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UPPER BOUNDS FOR COVERING TOTAL DOUBLE ROMAN DOMINATION

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ABSTRACT. Let $G = (V, E)$ be a finite simple graph where $V = V(G)$ and $E = E(G)$. Suppose that G has no isolated vertex. A covering total double Roman dominating function ($CTDRD$ function) f of G is a total double Roman dominating function ($TDRD$ function) of G for which the set $\{v \in V(G)|f(v) \neq 0\}$ is a covering set. The covering total double Roman domination number $\gamma_{ctdR}(G)$ is the minimum weight of a $CTDRD$ function on G. In this work, we present some contributions to the study of $\gamma_{ctdR}(G)$-function of graphs. For the non star trees T, we show that $\gamma_{ctdR}(T) \leq \frac{4n(T)+5s(T)-4l(T)}{3}$, where $n(T)$, $s(T)$ and $l(T)$ are the order, the number of support vertices and the number of leaves of T respectively. Moreover, we characterize trees T achieve this bound. Then we study the upper bound of the 2-edge connected graphs and show that, for a 2-edge connected graphs G, $\gamma_{ctdR}(G) \leq \frac{4n}{3}$ and finally, we show that, for a simple graph G of order n with $\delta(G) \geq 2$, $\gamma_{ctdR}(G) \leq \frac{4n}{3}$ and this bound is sharp.

Keywords: Total double Roman domination, covering, tree, upper bound.

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1. Introduction

Let G be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For the terminologies and notations which are not defined here explicitly, we may use [10] as a reference. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u : uv \in E(G)\}$. The *closed neighborhood* of a vertex $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$. The *closed neighborhood* of a set $S \subseteq V$ is the set $N[S] = N(S) \cup S = \cup_{v\in S} N[v]$. We denote the *degree* of v by $\deg(v) = \deg_G(v) = |N(v)|$. By $\Delta = \Delta(G)$ and $\delta = \delta(G)$, we denote the *maximum degree* and *minimum degree* of a graph G, respectively. We write K_n, P_n and C_n for the *complete graph*, *path* and *cycle* of order n, respectively.

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A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set in G, and a dominating set of G of cardinality $\gamma(G)$ is called a *γ-set* of G [5].

Let G be a connected graph. An *edge cut* of G is a subset F of $E(G)$ such that $G - E$ is disconnected. In the other words an edge cut is an edge set of the form $[S, \overline{S}]$, where S is a nonempty proper subset of $V(G)$ and $S = V(G) \setminus S$. A *k-edge cut* is an edge cut of k elements, i.e. $|[S, \overline{S}]| = k$. The minimum size of any edge cut is called the *edge-connectivity* of G and denoted by $\lambda(G)$. The graph G is said to be *k-edge connected* if $\lambda(G) \geq k$. All nontrivial connected graphs are 1-edge connected. An *(open) ear* of a graph G is a maximal path whose internal vertices has degree 2 in G. A *closed ear* of a graph G is a cycle, whose all vertices except one have degree 2 in G. A *closed ear decomposition* of G is a decomposition $G = \cup_{i=0}^{k} P_i$, where P_0 is an (initial) cycle and P_i for $i \geq 1$ is an (open) ear or a closed ear in G, [2, 10].

Given a graph G and a positive integer m, assume that $g : V(G) \to \{0, 1, 2, \ldots, m\}$ is a function, and suppose that $(V_0, V_1, V_2, \ldots, V_m)$ is the ordered partition of V induced by g, where $V_i = \{v \in V | g(v) = i\}$ for $i \in \{0, 1, \ldots, m\}$. So we can write $g = (V_0, V_1, V_2, \ldots, V_m)$.

A *double Roman dominating function* on a graph G is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions are met:
(a) if $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.
(b) if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \cup V_3$.

The weight of a double Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function (DRD function) f on G is called *double Roman domination number* of G. We denote this number with $\gamma_{dR}(G)$ and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a *$\gamma_{dR}(G)$-function* of G. Double Roman domination was studied in [1, 6, 8, 9] and elsewhere.
The *total double Roman dominating function* ($TDRD$ function) on a graph G with no isolated vertex is an DRD function f on G with the additional property that the subgraph of G induced by the set $\{v \in V | f(v) \neq 0\}$ has no isolated vertices. The *total double Roman domination number* $\gamma_{tdR}(G)$ is the minimum weight of a $TDRD$ function on G. A $TDRD$ function on G with weight $\gamma_{tdR}(G)$ is called a *$\gamma_{tdR}(G)$-function* [3]. About the parameters of domination one see the latest book authored by Haynes et al. [4]. Another invariant of double Roman dominating function is defined as follows.

