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ON THE M-FOLD PRODUCT OF FRACTIONAL OPERATORS

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Abstract. In this work, using the m-fold product of fractional integral and maximal operators, we prove that the boundedness of these fractional operators and their corresponding multilinear fractional operators under some conditions on weighted variable exponent Lorentz spaces.

Keywords: Multilinear fractional maximal operator, multilinear fractional integral operator, weighted variable exponent Lorentz spaces.

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1. Introduction

Throughout this paper, the space $L^1_{loc}(\mathbb{R}^n)$ consists of all (equivalence classes) measurable functions f on $\mathbb{R}^n$ such that $f.\chi_K \in L^1(\mathbb{R}^n)$ for every compact subset $K \subset \mathbb{R}^n$, where χ_K is the characteristic function of K. Let μ be a Borel measure on $\mathbb{R}^n$. The distribution function of f is defined as

$$\lambda_f(y) = \mu(\{x \in \mathbb{R}^n : |f(x)| > y\}) = \int\limits_{\{x\in\mathbb{R}^n:|f(x)|>y\}} d\mu(x), \ y \geq 0.$$

The rearrangement function of f is given by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} = \sup\{y > 0 : \lambda_f(y) > t\}, t \geq 0.$$

The average function of f^* is defined to be

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds$$

for $t > 0$, [6].

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Let $0 < l \le \infty$. We denote

$$p_- = \inf_{x\in[0,l]} p(x), \quad p^+ = \sup_{x\in[0,l]} p(x).$$

Moreover, we use the notation

$$P_a = \{p : a < p_- \le p^+ < \infty\}, \ a \in \mathbb{R}.$$

The set $\wp[0,l]$ is the family of $p \in L^\infty([0,l])$ such that there exist the limits $p(0) = \lim_{x\to 0} p(x)$, $p(\infty) = \lim_{x\to\infty} p(x)$ and we have

$$|p(x) - p(0)| \le \frac{C}{\ln\frac{1}{|x|}}, \qquad |x| \le \frac{1}{2} \quad (C > 0)$$

and

$$|p(x) - p(\infty)| \le \frac{C}{\ln(e + |x|)}, \quad (C > 0). \tag{1.1}$$

If $l = \infty$, then it's enough to the inequality (1.1) satisfies. We also denote $\wp_a([0,l]) = \wp([0,l]) \cap P_a([0,l])$.

Let Ω be open set in $\mathbb{R}^n$. We denote by $l = \mu(\Omega)$. Assume that $p, q \in \wp_0([0,l])$. The variable exponent Lorentz space $L^{p(.),q(.)}(\Omega)$ is defined as the set of all (equivalence classes) measurable functions f on Ω such that $\rho_{p,q}(f) < \infty$, where

$$\rho_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)}-1} (f^*(t))^{q(t)}\, dt. \tag{1.2}$$

We use the notation

$$\|f\|^1_{L^{p(.),q(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p,q}(\frac{f}{\lambda}) \le 1\right\}.$$

Let $p \in \wp_0([0,l])$ and $q \in \wp_1([0,l])$. If $l = \infty$, then the equality (1.2) is equivalent to the following sum

$$\int_0^1 t^{\frac{q(0)}{p(0)}-1} (f^*(t))^{q(t)}\, dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1} (f^*(t))^{q(t)}\, dt.$$

If $l < \infty$, then the equality (1.2) is equivalent to the integral

$$\int_0^l t^{\frac{q(0)}{p(0)}-1} (f^*(t))^{q(t)}\, dt.$$

The space $L^{p(.),q(.)}(\Omega)$ is normed vector space with norm

$$\|f\|_{L^{p(.),q(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p,q}(\frac{f}{\lambda}) \le 1\right\},$$

where $\rho_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)}-1} (f^{**}(t))^{q(t)}\, dt$, [3]. Also if $p(.) = p$ and $q(.) = q$ are constants, then the space $L^{p(.),q(.)}(\Omega)$ coincide with usual Lorentz space $L^{p,q}(\Omega)$. In the literature, there is another definition of variable exponent Lorentz spaces in which variable exponent Lorentz spaces coincide with variable exponent Lebesgue spaces when p(.)=q(.), [7]. But since the variable exponent $p(.)$ is defined on Ω and not on $[0,l]$ in the variable exponent Lebesgue spaces, the identity $L^{p(.),p(.)}(\Omega) = L^{p(.)}(\Omega)$ does not hold for this definition we used in our study.

A weight ω function is defined by nonnegative and measurable function on $[0,l]$. The weighted Lorentz space $L_{\omega}^{p(.),q(.)}(\Omega)$ with the weight ω consists of $f \in M(\Omega,\mu)$ such that

$$\|f\|^{1}_{L_{\omega}^{p(.),q(.)}(\Omega)} = \left\|\omega(t)\, t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^{*}(t)\right\|_{L^{q(.)}(\Omega)} < \infty.$$

Also, it is normed space with the norm

$$\|f\|_{L_{\omega}^{p(.),q(.)}(\Omega)} = \left\|\omega(t)\, t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^{**}(t)\right\|_{L^{q(.)}(\Omega)}.$$

Let $p,q \in \wp_1([0,l])$, $\gamma \in \wp([0,l])$ and $\omega(t) = t^{\gamma(t)}$. If γ satisfies the following conditions

$$\gamma(0) < \frac{1}{p'(0)}\ ,\ \gamma(\infty) < \frac{1}{p'(\infty)},$$

then

$$\|f\|^{1}_{L_{\omega}^{p(.),q(.)}} \le \|f\|_{L_{\omega}^{p(.),q(.)}} \le C\,\|f\|^{1}_{L_{\omega}^{p(.),q(.)}}$$

are obtained, where $C>0$ does not depend on f, [3]. In other words, we have $\|f\|^{1}_{L_{\omega}^{p(.),q(.)}} \approx \|f\|_{L_{\omega}^{p(.),q(.)}}$. In this paper, we will assume that $p,q \in \wp_1([0,l])$, $\gamma \in \wp([0,l])$, $\omega(t) = t^{\gamma(t)}$, $\gamma(0) < \frac{1}{p'(0)}$ and $\gamma(\infty) < \frac{1}{p'(\infty)}$. Also, we will use the notation $f \lesssim g$ to mean that there is a positive constant C such that $f(t) \le Cg(t)$ for all $t \in \Omega$.

