ be two ECPyFLS. Then their $P-$union $C_{P_{y_1}{}^I} \cup C_{P_{y_2}{}^I} = (\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}, P_{y_1} \cup P_{y_2})$ need not be an ECPyFLS.*

Proof. Let us consider the values of interval valued pythagorean fuzzy linear spaces and pythagorean fuzzy linear space as in the values tabulated below

TABLE 7. Values of interval valued pythagorean fuzzy sets and pythagorean fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.35, 0.55], [0.45, 0.8])$	$[0.29, 0.42]$
l_2	$([0.15, 0.25], [0.37, 0.64])$	$[0.34, 0.49]$
l_3	$([0.5, 0.7], [0.48, 0.76])$	$[0.26, 0.37]$

TABLE 8. Values of interval valued pythagorean fuzzy sets and pythagorean fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.45, 0.65], [0.48, 0.85])$	$[0.32, 0.47]$
l_2	$([0.48, 0.68], [0.34, 0.4])$	$[0.38, 0.55]$
l_3	$([0.3, 0.43], [0.2, 0.21])$	$[0.16, 0.25]$

$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.45, 0.65], [0.48, 0.85]), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2) = ([0.48, 0.68], [0.37, 0.64]),$
$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3) = ([0.5, 0.7], [0.48, 0.76])$

For $\alpha = \beta = 1$ 3.3 we have,

$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2 * l_3) \geq \min\left\{ \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3)\right\}$$

$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) \geq \min\left\{([0.48, 0.68], [0.37, 0.64]), ([0.5, 0.7], [0.48, 0.76])\right\}$$

$$=([0.48, 0.68], [0.37, 0.64])$$

$\Rightarrow \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.45, 0.65], [0.48, 0.85]) \geq ([0.48, 0.68], [0.37, 0.64])$ which is incorrect.

$$P_{y_1} \cup P_{y_2}(l_1) = [0.32, 0.47], P_{y_1} \cup P_{y_2}(l_2) = [0.38, 0.55], P_{y_1} \cup P_{y_2}(l_3) = [0.26, 0.37]$$

For $\alpha = \beta = 1$ 3.3 we have,

$$P_{y_1} \cup P_{y_2}(l_2 * l_1) \leq \max\left\{P_{y_1} \cup P_{y_2}(l_2), P_{y_1} \cup P_{y_2}(l_3)\right\}$$

$$P_{y_1} \cup P_{y_2}(l_1) \leq \max\left\{[0.38, 0.55], [0.26, 0.37]\right\}$$

$$= [0.38, 0.55]$$

$\Rightarrow P_{y_1} \cup P_{y_2}(l_1) = [0.32, 0.47] \leq [0.38, 0.55]$ which is correct.
Therefore, the $P-$union of two ECPyLS need not be ECPyLS. □

Proposition 4.7. *Let $C_{{P_{y_1}}^I} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ and $C_{{P_y}^I} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ be two ICPyFLS. Then their $R-$union $C_{{P_{y_1}}^I} \cup C_{{P_{y_2}}^I} = (\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}, P_{y_1} \cap P_{y_2})$ need not be an ICPyFLS.*

Proof. Let us consider the values of interval valued pythagorean fuzzy linear spaces and pythagorean fuzzy linear space as in the table 2 the values tabulated below

TABLE 9. Values of interval valued pythagorean fuzzy sets and pythagorean fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.5, 0.7], [0.46, 0.76])$	$[0.52, 0.8]$
l_2	$([0.55, 0.75], [0.53, 0.84])$	$[0.56, 0.81]$
l_3	$([0.46, 0.66], [0.52, 0.79])$	$[0.47, 0.64]$

$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.5, 0.7], [0.5, 0.8]), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2) = ([0.55, 0.75], [0.53, 0.84]),$
$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3) = ([0.6, 0.8], [0.53, 0.85])$

For $\alpha = \beta = 1$ 3.3 we have,

$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2 * l_3) \geq \min\left\{ \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3)\right\}$$

$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) \geq \min\left\{([0.55, 0.75], [0.53, 0.84]), ([0.6, 0.8], [0.53, 0.85])\right\}$$

$$=([0.55, 0.75], [0.53, 0.84])$$

$\Rightarrow \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.5, 0.7], [0.5, 0.8]) \geq ([0.55, 0.75], [0.53, 0.84])$ which is incorrect.

$$P_{y_1} \cap P_{y_2}(l_1) = [0.39, 0.66], P_{y_1} \cap P_{y_2}(l_2) = [0.19, 0.44], P_{y_1} \cap P_{y_2}(l_3) = [0.36, 0.64]$$