A covering total double Roman dominating function (CTDRD function) on a graph G is a TDRD function for which $\{v \in V | f(v) \neq 0\}$ *is a covering set or* $V_0 = V_0^f = \{v \in V | f(v) = 0\}$ *is an independent set. The covering total double Roman domination number* $\gamma_{ctdR}(G)$ *equals the minimum weight of a CTDRD function of G.*
It should be noted that the classic concept of vertex covering instead of outer independent in the graph, has so far been used for other domination parameter. For example, you can see the paper entitled: Covering Italian domination in graphs [7].

The paper is organized as follows. We study the upper bound for some families of graphs G in section 2, and next in section 3, we show that for a simple graph G of order n with $\delta(G) \geq 2$, $\gamma_{ctdR}(G) \leq \frac{4n}{3}$.

2. Trees

In this section we investigate the upper bound for covering total double Roman domination number of trees. We start by a path P_n.

Lemma 2.1. *Let $P_n = x_1, \cdots, x_n$ be a path of order $n \geq 3$ and $n \neq 4$. Then there exists a $CTDRD$ function f of P_n such that $\omega(f) \leq \frac{4n}{3}$ and $f(x_1) = f(x_n) = 1$. This bound is sharp for $n = 3$ and $n = 6$.*

Proof. The proof is given by using induction on n. The result is an immediate for $n < 8$. Let $n \geq 8$ and let the result hold for all paths of order less than n. Let P_n be a path of order $n \geq 8$. Assume that P' is the path obtained from P_n by removing the vertices x_1, x_2, x_3. From the induction hypothesis, there exists a $CTDRD$ function g of P' such that $\omega(g) \leq \frac{4(n-3)}{3}$ and $g(x_4) = g(x_n) = 1$. Define the function f on P_n by $f(x_1) = f(x_3) = 1$, $f(x_2) = 2$, and $f(x) = g(x)$ otherwise. Clearly, f is a $CTDRD$ function of P_n, with $f(x_1) = f(x_n) = 1$, and $\omega(f) \leq \frac{4(n-3)}{3} + 4 \leq \frac{4n}{3}$. □

As an immediate consequence we have.

Corollary 2.1. *Let $P_n = x_1, \cdots, x_n$ be a path of order $n \geq 3$ and $n \neq 4$. Then $\gamma_{ctdR}(P_n) \leq \lfloor \frac{4n}{3} \rfloor$.*

It is easy to see that $\gamma_{ctdR}(P_5) = 6 < \dfrac{20}{3}$, $\gamma_{ctdR}(P_7) = 9 < \dfrac{28}{3}$, $\gamma_{ctdR}(P_8) = 10 < \dfrac{32}{3}$, $\gamma_{ctdR}(P_9) = 11 < \dfrac{36}{3}$ and $\gamma_{ctdR}(P_{10}) = 12 < \dfrac{40}{3}$. Now using induction posed in Lemma 2.1 it follows that $\gamma_{ctdR}(P_n) < \dfrac{4.n}{3}$ for $n \neq 3, 6$.

Let $\mathcal{T}_1$ be a family of trees T such that every vertex of T is a support vertex or a leaf. Let $\mathcal{T}_2$ be a family of trees T obtained from two stars S_p and S_q with $p, q \geq 1$, such that the center of S_p is adjacent to an end vertex of a path P_3 and the center of S_q is adjacent to the other end vertex of this P_3. Let $S_n = K_{1,n-1}$ be a star of order n.

We remark the following immediately.

Remark 1

- If $T = S_n$ for $n \geq 3$, then $\gamma_{ctdR}(T) = 4 > \frac{4n+5s-4l}{3} = 3$. So $\gamma_{ctdR}(S_n) = \frac{4n+5s-4l}{3} + 1$.
- Let $T \neq S_n$ be a tree of order $n \geq 4$ and let v be a support vertex of T, and f be a CTDRD function of T. Then we can assume that $f(v) + f(L_v) = 3$ where L_v is the set of leaf u adjacent to v. In particular,
- if $|L_v| \geq 2$, then we may have $f(v) = 3$ and $f(L_v) = 0$;
- if $L_v = \{u\}$, and $\deg_T(v) = 2$, then we may have $f(v) = 2$ and $f(u) = 1$.

Here, our aim is to determine some bounds on the CTDRD number of trees. We bound the CTDRD number of trees from above and characterize all trees attaining the bound. Let $L(T)$ and $S(T)$ be the set of leaves and the set of support vertices of a tree T, respectively. Let $l(T) = |L(T)|$ and $s(T) = |S(T)|$. In order to characterize all trees T attaining the upper bound given in the next theorem, we introduce a partition of $V(T)$ as follows. Let F be the forest obtained from T by removing $L(T)$ and $S(T)$ from T and let T_0 be a tree as a component of the forest F. Let v be a leaf of T' with its distance from v mod 4. This produces four sets $A(T') = \{u : d_{T'}(u, v) = 0 (\text{mod } 4)\}$, $B(T') = \{u : d_{T'}(u, v) = 1 (\text{mod } 4)\}$, $C(T') = \{u : d_{T'}(u, v) = 2 (\text{mod } 4)\}$ and $D(T') = \{u : d_{T'}(u, v) = 3 (\text{mod } 4)\}$ that partition the vertices of T'. We now have the partition $P = \{S(T) \cup L(T), A(T'), B(T'), C(T'), D(T')\}_{T'}$ of the set of vertices of T. For the sake of convenience, we let $\mathcal{A}(T) = \bigcup_{T'} A(T')$, $\mathcal{B}(T) = \bigcup_{T'} B(T')$, $\mathcal{C}(T) = \bigcup_{T'} C(T')$ and $\mathcal{D}(T) = \bigcup_{T'} D(T')$.