Recently, the boundedness of various types of operators between the some function spaces, which play an important role in harmonic analysis and partial differential equations, have been studied. One of them, the fractional integral operator is given as

$$I_{\alpha}(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy,\ x \in \mathbb{R}^n, 0<\alpha<n$$

for any $f \in S$ that is Schwartz class. This operator plays an important role in the theory of Sobolev's embeddings, [11]. Also, L. Ephremidze, V. Kokilashvili and S. Samko (see [3, 9]) proved the boundedness of fractional integral from the space $L_{\omega}^{p(.),q(.)}(\Omega)$ with the weight $\omega(t) = t^{\gamma(t)}$ into the space $L_{\omega}^{p_{\alpha}(.),q(.)}(\Omega)$, where $\frac{1}{p_{\alpha}(t)} = \frac{1}{p(t)} - \frac{\alpha}{n}$, $p,q \in \wp_1([0,l])$, $\gamma \in \wp([0,l])$, $\gamma(0) < \frac{1}{p'(0)}$ and $\gamma(\infty) < \frac{1}{p'(\infty)}$.

The fractional integral operator of variable order $\alpha(.)$ is defined for any $f \in S$ by

$$I_{\alpha(.)}(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}}dy,\ x \in \mathbb{R}^n,$$

where $0 < \alpha(.) < n$. This operator was shown to be bounded from the variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^n)$ into the weighted variable exponent Lebesgue space $L_{\rho}^{q(.)}(\mathbb{R}^n)$, where $\rho(x) = (1+|x|)^{-\gamma}$ with some $\gamma > 0$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$, $p \in \wp([0,l])$ and $1 < p(\infty) \le p(x) \le P < \infty$, [10].

The Hardy-Littlewood maximal operator is defined for any $f \in L^{1}_{loc}(\mathbb{R}^n)$ by

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_{Q} |f(y)|\,dy,\ x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes. The boundedness of Hardy-littlewood maximal operator is considered on Lebesgue spaces, variable exponent Lebesgue spaces, variable exponent Wiener amalgam spaces, etc. L. Ephremidze, V. Kokilashvili and S. Samko (see [3, 9]) proved the boundedness of this operator in the variable exponent weighted Lorentz space $L_{\omega}^{p(.),q(.)}(\Omega)$, where $p,q\in\wp_1([0,l])$, $\omega(t)=t^{\gamma(t)}$, $\gamma\in\wp([0,l])$, $\gamma(0)<\frac{1}{p'(0)}$ and $\gamma(\infty)<\frac{1}{p'(\infty)}$. Moreover, A. Kucukaslan, V.S. Guliyev, C. Aykol and A. Serbetci proved the boundedness of the Hardy–Littlewood maximal operator on local variable Morrey–Lorentz spaces, [12].

Another important operator is the called fractional maximal operator M_α is defined by

$$M_\alpha(f)(x)=\sup_{x\in Q}\frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int\limits_Q |f(y)|\,dy,\ x\in\mathbb{R}^n,$$

for any $f\in L_{loc}^1(\mathbb{R}^n)$. Similarly under some conditions, using the boundedness of fractional integral operator, this operator is bounded from the space $L_{\omega}^{p(.),q(.)}(\Omega)$ with the weight $\omega(t)=t^{\gamma(t)}$ into the space $L_{\omega}^{p_\alpha(.),q(.)}(\Omega)$, where $\frac{1}{p_\alpha(t)}=\frac{1}{p(t)}-\frac{\alpha}{n}$, [3].

The fractional maximal operator of variable order $\alpha(.)$ is given for any $f\in L_{loc}^1(\mathbb{R}^n)$ by

$$M_{\alpha(.)}(f)(x)=\sup_{x\in Q}\frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}}\int\limits_Q |f(y)|\,dy,\ x\in\mathbb{R}^n.$$

The fractional maximal operator of variable order $\alpha(.)$ is bounded from the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ into the weighted variable exponent Lebesgue space $L_\rho^{q(.)}(\mathbb{R}^n)$, where $\rho(x)=(1+|x|)^{-\gamma}$ with some $\gamma>0$, $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n}$, $p\in\wp([0,l])$ and $1<p(\infty)\le p(x)\le P<\infty$, [10].

In many research, the boundedness of multilinear integral operators (multilinear fractional integral operators, multisublinear maximal operators etc.) in weighted function spaces is considered. These operators play an important role in Harmonic Analysis. Historically, the multilinear fractional integrals were introduced by L. Grafakos [4], C. Kenig and E. Stein [8], L. Grafakos and N. Kalton [5]. They make with the operator

$$B_K(f,g)(x)=\int\limits_{\mathbb{R}^n} f(x+t)\,g(x-t)\,K(t)\,dt,\ x\in\mathbb{R}^n,$$

where $K(t)=\frac{1}{|t|^{n-\alpha}}$, $0<\alpha<n$. They obtained that B_K is bounded from $L^{p_1}\times L^{p_2}$ to L^q under the conditions $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $1<p_1,\ p_2,\ q<\infty$. This boundedness follows from the pointwise estimate

$$B_K(f,g)(x)\le I_\alpha(f^r)^{\frac{1}{r}}\,I_\alpha(g^s)^{\frac{1}{s}},$$

where $r=\frac{p_1}{p}$, $s=\frac{p_2}{p}$, $f,g\ge 0$, [17]. As a tool to understand B_K, the following operator is defined.

Let $m\ge 1$ and $0<\alpha<nm$. For $\overrightarrow{f}\in S\times\ldots\times S$, the multilinear fractional integral operator is defined by

$$I_\alpha\left(\overrightarrow{f}\right)(x)=\int\limits_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\ldots f_m(y_m)}{(|x-y_1|+\ldots+|x-y_m|)^{nm-\alpha}}d\overrightarrow{y},$$

where $d\overrightarrow{y}=dy_1\ldots dy_m$, $\overrightarrow{f}=(f_1,\ldots,f_m)$. If the functions α_i $(1\le i\le m)$ satisfy the statements $0<\alpha_i<n$ and $\alpha_1+\ldots+\alpha_m=\alpha$, then $I_\alpha\left(\overrightarrow{f}\right)\le\prod\limits_i^m I_{\alpha_i}f_i$, [17, 18].