For $\alpha = \beta = 1$ 3.3 we have,

$$P_{y_1} \cap P_{y_2}(l_2 * l_3) \leq \max\Big\{P_{y_1} \cap P_{y_2}(l_2), P_{y_1} \cap P_{y_2}(l_3)\Big\}$$
$$P_{y_1} \cap P_{y_2}(l_1) \leq \max\Big\{[0.19, 0.44], [0.36, 0.64]\Big\}$$
$$= [0.36, 0.64]$$

$\Rightarrow P_{y_1} \cap P_{y_2}(l_1) = [0.39, 0.66] \leq [0.36, 0.64]$ which is incorrect.
Therefore, the $R-$union of two ICPyLS need not be ICPyLS. □

Proposition 4.8. *Let* $C_{P_{y_1}{}^I} = \{\widetilde{P_{y_1}}, P_{y_1}\}$ *and* $C_{P_y{}^I} = \{\widetilde{P_{y_2}}, P_{y_2}\}$ *be two ECPyFLS. Then their* $R-$*union* $C_{P_{y_1}{}^I} \cup C_{P_{y_2}{}^I} = (\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}, P_{y_1} \cap P_{y_2})$ *need not be an ECPyFLS.*

Proof. Let us consider the values of interval valued pythagorean fuzzy linear spaces and pythagorean fuzzy linear space as in the values tabulated below

TABLE 10. Values of interval valued Pythagorean fuzzy sets and Pythagorean Fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.45, 0.65], [0.45, 0.8])$	$[0.26, 0.41]$
l_2	$([0.25, 0.35], [0.47, 0.74])$	$[0.25, 0.37]$
l_3	$([0.57, 0.77], [0.48, 0.76])$	$[0.36, 0.53]$

TABLE 11. Values of interval valued pythagorean fuzzy sets and pythagorean fuzzy sets

Ł	$\widetilde{P_y}$	P_y
l_1	$([0.55, 0.75], [0.48, 0.85])$	$[0.36, 0.54]$
l_2	$([0.56, 0.76], [0.44, 0.5])$	$[0.45, 0.65]$
l_3	$([0.4, 0.53], [0.3, 0.31])$	$[0.06, 0.18]$

$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.55, 0.75], [0.48, 0.85]), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2) = ([0.56, 0.76], [0.47, 0.74]),$
$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3) = ([0.57, 0.77], [0.48, 0.76])$

For $\alpha = \beta = 1$ 3.3 we have,

$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2 * l_3) \geq \min\Big\{\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_2), \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_3)\Big\}$$
$$\widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) \geq \min\Big\{([0.56, 0.76], [0.47, 0.74]), ([0.57, 0.77], [0.48, 0.76])\Big\}$$
$$=([0.56, 0.76], [0.47, 0.74])$$

$\Rightarrow \widetilde{P_{y_1}} \cup \widetilde{P_{y_2}}(l_1) = ([0.55, 0.75], [0.48, 0.85]) \geq ([0.67, 0.87], [0.47, 0.74])$ which is incorrect.

$P_{y_1} \cap P_{y_2}(l_1) = [0.26, 0.41], P_{y_1} \cup P_{y_2}(l_2) = [0.25, 0.37], P_{y_1} \cap P_{y_2}(l_3) = [0.06, 0.18]$

For $\alpha = \beta = 1$ 3.3 we have,

$$P_{y_1} \cap P_{y_2}(l_2 * l_1) \leq \max\Big\{P_{y_1} \cap P_{y_2}(l_2), P_{y_1} \cap P_{y_2}(l_3)\Big\}$$
$$P_{y_1} \cap P_{y_2}(l_1) \leq \max\Big\{[0.25, 0.37], [0.06, 0.18]\Big\}$$
$$= [0.25, 0.37]$$

$\Rightarrow P_{y_1} \cap P_{y_2}(l_1) = [0.26, 0.41] \leq [0.25, 0.37]$ which is incorrect.
Therefore, the $R-$union of two ECPyLS need not be ECPyLS. □

5. Conclusions

With the introduction of fuzzy linear spaces by G.Lubczonok and V.Murali we observe various extensions on these fuzzy linear spaces like interval valued fuzzy linear spaces, cubic fuzzy linear spaces, N-cubic fuzzy linear spaces.In the present paper we introduced the idea of cubic pythagorean fuzzy linear spaces. The purpose of this paper is to extend the structure of cubic sets to pythagorean fuzzy linear spaces and study in detail few basic operations explicitly along with examples. In case of future work we try introducing methods to solve multiple decision making problems on these spaces.

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