Theorem 2.1. *Let T be a tree of order at least 2 with $s(T)$ support vertices and $l(T)$ leaves. If $T \neq S_n$, $n \geq 3$, then*

$$\gamma_{ctdR}(T) \leq \frac{4n(T) + 5s(T) - 4l(T)}{3}.$$

The equality holds if and only if $T \in \mathcal{T}_1 \cup \mathcal{T}_2$.

Proof. We make use of the notations which were introduced earlier. Clearly, $n(F) = n(T) - s(T) - l(T)$. Suppose that g assigns 1 to the vertices in $\mathcal{A}(T)$ and the vertices u such that u is a leaf of some T' in $\mathcal{B}(T) \cup \mathcal{C}(T) \cup \mathcal{D}(T)$, and 2 to the non leaf vertices in $\mathcal{B}(T) \cup \mathcal{D}(T)$, and 0 to the non leaf vertices in $\mathcal{C}(T)$. Iterate this process for all components T' of F. It is not difficult to check that for any T' in F, $w(g|_{T'}) \leq \frac{4n(T')}{3}$ and $w(g|_{T'}) = \frac{4n(T')}{3}$ if and only if $T' = P_3$. Therefore $w(g) = \frac{4n(F)}{3}$. We now define $f : V(T) \to \{0, 1, 2, 3\}$ by

$$f(u) = \begin{cases} g(u), & \text{if } u \in \mathcal{A}(T) \cup \mathcal{B}(T) \cup \mathcal{C}(T) \cup \mathcal{D}(T), \\ 3, & \text{if } u \in S(T), \\ 0, & \text{if } u \in L(T). \end{cases}$$

It is easy to check that f is a CTDRD function of T. Therefore, $\gamma_{ctdR}(T) \leq w(f) = \frac{4n(F)}{3} + 3s(T) = \dfrac{4n(T) + 5s(T) - 4l(T)}{3}$.

Let $\gamma_{ctdR}(T) = \dfrac{4n(T) + 5s(T) - 4l(T)}{3}$. By the definition of function g on F, if F contains a component T' of order $m \neq 3$, then we can see that $\gamma_{ctdR}(T) < \frac{4n(F)}{3} + 3s(T) = \dfrac{4n(T) + 5s(T) - 4l(T)}{3}$; or if F contains at least two components $P_3 = vwu$ and $P_3 = v'w'u'$, then by changing $f(v) = 0$ or $f(v') = 0$, it follows that the resulted function is a CTDRD function of T and denotes $\gamma_{ctdR}(T) < \frac{4n(F)}{3} + 3s(T) = \dfrac{4n(T) + 5s(T) - 4l(T)}{3}$. On the other hand, if F has exactly one component $T' = P_3$, then $w(g) = 4$ and $\gamma_{ctdR}(T) = \frac{4.3}{3} + 3s(T) = \dfrac{4n(T) + 5s(T) - 4l(T)}{3}$. This tree T is in $\mathcal{T}_2$. If F is an empty set, then every vertex of T is a leaf or a support vertex. It follows that $n(F) = 0$ and $\gamma_{ctdR}(T) = \frac{4n(F)}{3} + 3s(T) = 3s(T) = \dfrac{4n(T) + 5s(T) - 4l(T)}{3}$. This tree T is in $\mathcal{T}_1$. On the other hand, if $T \in \mathcal{T}_1 \cup \mathcal{T}_1$, then the equality is clear. □

3. 2-edge connected graphs

In this section we investigate the upper bound of covering total double Roman domination number of some connected graphs. For integers m and k where $m \geq 3$ and $k \geq 3$, let $C_{m,k}$ be the graph obtained from a cycle $C_m : x_1x_2 \cdots x_mx_1$ and a path $y_1y_2 \cdots y_k$ where $y_1 = x_i$ and $y_k = x_j$ and the order of C_{m+k} is $n = m + k - 2$.