Let $\overrightarrow{f} \in L^1_{loc}(\mathbb{R}^n) \times ... \times L^1_{loc}(\mathbb{R}^n)$ and $0 \le \alpha < nm$. the multisublinear fractional maximal operator (multilinear for brevity) is given by

$$M_\alpha\left(\overrightarrow{f}\right)(x) = \sup_{x \in Q} \frac{1}{|Q|^{m-\frac{\alpha}{n}}} \prod_i^m \int_Q |f_i(y_i)|\, dy_i.$$

Also if $\alpha_1 + ... + \alpha_m = \alpha$, then $M_\alpha\left(\overrightarrow{f}\right) \le \prod_i^m M_{\alpha_i} f_i$. Let $0 < \alpha < nm$. Then, there exists a positive C such that $M_\alpha\left(\overrightarrow{f}\right) \le C I_\alpha\left(\left|\overrightarrow{f}\right|\right)$, [17, 18].

2. Main Results

In this section, we will first give the boundedness of the linear fractional integral operator and maximal operator on weighted variable exponent lorentz spaces under some conditions.

Theorem 2.1. *Let* $0 < \alpha_-$, $\alpha_+ < n$. *If* $-\frac{1}{p(0)} < \gamma(0)$, $\gamma(0) + \frac{\alpha_+}{n} < \frac{1}{p'(0)}$, $-\frac{1}{p(\infty)} < \gamma(\infty)$ *and* $\gamma(\infty) + \frac{\alpha_+}{n} < \frac{1}{p'(\infty)}$, *then* $I_{\alpha(.)}$ *is bounded from the space* $\left(L^{p(.),q(.)}_{\omega_{\alpha_-}} \cap L^{p(.),q(.)}_{\omega_{\alpha_+}}\right)(\Omega)$ *into the space* $L^{p(.),q(.)}_{\omega}(\Omega)$, *where* $\omega_{\alpha_-}(t) = t^{\gamma(t)+\frac{\alpha_-}{n}}$ *and* $\omega_{\alpha_+}(t) = t^{\gamma(t)+\frac{\alpha_+}{n}}$. *The condition at infinity being needed in the case* $l = \infty$.

Proof. Take any $f \in \left(L^{p(.),q(.)}_{\omega_{\alpha_-}} \cap L^{p(.),q(.)}_{\omega_{\alpha_+}}\right)(\Omega)$. Since $\alpha_- < \alpha(.) < \alpha_+$, we have

$$\begin{aligned}
\left|I_{\alpha(.)}(f)(x)\right| &= \left|\int_\Omega \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy\right| \\
&= \left|\int_{\{x\in\Omega:\,|x-y|\le 1\}} \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy + \int_{\{x\in\Omega:\,|x-y|>1\}} \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy\right| \\
&\le \left|\int_{\{x\in\Omega:\,|x-y|\le 1\}} \frac{f(y)}{|x-y|^{n-\alpha_+}} dy\right| + \left|\int_{\{x\in\Omega:\,|x-y|>1\}} \frac{f(y)}{|x-y|^{n-\alpha_-}} dy\right| \\
&< \int_\Omega \frac{|f(y)|}{|x-y|^{n-\alpha_+}} dy + \int_\Omega \frac{|f(y)|}{|x-y|^{n-\alpha_-}} dy \\
&= I_{\alpha_+}(|f|)(x) + I_{\alpha-}(|f|)(x)
\end{aligned} \tag{2.1}$$

for all $x \in \Omega$. Then by (2.1), we can write

$$\begin{aligned}
\left(I_{\alpha(.)}(f)\right)^*(t) &\le \left(I_{\alpha_-}(|f|) + I_{\alpha+}(|f|)\right)^*(t) \\
&\le \left(I_{\alpha_-}(|f|)\right)^*\left(\frac{t}{2}\right) + \left(I_{\alpha+}(|f|)\right)^*\left(\frac{t}{2}\right), \; t > 0.
\end{aligned} \tag{2.2}$$

By (2.2), we find

$$\begin{aligned}&\left\|I_{\alpha(.)}(f)\right\|_{L_{\omega}^{p(.),q(.)}(\Omega)}\approx\left\|I_{\alpha(.)}(f)\right\|^{1}_{L_{\omega}^{p(.),q(.)}(\Omega)}=\left\|\omega(t)\,t^{\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha(.)}(f)\right)^{*}(t)\right\|_{L^{q(\cdot)}(\Omega)}\\&\leq\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(\left(I_{\alpha_-}(|f|)\right)^{*}+\left(I_{\alpha+}(|f|)\right)^{*}\right)\left(\frac{t}{2}\right)\right\|_{L^{q(\cdot)}(\Omega)}\\&\leq\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha_-}(|f|)\right)^{*}\left(\frac{t}{2}\right)\right\|_{L^{q(\cdot)}(\Omega)}+\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha+}(|f|)\right)^{*}\left(\frac{t}{2}\right)\right\|_{L^{q(\cdot)}(\Omega)}\\&\leq\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha_-}(|f|)\right)^{**}\left(\frac{t}{2}\right)\right\|_{L^{q(\cdot)}(\Omega)}+\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha+}(|f|)\right)^{**}\left(\frac{t}{2}\right)\right\|_{L^{q(\cdot)}(\Omega)}.\end{aligned}\tag{2.3}$$

Also, we have

$$\begin{aligned}\left(I_{\alpha_-}(|f|)\right)^{**}\left(\frac{t}{2}\right)&=\frac{2}{t}\int_{0}^{\frac{t}{2}}\left(I_{\alpha_-}(|f|)\right)^{*}(s)\,ds\\&\leq\frac{2}{t}\int_{0}^{t}\left(I_{\alpha_-}(|f|)\right)^{*}(s)\,ds=2\left(I_{\alpha_-}(|f|)\right)^{**}(t).\end{aligned}\tag{2.4}$$

Similarly, we write

$$\left(I_{\alpha+}(|f|)\right)^{**}\left(\frac{t}{2}\right)\leq2\left(I_{\alpha+}(|f|)\right)^{**}(t)\tag{2.5}$$

Hence by (2.3), (2.4) and (2.5), we get

$$\begin{aligned}&\left\|I_{\alpha(.)}(f)\right\|_{L_{\omega}^{p(.),q(.)}(\Omega)}\lesssim\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha_-}(|f|)\right)^{**}(t)\right\|_{L^{q(\cdot)}(\Omega)}+\\&+\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha+}(|f|)\right)^{**}(t)\right\|_{L^{q(\cdot)}(\Omega)}.\end{aligned}\tag{2.6}$$