Lemma 3.1. *Let $C_n = x_1, \cdots, x_nx_1$ be a cycle of order $n \geq 3$. Then there exists a CTDRD function f of C_n such that $f(x_1) \geq 1$ and $\omega(f) \leq \frac{4n}{3}$. This bound is sharp for $n = 3$ and $n = 6$.*

Proof. The proof is given by induction on n. The result is obvious for $n < 6$. Let $n \geq 6$ and let the result hold for all cycles of order less than n. Assume that C' is the graph obtained from C_n by removing the vertices x_2, x_3, x_4 and joining x_1 to x_5. By the induction hypothesis, there exists a *CTDRD* function g of C' such that $\omega(g) \leq \frac{4(n-3)}{3}$ and $g(x_1) \geq 1$. If $g(x_1) = 3$, $g(x_5) = 0$. Define the function f by $f(x_2) = 0$, $f(x_3) = 1$,

$f(x_4) = 3$, and $f(x) = g(x)$ otherwise. If $g(x_1) = 2, 3$ and $g(x_5) \geq 1$. Define the function f by $f(x_2) = f(x_3) = 1$, $f(x_4) = 2$, and $f(x) = g(x)$ otherwise. If $g(x_1) = 2$, $g(x_5) = 0$. Define the function f by $f(x_2) = 0$, $f(x_3) = f(x_4) = 2$, and $f(x) = g(x)$ otherwise. If $g(x_1) = 1$, $g(x_5) = 0, 1$. Define the function f by $f(x_2) = f(x_4) = 1$, $f(x_3) = 2$, and $f(x) = g(x)$ otherwise. All in all show that, f is a $CTDRD$ function of C_n such that $f(x_1) \geq 1$, and $\omega(f) \leq \frac{4(n-3)}{3} + 4 \leq \frac{4n}{3}$.

□

As an immediate consequence we have.

Corollary 3.1. *Let $C_n = x_1, \cdots, x_n x_1$ be a cycle of order $n \geq 3$. Then $\gamma_{ctdR}(C_n) \leq \lfloor \frac{4n}{3} \rfloor$.*

We have the classical result as follows.

Theorem 3.1. ([10]) *Let G be a graph of order at least 3. G is a 2-edge connected if and only if it has a closed ear decomposition. Moreover Every cycle in 2-edge connected graph is the initial cycle in some such decomposition.*

Now we have.

Proposition 3.1. *Let $G = C_{m,k}$ for integers $m \geq 3$ and $k \geq 3$. Then there exists a $CTDRD$ function f of $C_{m,k}$ such that $f(x_1) = 1 = f(y_k)$ and $\omega(f) \leq \frac{4(m+k-2)}{3}$.*

Proof. Proof. Let $C_m : x_1x_2 \cdots x_m x_1$ and a path $y_1y_2 \cdots y_k$ where $y_1 = x_i$ and $y_k = x_j$. If $x_i = x_j$, then we have two cycles with one common vertex $v = x_i = y_1 = y_k$. Without lose of generality we assume that $f(v) = 2$ and then it is easy to see that $\gamma_{ctdR}(C_{m,k}) \leq \frac{4(m+k-2)}{3}$. Now let $x_i \neq x_j$. If $k = 3$, then in one of assignments, we can assign 2 to $x_i = y_1$ and 1 to y_2. It follows that $\gamma_{ctdR}(C_{m,1}) \leq \frac{4(m+1)}{3}$. If $k \notin \{3, 4, 6\}$, then using Lemmas 2.1and 3.1 it follows that $\gamma_{ctdR}(C_{m,k}) \leq \frac{4(m+k-2)}{3}$. Let $k = 4$. If $m \neq 6$, then the graph $C_{m,k}$ finds one of the situations of the case $k = 3$ or the case $k \notin \{3, 4, 6\}$. If $m = 6$ and one of the path between x_i and x_j is other than of length 3, then the graph $C_{m,k}$ finds one of the situations of the case $k = 3$, or C_8. If all paths between x_i and x_j is of length 3, then we assume that $x_i = x_1$ and $x_j = x_4$. By assigning 2 to vertices x_1, x_2, x_5, y_2, and 1 to x_4, x_6 and 0 otherwise. It follows that $\gamma_{ctdR}(C_{m,k}) \leq \frac{4(m+k-2)}{3}$. Let $k = 6$. If $m \notin 10$, then $C_{m,k}$ finds one of the above situations. If $m = 10$, and one of the path between x_i and x_j is other than of length 5, then the graph $C_{m,k}$ finds one of the above situations, or C_{14}. If all paths between x_i and x_j is of length 5, then we assume that $x_i = x_1$ and $x_j = x_6$. By assigning 2 to vertices $x_2, x_4, x_6, x_8, x_{10}, y_2$, and 1 to $x_3, x_7, x_9, y_1, y_3, y_4$ and 0 otherwise. It follows that$\gamma_{ctdR}(C_{m,k}) \leq \frac{4(m+k-2)}{3}$. Therefore the desired result holds. □

Now we investigate the covering total double Roman dominating of 2-edge connected graph G.