On the other hand by [19], there exist $C_1>0$ and $C_2>0$ such that

$$\left(I_{\alpha_-}(|f|)\right)^{*}(t)\leq C_1\left(t^{-1+\frac{\alpha_-}{n}}\int_{0}^{t}f^{*}(s)\,ds+\int_{t}^{l}f^{*}(s)\,s^{-1+\frac{\alpha_-}{n}}ds\right)\tag{2.7}$$

and

$$\left(I_{\alpha_+}(|f|)\right)^{*}(t)\leq C_2\left(t^{-1+\frac{\alpha_+}{n}}\int_{0}^{t}f^{*}(s)\,ds+\int_{t}^{l}f^{*}(s)\,s^{-1+\frac{\alpha_+}{n}}ds\right).\tag{2.8}$$

Now let $\alpha_1(t)=\gamma(t)+\frac{\alpha_-}{n}+\frac{1}{p(t)}-\frac{1}{q(t)}$, $\alpha_2(t)=\gamma(t)+\frac{\alpha_+}{n}+\frac{1}{p(t)}-\frac{1}{q(t)}$, $\beta(t)=\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}$ and $v(t)=0$. Since $\gamma\in\wp([0,l])$ and $p,q\in\wp_1([0,l])$, it is clear that $\alpha_1,\alpha_2,\beta,v\in\wp([0,l])$. If we use (2.7) and Hardy-type inequalities (see [2]), we achieve

$$\begin{aligned}&\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha_-}(|f|)\right)^{**}(t)\right\|_{L^{q(\cdot)}(\Omega)}\approx\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(I_{\alpha_-}(|f|)\right)^{*}(t)\right\|_{L^{q(\cdot)}(\Omega)}\\&\leq C_1\left\|t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\left(t^{-1+\frac{\alpha_-}{n}}\int_{0}^{t}f^{*}(s)\,ds+\int_{t}^{l}f^{*}(s)\,s^{-1+\frac{\alpha_-}{n}}ds\right)\right\|_{L^{q(\cdot)}(\Omega)}\end{aligned}$$

$$\lesssim \left\| t^{\gamma(t)+\frac{\alpha_-}{n}+\frac{1}{p(t)}-\frac{1}{q(t)}-1}\int_0^t f^*(s)\,ds \right\|_{L^{q(.)}(\Omega)} + \left\| t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}\int_t^l f^*(s)\,s^{-1+\frac{\alpha_-}{n}}ds \right\|_{L^{q(.)}(\Omega)}$$

$$= \left\| t^{\alpha_1(t)+v(t)-1}\int_0^t \frac{f^*(s)\,s^{\alpha_1(s)}}{s^{\alpha_1(s)}}ds \right\|_{L^{q(.)}(\Omega)} + \left\| t^{\beta(t)+v(t)}\int_t^l \frac{f^*(s)\,s^{\beta(s)+\frac{\alpha_-}{n}}}{s^{\beta(s)+1}}ds \right\|_{L^{q(.)}(\Omega)}$$

$$\lesssim \left\| f^* t^{\alpha_1(t)} \right\|_{L^{q(.)}(\Omega)} + \left\| f^* t^{\beta(t)+\frac{\alpha_-}{n}} \right\|_{L^{q(.)}(\Omega)}$$

$$= \left\| \omega_{\alpha_-} t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^* \right\|_{L^{q(.)}(\Omega)} + \left\| \omega_{\alpha_-} t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^* \right\|_{L^{q(.)}(\Omega)} = 2\left\| f \right\|_{L^{p(.),q(.)}_{\omega_-}(\Omega)}. \tag{2.9}$$

Similarly if we use (2.8) and Hardy-type inequalities, we get

$$\left\| t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}} \left(I_{\alpha+}(|f|)\right)^{**}(t) \right\|_{L^{q(.)}(\Omega)} \approx \left\| t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}} \left(I_{\alpha_+}(|f|)\right)^{*}(t) \right\|_{L^{q(.)}(\Omega)}$$

$$\lesssim \left\| t^{\alpha_2(t)+v(t)-1}\int_0^t \frac{f^*(s)\,s^{\alpha_2(s)}}{s^{\alpha_2(s)}}ds \right\|_{L^{q(.)}(\Omega)} + \left\| t^{\beta(t)+v(t)}\int_t^l \frac{f^*(s)\,s^{\beta(s)+\frac{\alpha_+}{n}}}{s^{\beta(s)+1}}ds \right\|_{L^{q(.)}(\Omega)}$$

$$= \left\| \omega_{\alpha_+} t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^* \right\|_{L^{q(.)}(\Omega)} + \left\| \omega_{\alpha_+} t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^* \right\|_{L^{q(.)}(\Omega)} = 2\left\| f \right\|_{L^{p(.),q(.)}_{\omega_+}(\Omega)}. \tag{2.10}$$

Combining (2.6), (2.9) and (2.10), we obtain

$$\left\| I_{\alpha(.)}(f) \right\|_{L^{p(.),q(.)}_{\omega}(\Omega)} \lesssim \left\| f \right\|_{L^{p(.),q(.)}_{\omega_-}(\Omega)} + \left\| f \right\|_{L^{p(.),q(.)}_{\omega_+}(\Omega)} = \left\| f \right\|_{L^{p(.),q(.)}_{\omega_-}\cap L^{p(.),q(.)}_{\omega_+}}.$$

This completes the proof. □

Corollary 2.1. *Let $0 < \alpha_-, \alpha_+ < n$, $\gamma \in \wp([0,l])$. If $-\frac{1}{p(0)} < \gamma(0)$, $\gamma(0) + \frac{\alpha_+}{n} < \frac{1}{p'(0)}$, $-\frac{1}{p(\infty)} < \gamma(\infty)$ and $\gamma(\infty) + \frac{\alpha_+}{n} < \frac{1}{p'(\infty)}$, then $M_{\alpha(.)}$ is bounded from the space $\left(L^{p(.),q(.)}_{\omega_{\alpha_-}} \cap L^{p(.),q(.)}_{\omega_{\alpha_+}}\right)(\Omega)$ into the space $L^{p(.),q(.)}_{\omega}(\Omega)$. The condition at infinity being needed in the case $l = \infty$.*

Proof. Let $f \in \left(L^{p(.),q(.)}_{\omega_{\alpha_-}} \cap L^{p(.),q(.)}_{\omega_{\alpha_+}}\right)(\Omega)$ be given. It is known that there exist $C > 0$ such that

$$M_{\alpha(.)}(f)(x) \leq C I_{\alpha(.)}(|f|)(x), \tag{2.11}$$

for all $x \in \Omega$, [11]. Then by Theorem 2.1 and (2.11), we conclude

$$\left\| M_{\alpha(.)}(f) \right\|_{L^{p(.),q(.)}_{\omega}(\Omega)} \lesssim \left\| I_{\alpha(.)}(|f|) \right\|_{L^{p(.),q(.)}_{\omega}(\Omega)} \lesssim \left\| f \right\|_{L^{p(.),q(.)}_{\omega_-}\cap L^{p(.),q(.)}_{\omega_+}}. \tag{2.12}$$

□

Lemma 2.1. *Let $l = \mu(\Omega) \leq 1$, $0 \leq \alpha_1 \leq \alpha_2$. Then*

$$L^{p(.),q(.)}_{\omega_{\alpha_1}}(\Omega) \hookrightarrow L^{p(.),q(.)}_{\omega_{\alpha_2}}(\Omega),$$

where $\omega_{\alpha_i}(t) = t^{\gamma(t)+\frac{\alpha_i}{n}}$, $i = 1, 2$.

Proof. By assumptions, we can write $\omega_{\alpha_2}(t) = t^{\gamma(t)+\frac{\alpha_2}{n}} \leq t^{\gamma(t)+\frac{\alpha_1}{n}} = \omega_{\alpha_1}(t)$. Then, we have $\|.\|_{L^{p(.),q(.)}_{\omega_{\alpha_2}}(\Omega)} \leq \|.\|_{L^{p(.),q(.)}_{\omega_{\alpha_1}}(\Omega)}$. So by Closed Mapping Theorem, we obtain $L^{p(.),q(.)}_{\omega_{\alpha_1}}(\Omega) \hookrightarrow L^{p(.),q(.)}_{\omega_{\alpha_2}}(\Omega)$. □

Theorem 2.2. *Let $l=\mu(\Omega)\leq 1$, $0<\alpha_-,\alpha_+<n$. Then $I_{\alpha(.)}$ is bounded from the space $L^{p(.),q(.)}_{\omega_{\alpha_-}}(\Omega)$ into the space $L^{p(.),q(.)}_{\omega}(\Omega)$, if $-\frac{1}{p(0)}<\gamma(0)$, $\gamma(0)+\frac{\alpha_+}{n}<\frac{1}{p'(0)}$.*

Proof. Since $\omega_{\alpha_+}\leq\omega_{\alpha_-}$ and by Lemma 2.1, we have $L^{p(.),q(.)}_{\omega_{\alpha_-}}(\Omega)\hookrightarrow L^{p(.),q(.)}_{\omega_{\alpha_+}}(\Omega)$. Now, take any $f\in L^{p(.),q(.)}_{\omega_{\alpha_-}}(\Omega)$. Then, by (2.12)

$$\begin{aligned}\left\|I_{\alpha(.)}(f)\right\|_{L^{p(.),q(.)}_{\omega}(\Omega)} &\lesssim \|f\|_{L^{p(.),q(.)}_{\omega_-}\cap L^{p(.),q(.)}_{\omega_+}}\\ &=\|f\|_{L^{p(.),q(.)}_{\omega_-}(\Omega)}+\|f\|_{L^{p(.),q(.)}_{\omega_+}(\Omega)}\leq 2\|f\|_{L^{p(.),q(.)}_{\omega_-}(\Omega)}.\end{aligned}$$

Therefore, the desired is achieved. □

The following Corollary is easily proved by Theorem 2.2 and the inequality (2.11).

Corollary 2.2. *Let $l=\mu(\Omega)\leq 1$, $0<\alpha_-,\alpha_+<n$. Then, $M_{\alpha(.)}$ is bounded from the space $L^{p(.),q(.)}_{\omega_{\alpha_-}}(\Omega)$ into the space $L^{p(.),q(.)}_{\omega}(\Omega)$, if $-\frac{1}{p(0)}<\gamma(0)$, $\gamma(0)+\frac{\alpha_+}{n}<\frac{1}{p'(0)}$.*

Definition 2.1. *Let $0\leq\alpha_i(.)<n$, $i=1,...,m$ and $\overrightarrow{f}\in S\times...\times S$. The m-fold product of fractional integral operator of variable order $\alpha(.)$ is given by*

$$I^{\otimes}_{\alpha(.)}\left(\overrightarrow{f}\right)(x)=I_{\alpha_1(.)}(f_1)(x)\,I_{\alpha_2(.)}(f_2)(x)\,...I_{\alpha_m(.)}(f_m)(x)\,,\ x\in\Omega,$$

where $\alpha(.)=\alpha_1(.)+...+\alpha_m(.)$ and $\overrightarrow{f}=(f_1,...,f_m)$.

Definition 2.2. *Let $\overrightarrow{f}\in L^1_{loc}(\mathbb{R}^n)\times...\times L^1_{loc}(\mathbb{R}^n)$ and $0\leq\alpha_i(.)<n$, $i=1,...,m$. The m-fold product of fractional maximal operator of variable order $\alpha(.)$ is defined by*

$$M^{\otimes}_{\alpha(.)}\left(\overrightarrow{f}\right)(x)=M_{\alpha_1(.)}(f_1)(x)\,M_{\alpha_2(.)}(f_2)(x)\,...M_{\alpha_m(.)}(f_m)(x)\,,\ x\in\Omega,$$

where $\alpha(.)=\alpha_1(.)+...+\alpha_m(.)$ and $\overrightarrow{f}=(f_1,...,f_m)$.

M. Carro and E. Roure (see [1]) defined the 2-fold product of Hardy-Littlewood maximal operator. They study boundedness of this operator from the space $L^{p_1,1}(\omega_1)\times L^{p_2,1}(\omega_2)$ into $L^{p,\infty}\left(\omega_1^{\frac{p}{p_1}}\omega_2^{\frac{p}{p_2}}\right)$ under some conditions. They so proved the boundednes of the bilinear Hardy-Littlewood maximal operator from the space $L^{p_1,1}(\omega_1)\times L^{p_2,1}(\omega_2)$ into $L^{p,\infty}\left(\omega_1^{\frac{p}{p_1}}\omega_2^{\frac{p}{p_2}}\right)$.

The proof of the following lemma is obtained immediately from the inequality (2.11).

Lemma 2.2. *Let $0\leq\alpha_i(.)<n$, $i=1,...,m$. Then*

$$M^{\otimes}_{\alpha(.)}\left(\overrightarrow{f}\right)\lesssim I^{\otimes}_{\alpha(.)}\left(\left|\overrightarrow{f}\right|\right),$$

where $\alpha=\alpha_1(.)+...+\alpha_m(.)$, $\overrightarrow{f}=(f_1,...,f_m)$ and $\left|\overrightarrow{f}\right|=(|f_1|,...,|f_m|)$.