Theorem 3.2. *Let G be a 2-edge connected graph. Then $\gamma_{ctdR}(G) \leq \dfrac{4n(G)}{3}$.*

Proof. Let H be a 2-edge connected graph and G be a graph obtain from H by adding a path P_k with end vertices v_1, v_k in H such that the vertices $\deg_G(v_i) = 2$ for $2 \leq i \leq k-1$. Let the path P_k be added to H as an (open) ear. If $k \geq 5 (k \neq 6)$, then $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(H) + \gamma_{ctdR}(P_{k-2}) \leq \frac{4n(H)}{3} + \frac{4(k-2)}{3} = \frac{4n(G)}{3}$. Now assume that $k \in \{3, 4, 6\}$. For $k = 4$, we delete the vertices v_1, v_2, v_3 and connect v_4 to the all neighbors of v_1, the resulted graph H is 2-edge connected. In the resulted graph H, if the weight of v_1 is 0, then in graph G we assign 0 to v_4, 1 to v_3 and 3 to v_2 or 0 to v_4, 2 to v_3 and 2 to v_2 object to the weight of neighbors of v_4. If the weight of v_1 is 1, then in graph G, we assign 1 to v_4, 1 to v_3 and 2 to v_2. If in the resulted graph, the weight of v_1 is 2, then in graph G, we assign 2 to v_4, 1 to v_3 and v_2. If in the resulted graph, the weight of v_1 is 3, then in graph G, we assign 3 to v_4, 1 to v_3 and 0 to v_2, or we assign 3 to v_4, 0 to v_3 and 1 to v_2. For $k = 6$, we delete the vertices v_2, v_3, v_4 and connect v_5 to v_1, the resulted graph H is 2-edge connected and $\gamma_{ctdR}(H) \leq \frac{4n(H)}{3}$. In H if we assign 0 to v_5, then by assignment $1, 2, 1$ to the v_4, v_3, v_2 respectively, or $2, 1, 1$ to the v_4, v_3, v_2 respectively, or $3, 1, 0$ to the v_4, v_3, v_2 respectively. If we assign 1 to v_5, then by assignment $0, 3, 1$ to v_4, v_3, v_2 respectively, or $1, 2, 1$ to the v_4, v_3, v_2 respectively, or $2, 1, 1$ to the v_4, v_3, v_2 respectively, or $3, 0, 1$ to the v_4, v_3, v_2 respectively. If we assign 2 to v_5, then by assignment $1, 1, 2$ to v_4, v_3, v_2 respectively. If we assign 3 to v_5, then by assignment $0, 1, 3$ to v_4, v_3, v_2 respectively, or $1, 0, 3$ to the v_4, v_3, v_2 respectively, or $1, 2, 1$ to v_4, v_3, v_2 respectively. Let $k = 3$. Then in one of assignments, we can assign 2 to v_1 or v_3, and 1 to two of them. The proof will be ended, whenever we investigate the case of adding a path P_k as a closed ear. Let G be a graph obtained from H by adding a path P_k as a closed ear, with end vertices v_1, v_k in H such that the vertices $\deg_G(v_i) = 2$ for $2 \leq i \leq k-1$. The proof is similar to the proof of the case, whenever P_k is an open ear. □

Let $\mathcal{Q}$ be a family of connected graphs with this property, $\gamma_{ctdR}(Q) \leq \frac{4n(Q)}{3}$ for any $Q \in \mathcal{Q}$.

Proposition 3.2. *Let $Q \in \mathcal{Q}$ and $u \in V(Q)$. Let H be a 2-edge connected graph and y_k be a vertex in H. If G is a graph obtained from Q and and H, by adding the edge uy_k, then $\gamma_{ctdR}(G) \leq \frac{4n(G)}{3}$.*

Proof. Let f be a $\gamma_{tdR}(Q)$-function and g be a $\gamma_{ctdR}(H)$-function with $g(y_k) = 1$. Then the function h defined by $h(x) = f(x)$ for $x \in V(Q)$ and $h(x) = g(x)$ otherwise, is a $CTDRD$ function of G. By Theorem 3.2 we can show that $\gamma_{ctdR}(H) \leq \frac{4n(H)}{3}$ and $g(y_k) = 1$. The fact of $Q \in \mathcal{Q}$ and $\gamma_{ctdR}(H) \leq \frac{4n(H)}{3}$ conclude that $\gamma_{ctdR}(G) \leq w(f) + w(g) \leq \frac{4n(Q)}{3} + \frac{4n(H)}{3} = \frac{4n(G)}{3}$. □

In later result we have shown that some family of graphs with connectivity 1 satisfy in this bound. In the next section we study the CTDRD of graphs with $\delta(G) \geq 2$, which includes all connected graph with connectivity 1.

4. Graphs G with $\delta(G) \geq 2$

For upper bound, we show that for any graph with minimum degree 2, it follows that$\gamma_{ctdR}(G) \leq \frac{4n}{3}$ and this bound is sharp.

Theorem 4.1. *Let G be a simple graph of order n with $\delta(G) \geq 2$. Then*

$$\gamma_{ctdR}(G) \leq \frac{4n}{3}.$$

This bound is sharp for graphs $(\bigcup_{i\geq 1} C_6) \cup (\bigcup_{j\geq 1} C_3)$ where $i + j \geq 1$.

Proof. Suppose that G is a simple graph of order n with $\delta(G) \geq 2$. For $n \leq 10$, It is not difficult to show that the result holds for any graph G for which $\delta(G) \geq 2$. We proceed the proof by induction on n. Suppose that $n \geq 3$ and the result hold for all graphs of order less than n with $\delta(G) \geq 2$. To prove the inductive verdict, we investigate the graphs with $\delta(G) = 2$ and the graphs with $\delta(G) \geq 3$ separately. For the simple graphs with $\delta(G) = 2$ there are the following cases.

Case 1. Suppose that G has an induced path $P_k = v_1, v_2, \cdots, v_k$ with $k \geq 3$. Let u, w be two vertices of degree at least three for which v_1, v_k are adjacent to u, w respectively. We delete three vertices v_1, v_2, v_3 and add the edge v_4u if $k \geq 4$, otherwise, add wu. The resulted graph G' has degree at least 2. By using induction, $\gamma_{ctdR}(G') \leq \dfrac{4(n-3)}{3}$. Now by deleting the edge v_4u (wu) and assigning 1 to v_1, v_3 and 2 to v_2, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 \leq \dfrac{4n}{3}$.

Case 2. Suppose that G has an induced path $P_2 = v_1, v_2$ as a maximal path. Let u, w be two vertices of degree at least three for which v_1, v_2 are adjacent to u, w respectively and there do not exist another path between u and w. For graph G we consider the assignment as follows.
We delete three vertices v_1, v_2, w, and join u to each vertex $x \in N_{G-v_2}(w)$. Let G' be the resulted graph. Then $\delta(G) \geq 2$. By using induction, $\gamma_{ctdR}(G') \leq \dfrac{4(n-3)}{3}$. We assign 1 to v_1, u, 0 to w and 2 to v_2 , or 1 to w, v_2, 2 to v_1 and 0 to u if the weight of u in G' is 0. We assign 1 to one of v_1 or v_2 and 2 to one of the others, and 1 to v_2, if the weight of u in G' is 1. We assign 1 to v_1, and v_2 and 2 to w, if the weight of u in G' is 2. We assign 3 to w and 0 to one of v_1 or v_2 and 1 to one of the others if the weight of u in G' is 3. All in all, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 \leq \dfrac{4n}{3}$.

Case 3. Suppose that G has two paths $P_1 = x_1, x_2$ and $P_2 = y_1, y_2$ as maximal paths. Let u, w be two vertices of degree at least three for which x_1, y_1 and x_2, y_2 are adjacent to u, w respectively and there have no other path between u and w. Let $H = G - \{x_1, x_2, y_1, y_2, u\}$ be a subgraph of G and G' be a graph obtained from H by joining w to the vertices in $N(u) \setminus \{x_1, y_1\}$. Let f be the γ_{ctdR}-function on G'.
If $f(w) = 0$, we extend f to the γ_{ctdR}-function g on G by assigning, $g(u) = 0$, ($g(x_1) = g(y_1) = 2$, $g(x_2) = g(y_2) = 1$, or $g(x_1) = g(y_2) = 2$, $g(x_2) = g(y_1) = 1$), and $g(v) = f(v)$ for $v \in G'$.
If $f(w) = 1$ and its adjacent vertex z in G' with $f(z) = 2$ is not in $N_G(u)$ (is in $N_G(u)$), then we extend f the γ_{ctdR}-function g on G by assigning, $g(u) = 1$, ($g(x_1) = g(y_1) = 2$, $g(x_2) = g(y_2) = 0, g(w) = 2$, and $g(v) = f(v)$ for $v \in G' - w$ ($g(u) = g(x_2) = g(y_2) = 2$, $g(x_1) = g(y_1) = 0$, and $g(v) = f(v)$ for $v \in G'$).
If $f(w) = 2$ and its adjacent vertex z in G' with $f(z) \geq 1$ is not in $N_G(u)$ (is in $N_G(u)$), then we extend the γ_{ctdR}-function g on G by assigning, $g(u) = g(x_1) = g(y_1) = 2$, $g(x_2) = g(y_2) = 0$, and $g(v) = f(v)$ for $v \in G'$ ($g(u) = g(x_2) = g(y_2) = 2$, $g(x_1) = g(y_1) = 0$, and $g(v) = f(v)$ for $v \in G'$).
If $f(w) = 3$ and its adjacent vertex z in G' with $f(z) \geq 1$ is not in $N_G(u)$ (is in $N_G(u)$), then we extend the γ_{ctdR}-function g on G by assigning, $g(u) = 3$, $g(x_1) = g(y_1) = 1$, $g(x_2) = g(y_2) = 0$ ($g(u) = 3$, $g(x_1) = g(y_1) = 0$, $g(x_2) = g(y_2) = 1$), and $g(v) = f(v)$ for $v \in G'$.
All in all, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 6 \leq \dfrac{4(n'+5)}{3} = \dfrac{4n}{3}$.