In the following Lemmas and Theorems, we will assume $\gamma_i\in\wp([0,l])$, $p_i,q_i\in\wp_1([0,l])$, $(i=1,...,m)$ and take $\omega(t)=\omega_1(t)\,...\,\omega_m(t)$, $\frac{1}{p(.)}=\frac{1}{p_1(.)}+...+\frac{1}{p_m(.)}$, $\frac{1}{q(.)}=\frac{1}{q_1(.)}+...+\frac{1}{q_m(.)}$, $\omega_i(t)=t^{\gamma_i(t)}$, $\omega_{\alpha^i_-}(t)=t^{\gamma_i(t)+\frac{\alpha^i_-}{n}}$, $\omega_{\alpha^i_+}(t)=t^{\gamma_i(t)+\frac{\alpha^i_+}{n}}$ (i=1,2,...,m).

Lemma 2.3. *If* $\gamma_i(0) < \frac{1}{p_i'(0)}$, $\gamma_i(\infty) < \frac{1}{p_i'(\infty)}$ $(i = 1, ..., m)$, *then for all* $f_i \in L_{\omega_i}^{p_i(.),q_i(.)}(\Omega)$, *there exists* $C > 0$ *such that*

$$\left\| \prod_{i=1}^{m} f_i \right\|_{L_{\omega}^{p(.),q(.)}} \leq C \prod_{i=1}^{m} \|f_i\|_{L_{\omega_i}^{p_i(.),q_i(.)}}.$$

The condition at infinity being needed in the case $l = \infty$.

Proof. Let $f_i \in L_{\omega_i}^{p_i(.),q_i(.)}(\Omega)$ be given for $i = 1, ..., m$. We easily write,

$$\left(\prod_{i=1}^{m} f_i \right)^* \lesssim \prod_{i=1}^{m} f_i^{**}$$

by (2.14) in [15]. If we use the Hölder inequality for variable exponent Lebesgue spaces, then there exists $C > 0$ such that

$$\begin{aligned} \left\| \prod_{i=1}^{m} f_i \right\|_{L_{\omega}^{p(.),q(.)}(\Omega)} &\approx \left\| \prod_{i=1}^{m} f_i \right\|_{L_{\omega}^{p(.),q(.)}(\Omega)}^{1} = \left\| \omega(t)\, t^{\frac{1}{p(t)} - \frac{1}{q(t)}} \left(\prod_{i=1}^{m} f_i \right)^* \right\|_{L^{q(.)}(\Omega)} \\ &\lesssim \left\| \omega_1(t) ... \omega_m(t)\, t^{\left(\frac{1}{p_1(.)} - \frac{1}{q_1(.)}\right)} ... t^{\left(\frac{1}{p_m(.)} - \frac{1}{q_m(.)}\right)} \prod_{i=1}^{m} f_i^{**} \right\|_{L^{q(.)}(\Omega)} \\ &\leq C \prod_{i=1}^{m} \left\| \omega_i(t)\, t^{\left(\frac{1}{p_i(.)} - \frac{1}{q_i(.)}\right)} f_i^{**} \right\|_{L^{q(.)}(\Omega)} = \prod_{i=1}^{m} \|f_i\|_{L_{\omega_i}^{p_i(.),q_i(.)}(\Omega)}. \end{aligned}$$

□

Theorem 2.3. *Let* $0 < \alpha_-^i$, $\alpha_+^i < n$ $(i = 1, ..., m)$ *and* $\alpha(.) = \alpha_1(.) + ... + \alpha_m(.)$. *Then,* $I_{\alpha(.)}^{\otimes}$ *is bounded from the space* $\left(L_{\omega_{\alpha_-^1}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^1}}^{p_1(.),q_1(.)} \right)(\Omega) \times ... \times \left(L_{\omega_{\alpha_-}^m}^{p_m(.),q_m(.)} \cap L_{\omega_{\alpha_+^m}}^{p_m(.),q_m(.)} \right)$ (Ω) *into the space* $L_{\omega}^{p(.),q(.)}(\Omega)$, *if* $-\frac{1}{p_i(0)} < \gamma_i(0)$, $\gamma_i(0) + \frac{\alpha_+^i}{n} < \frac{1}{p_i'(0)}$ *and* $-\frac{1}{p_i(\infty)} < \gamma_i(\infty)$, $\gamma_i(\infty) + \frac{\alpha_+^i}{n} < \frac{1}{p_i'(\infty)}$, $(i = 1, ..., m)$. *The condition at infinity being needed in the case* $l = \infty$.

Proof. Let $\overrightarrow{f} = (f_1, ..., f_m) \in \left(L_{\omega_{\alpha_-^1}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^1}}^{p_1(.),q_1(.)} \right)(\Omega) \times ... \times \left(L_{\omega_{\alpha_-}^m}^{p_m(.),q_m(.)} \cap L_{\omega_{\alpha_+^m}}^{p_m(.),q_m(.)} \right)$ (Ω) be given. From Theorem 2.1 and Lemma 2.3, there exists $C > 0$ such that

$$\begin{aligned} &\left\| I_{\alpha(.)}^{\otimes}\left(\overrightarrow{f}\right) \right\|_{L_{\omega}^{p(.),q(.)}(\Omega)} = \left\| I_{\alpha_1(.)}(f_1)\, I_{\alpha_2(.)}(f_2) ... I_{\alpha_m(.)}(f_m) \right\|_{L_{\omega}^{p(.),q(.)}(\Omega)} \\ &\leq C \prod_{i=1}^{m} \left\| I_{\alpha_i(.)}(f_i) \right\|_{L_{\omega_i}^{p_i(.),q_i(.)}(\Omega)} \lesssim C \prod_{i=1}^{m} \|f_i\|_{L_{\omega_{\alpha_-^i}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^i}}^{p_1(.),q_1(.)}} \\ &= C \left\| \overrightarrow{f} \right\|_{\left(L_{\omega_{\alpha_-^1}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^1}}^{p_1(.),q_1(.)} \right) \times ... \times \left(L_{\omega_{\alpha_-}^m}^{p_m(.),q_m(.)} \cap L_{\omega_{\alpha_+^m}}^{p_m(.),q_m(.)} \right)}. \end{aligned} \tag{2.13}$$