Case 4. Suppose that G has $k \geq 3$ paths $P_{i2} = v_{i1}, v_{i2}$ as maximal paths for $1 \leq i \leq k$. Let u, w be two vertices of degree at least three for which v_{i1}, v_{i2} are adjacent to u, w respectively and there have no path $P_1 = v$ such that v be adjacent to u and w. If u and w have no another neighbors, then the result is clear. Therefore we assume that u or w has another neighbors. Let $H = G - \{v_{i1}, v_{i2} : i = 1, 2 \cdots k\}$ be a subgraph of G and G' be a graph obtained from H by joining w to the vertices in $N_G(u) \setminus \{v_{i1} : i = 1, 2 \cdots k\}$. Let f be the γ_{ctdR}-function on G'.

Henceforth, a similar discussion of the proof of Case 3, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G')+ 2k+3 \leq \dfrac{4(n-(2k+1))}{3} + 2k+3 \leq \dfrac{4n}{3}$, because $k \geq 3$.

Case 5. Suppose that G has an induced path $P_1 = v_1$ as a maximal path. Let u, w be two vertices of degree at least three for which v_1 be adjacent to u, w and there have no other path of length 1 between u and w. If u, w have neighbors of degree at least three, we delete three vertices v_1, u, w, then the resulted graph G' has $\delta(G') \geq 2$. By using induction, $\gamma_{ctdR}(G') \leq \dfrac{4(n-3)}{3}$. By assigning 1 to u, w and 2 to v_1, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 \leq \dfrac{4n}{3}$.

If u or w has at least another neighbor of degree 2, and assume that $v_2, v_3, \cdots, v_k$ are vertices of degree 2 in $N(w)$. We delete the vertices $w, v_1, \cdots, v_k$ from G and G' is the resulted graph. Then $\delta(G') \geq 2$. By using induction, $\gamma_{ctdR}(G') \leq \dfrac{4(n-k-1)}{3}$. By assigning 1 to $v_1, \cdots, v_k$ and 2 to w, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + k + 2 \leq \dfrac{4n}{3}$, because $3(k+2) \leq 4(k+1)$ for $k \geq 2$.

Case 6. Suppose that G has an induced path $P_1 = v_1$ as a maximal path. Assume that, for any two vertices u, w of degree at least three, there dose not exist a path of length 1 adjacent to u, w or there exist at least two paths of length 1 between u, w. Let there exist $t \geq 2$ paths $P_3 = x_3, \cdots, P_{t1} = x_{t1}$ where x_{i1} be adjacent to u and w. If the other neighbors of both u and w are of degree at least three, then we delete the vertices $w, x_3, \cdots, x_{t1}, u$ from G and G' is the resulted graph with $\delta(G') \geq 2$. By using induction, $\gamma_{ctdR}(G') \leq \dfrac{4(n-t-2)}{3}$. By assigning 1 to x_3, 2 to w and u, 0 to $x_{21}, \cdots, x_{t1}$, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 5 \leq \dfrac{4(n-(t+2))}{3} + 5 \leq \dfrac{4n}{3}+$, because $t \geq 2$.

If the vertices u or w has neighbors $x_1, x_2, \cdots, x_m$ or $y_1, y_2, \cdots, y_k$ of degree 2 and since there is no path P_2 in G, then for every vertex $x \in N(x_i)$ or $y \in N(y_j)$, $\deg(x) \geq 3$, $\deg(y) \geq 3$. We delete the vertices $y_1, y_2, \cdots, y_k, w, x_3, \cdots, x_{t1}, u, x_1, x_2, \cdots, x_m$ from G and G' is the resulted graph with $\delta(G') \geq 2$. By assigning 1 to x_3, y_js, x_is, 2 to w and u, 0 to $x_{21}, \cdots, x_{t1}$, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 5 + k + m + \leq \dfrac{4(n-(t+2+k+m))}{3} + 5 + k + m \leq \dfrac{4n}{3}+$, because $t \geq 2, k$ or $m \geq 1$.