□

Corollary 2.3. *Let* $0 < \alpha_-^i$, $\alpha_+^i < n$ $(i = 1, ..., m)$ *and* $\alpha(.) = \alpha_1(.) + ... + \alpha_m(.)$. *Then, the operator* $M_{\alpha(.)}^{\otimes}$ *is bounded from the space* $\left(L_{\omega_{\alpha_-^1}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^1}}^{p_1(.),q_1(.)} \right)(\Omega) \times ... \times \left(L_{\omega_{\alpha_-}^m}^{p_m(.),q_m(.)} \cap L_{\omega_{\alpha_+^m}}^{p_m(.),q_m(.)} \right)(\Omega)$ *into the space* $L_{\omega}^{p(.),q(.)}(\Omega)$, *if* $-\frac{1}{p_i(0)} < \gamma_i(0)$, $\gamma_i(0) + \frac{\alpha_+^i}{n} <$

$\frac{1}{p_i'(0)}$ *and* -$\frac{1}{p_i(\infty)} < \gamma_i(\infty)$, $\gamma_i(\infty) + \frac{\alpha_+^i}{n} < \frac{1}{p_i'(\infty)}$, $(i = 1, ..., m)$. *The condition at infinity being needed in the case* $l = \infty$.

Proof. By Lemma 2.2 and Theorem 2.3, the proof is clear. □

Definition 2.3. *Let* $m \geq 1$ *and* $0 < \alpha(.) < nm$. *For* $\overrightarrow{f} \in S \times ... \times S$, *the multilinear fractional integral operator of variable order* $\alpha(.)$ *is defined by*

$$I_{\alpha(.)}\left(\overrightarrow{f}\right)(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1)...f_m(y_m)}{(|x-y_1| + ... + |x-y_m|)^{nm-\alpha(x)}} d\overrightarrow{y},$$

where $d\overrightarrow{y} = dy_1...dy_m$ *and* $\overrightarrow{f} = (f_1, ..., f_m)$. *If the functions* $\alpha_i(.)$ $(1 \leq i \leq m)$ *satisfy the conditions* $0 < \alpha_i(.) < nm$ *and* $\alpha_1(.) + ... + \alpha_m(.) = \alpha(.)$, *then* $I_{\alpha(.)}\left(\overrightarrow{f}\right) \leq \prod\limits_{i=1}^{m} I_{\alpha_i(.)} f_i = I_{\alpha(.)}^{\otimes}\left(\overrightarrow{f}\right)$.

Definition 2.4. *Let* $\overrightarrow{f} \in L_{loc}^1(\mathbb{R}^n) \times ... \times L_{loc}^1(\mathbb{R}^n)$ *and* $0 \leq \alpha(.) < nm$. *The multisublinear fractional maximal operator (multilinear for brevity) of variable order* $\alpha(.)$ *is given by*

$$M_{\alpha(.)}\left(\overrightarrow{f}\right)(x) = \sup_{x \in Q} \frac{1}{|Q|^{m - \frac{\alpha(.)}{n}}} \prod_{i=1}^{m} \int\limits_{Q} |f_i(y_i)| \, dy_i.$$

Also if $\alpha_1(.) + ... + \alpha_m(.) = \alpha(.)$, *then* $M_{\alpha(.)}\left(\overrightarrow{f}\right) \leq \prod\limits_{i=1}^{m} M_{\alpha_i(.)} f_i = M_{\alpha(.)}^{\otimes}\left(\overrightarrow{f}\right)$. *Also it is clear by (2.11) that there exists a positive* C *such that* $M_{\alpha(.)}\left(\overrightarrow{f}\right) \leq C I_{\alpha(.)}\left(\left|\overrightarrow{f}\right|\right)$.

Theorem 2.4. *Let* $0 < \alpha_-^i, \alpha_+^i < n$ $(i = 1, ..., m)$ *and* $\alpha(.) = \alpha_1(.) + ... + \alpha_m(.)$. *Then, the multilinear operators* $M_{\alpha(.)}$ *and* $I_{\alpha(.)}$ *are bounded from the space* $\left(L_{\omega_{\alpha_-^1}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^1}}^{p_1(.),q_1(.)}\right)(\Omega)$ $\times ... \times \left(L_{\omega_{\alpha_-^m}}^{p_m(.),q_m(.)} \cap L_{\omega_{\alpha_+^m}}^{p_m(.),q_m(.)}\right)(\Omega)$ *into the space* $L_{\omega}^{p(.),q(.)}(\Omega)$, *if* -$\frac{1}{p_i(0)} < \gamma_i(0)$, $\gamma_i(0) + \frac{\alpha_+^i}{n} < \frac{1}{p_i'(0)}$ *and* -$\frac{1}{p_i(\infty)} < \gamma_i(\infty)$, $\gamma_i(\infty) + \frac{\alpha_+^i}{n} < \frac{1}{p_i'(\infty)}$, $(i = 1, ..., m)$. *The condition at infinity being needed in the case* $l = \infty$.

Proof. Let $\overrightarrow{f} \in \left(\left(L_{\omega_{\alpha_-^1}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^1}}^{p_1(.),q_1(.)}\right) \times ... \times \left(L_{\omega_{\alpha_-^m}}^{p_m(.),q_m(.)} \cap L_{\omega_{\alpha_+^m}}^{p_m(.),q_m(.)}\right)\right)(\Omega)$ be given. Since $I_{\alpha(.)}\left(\overrightarrow{f}\right) \leq I_{\alpha(.)}^{\otimes}\left(\overrightarrow{f}\right)$ and by (2.13), we have

$$\begin{aligned} \left\|I_{\alpha(.)}\left(\overrightarrow{f}\right)\right\|_{L_{\omega}^{p(.),q(.)}(\Omega)} &\leq \left\|I_{\alpha(.)}^{\otimes}\left(\overrightarrow{f}\right)\right\|_{L_{\omega}^{p(.),q(.)}(\Omega)} \\ &\lesssim \left\|\overrightarrow{f}\right\|_{\left(L_{\omega_{\alpha_-^1}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^1}}^{p_1(.),q_1(.)}\right) \times ... \times \left(L_{\omega_{\alpha_-^m}}^{p_m(.),q_m(.)} \cap L_{\omega_{\alpha_+^m}}^{p_m(.),q_m(.)}\right)}. \end{aligned}$$

Similarly, using the inequality $M_{\alpha(.)}\left(\overrightarrow{f}\right) \leq M_{\alpha(.)}^{\otimes}\left(\overrightarrow{f}\right)$, by Lemma 2.2 and (2.13), we get

$$\begin{aligned} \left\|M_{\alpha(.)}\left(\overrightarrow{f}\right)\right\|_{L_{\omega}^{p(.),q(.)}(\Omega)} &\leq \left\|M_{\alpha(.)}^{\otimes}\left(\overrightarrow{f}\right)\right\|_{L_{\omega}^{p(.),q(.)}(\Omega)} \lesssim \left\|I_{\alpha(.)}^{\otimes}\left(\overrightarrow{f}\right)\right\|_{L_{\omega}^{p(.),q(.)}(\Omega)} \\ &\lesssim \left\|\overrightarrow{f}\right\|_{\left(L_{\omega_{\alpha_-^1}}^{p_1(.),q_1(.)} \cap L_{\omega_{\alpha_+^1}}^{p_1(.),q_1(.)}\right) \times ... \times \left(L_{\omega_{\alpha_-^m}}^{p_m(.),q_m(.)} \cap L_{\omega_{\alpha_+^m}}^{p_m(.),q_m(.)}\right)}. \end{aligned}$$

Hence we say that the multilinear operators $M_{\alpha(.)}$ and $I_{\alpha(.)}$ are bounded from the space $\left(L^{p_1(.),q_1(.)}_{\omega_{\alpha^1_-}} \cap L^{p_1(.),q_1(.)}_{\omega_{\alpha^1_+}}\right)(\Omega)\times...\times\left(L^{p_m(.),q_m(.)}_{\omega^m_{\alpha_-}} \cap L^{p_m(.),q_m(.)}_{\omega_{\alpha^m_+}}\right)(\Omega)$ into the space $L^{p(.),q(.)}_{\omega}(\Omega)$. □

Theorem 2.5. *Let* $l = \mu(\Omega) \le 1$, $0 < \alpha^i_-, \alpha^i_+ < n$ $(i = 1, ..., m)$ *and* $\alpha(.) = \alpha_1(.) + ... + \alpha_m(.)$. *Then,* $I^{\otimes}_{\alpha(.)}$ *is bounded from the space* $\left(L^{p_1(.),q_1(.)}_{\omega_{\alpha^1_-}} \times ... \times L^{p_m(.),q_m(.)}_{\omega_{\alpha^m_-}}\right)(\Omega)$ *into the space* $L^{p(.),q(.)}_{\omega}(\Omega)$, *if* $-\frac{1}{p_i(0)} < \gamma_i(0)$ *and* $\gamma_i(0) + \frac{\alpha^i_+}{n} < \frac{1}{p_i'(0)}$, $(i = 1, ..., m)$.

Proof. Take any $\overrightarrow{f} \in \left(L^{p_1(.),q_1(.)}_{\omega_{\alpha^1_-}} \times ... \times L^{p_m(.),q_m(.)}_{\omega_{\alpha^m_-}}\right)(\Omega)$. Then by Theorem 2.2 and Lemma 2.3

$$\left\|I^{\otimes}_{\alpha(.)}\left(\overrightarrow{f}\right)\right\|_{L^{p(.),q(.)}_{\omega}(\Omega)} \lesssim \prod_{i=1}^{m} \left\|I_{\alpha_i(.)}(f_i)\right\|_{L^{p_i(.),q_i(.)}_{\omega_i}(\Omega)} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{p(.),q(.)}_{\omega^i_-}(\Omega)}$$

$$= \left\|\overrightarrow{f}\right\|_{L^{p_1(.),q_1(.)}_{\omega_{\alpha^1_-}} \times ... \times L^{p_m(.),q_m(.)}_{\omega_{\alpha^m_-}}}$$

is obtained. Therefore, we completed the proof. □

Following Corollary is obtained easily by Lemma 2.2 and Theorem 2.5.

Corollary 2.4. *Let* $l = \mu(\Omega) \le 1$, $0 < \alpha^i_-, \alpha^i_+ < n$ $(i = 1, ..., m)$ *and* $\alpha(.) = \alpha_1(.) + ... + \alpha_m(.)$. *Then,* $M^{\otimes}_{\alpha(.)}$ *is bounded from the space* $\left(L^{p_1(.),q_1(.)}_{\omega_{\alpha^1_-}} \times ... \times L^{p_m(.),q_m(.)}_{\omega_{\alpha^m_+}}\right)(\Omega)$ *into the space* $L^{p(.),q(.)}_{\omega}(\Omega)$, *if* $-\frac{1}{p_i(0)} < \gamma_i(0)$*and* $\gamma_i(0) + \frac{\alpha^i_+}{n} < \frac{1}{p_i'(0)}$, $(i = 1, ..., m)$.

Corollary 2.5. *Let* $l = \mu(\Omega) \le 1$, $0 < \alpha^i_-, \alpha^i_+ < n$ $(i = 1, ..., m)$ *and* $\alpha(.) = \alpha_1(.) + ... + \alpha_m(.)$. *Then, the multilinear operators* $I_{\alpha(.)}$ *and* $M_{\alpha(.)}$ *are bounded from the space* $\left(L^{p_1(.),q_1(.)}_{\omega_{\alpha^1_-}} \times ... \times L^{p_m(.),q_m(.)}_{\omega_{\alpha^m_+}}\right)(\Omega)$ *into the space* $L^{p(.),q(.)}_{\omega}(\Omega)$, *if* $-\frac{1}{p_i(0)} < \gamma_i(0)$*and* $\gamma_i(0) + \frac{\alpha^i_+}{n} < \frac{1}{p_i'(0)}$, $(i = 1, ..., m)$.

Proof. If we use the inequalities $I_{\alpha(.)}\left(\overrightarrow{f}\right) \le I^{\otimes}_{\alpha(.)}\left(\overrightarrow{f}\right)$, $M_{\alpha(.)}\left(\overrightarrow{f}\right) \le M^{\otimes}_{\alpha(.)}\left(\overrightarrow{f}\right)$, by Theorem 2.5 and Corollary 2.4 we find the desired. □

3. Conclusions

The bilinear and multilinear operators have been been studied in a number of papers [4, 5, 8] and have been proved the boundedness on some function spaces, [13, 14, 15, 16, 17, 18]. In this study, we are primarily motivated to boundedness of the n-fold product of fractional operators of variable order $\alpha(.)$ under some conditions. In this way, we had the opportunity to consider the boundedness of the multilinear fractional operators of variable order $\alpha(.)$ on weighted variable exponent Lorentz spaces.

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