Case 7. Suppose that G has $k \geq 1$ paths $P_{i2} = x_{i1}, x_{i2}$ for $1 \leq i \leq k$ and $m \geq 1$ paths $P_{j1} = y_{j1}$ for $1 \leq j \leq m$ with common end vertices u, w. Assume that other neighbors of u and w has degree at least three. We delete the vertices $u, w, x_{i1}s, x_{i2}s, y_{j1}s$. Then the resulted graph G' is a graph with $\delta(G') \geq 2$. By assigning 1 to x_{i1}s, x_{i2}s, 2 to w and u, 0 to y_{j1}s, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 + 2k \leq \dfrac{4(n-(2k+m+2))}{3} + 4 + 2k \leq \dfrac{4n}{3}+$, because $k, m \geq 1$.

Let $w_1, w_2, \cdots, w_t$ be vertices of degree at least three such that there exist $k_i \geq 1$ paths of length 2 and $m_i \geq 1$ paths of length 1 with end vertices w_i, w_{i+1}. Without loss of generality we can assume that the other neighbors of w_is (if exists) are of degree at least three. Let H be the subgraph induced by w_is and the paths between them and $G' = G - H$ be the resulted graph. Let H contains q paths of length 2 and r paths of length 1, where $q, r \geq t-1$. By using induction, $\gamma_{ctdR}(G') \leq \dfrac{4(n-(t+2q+r))}{3}$. By assigning 1 to the vertices of q paths of length two, 0 to the vertices of r paths of length one, and 2 to w_is, it follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 2t + 2q \leq \dfrac{4(n-(t+2q+r))}{3} + 2t + 2q \leq \dfrac{4n}{3}$, because $q, r \geq t-1$.

For the simple graphs with $\delta(G) \geq 3$ there are the following case and subcases.

Case 8. Let $\delta(G) \geq 3$ and $x, y, z \in V(G)$ be three vertices such that y be adjacent to x, z. Assume that $Q = \{x, y, z\}$ and $H = G - Q$. Then there exist the following subcases.

Subcase 8.1 Let $\delta(H) \geq 2$. Say $G' = H$, by using induction, $\gamma_{ctdR}(G') \leq \frac{4(n-3)}{3}$. We assign 2 to y, and 1 to x, z. It follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 \leq \frac{4(n-3)}{3} + 4 \leq \frac{4n}{3}$.

Now let for every such Q and $H = G - Q$, $\delta(H) \leq 1$. Then there exist vertices $v_j^i, u_j \in V(G)$ such that v_j^is are adjacent to vertices u_j, two vertices of Q where $\deg(v_j^i) = 3$ for $1 \leq i \leq k$, $1 \leq j \leq k'$ and $k \neq 1$, or there exist vertices $v'_r, u'_r \in V(G)$ such that vertex v'_r is adjacent to u'_r, two vertices of Q such that u'_r is adjacent to a vertex of Q and $\deg(v'_r) = \deg(u'_r) = 3$ for $1 \leq r \leq k''$, or there exist vertices w_ss $\in V(G)$ such that those are adjacent to all vertices of Q and $\deg(w_s) = 3$ for $1 \leq s \leq l$, or there exist vertices w'_ts $\in V(G)$ such that those are adjacent to at least two vertices of Q and $\deg_H(w'_t) = 1$ for $1 \leq t \leq l'$. Then there exist the following subcases.

Subcase 8.2 Let $k'' = 1$ and $l \geq 1$ or $l' \geq 1$. First let $l \geq 1$. Then G' be the graph obtained from G by removing the vertex y. The induction hypothesis implies that, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. Then we extend f to g on G by assigning 1 to y and $g(t) = f(t)$ for otherwise. It follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 1 \leq \dfrac{4n}{3}$.

If $l' = 1$, then G' is the graph obtained from G by removing the vertices x, y, z, and joining v'_1 to w'_1 and joining the vertex v_j^1 to v_j^is. By using induction, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. Then we extend f to g on G by assigning $1, 2, 1$ to the vertices x, y, z respectively, if $f(v'_1) \in \{0, 1\}$, $f(w'_1) \in \{0, 1\}$ or by assigning 2 to x, y, z, 0 to v'_1, if $f(v'_1) = 2$ and $f(w'_1) = 0$, by assigning 2 to two the vertices of $\{x, y, z\}$, 1 to one of the others, v'_1 if $f(v'_1) \in \{2, 3\}$ and $f(w'_1) \in \{1, 2, 3\}$ or by assigning 2 to x, y, z, 1 to v'_1, if $f(v'_1) = 3$ and $f(w'_1) \in \{0, 1\}$ or by assigning 1 to two the vertices of $\{x, y, z\}$, 2 to one of the others if $f(v'_1) \in \{0, 1\}$, $f(w'_1) = 2$ or $f(v'_1) = 1$, $f(w'_1) = 3$ or by assigning 2 to w'_1, to two the vertices of $\{x, y, z\}$, 1 to one of the others if $f(v'_1) = 0$, $f(w'_1) = 3$ and $g(t) = f(t)$ for otherwise. It follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 \leq \dfrac{4n}{3}$.

Now let $l' > 1$. let G' be the graph obtained from G by removing the vertices x, y, z, v'_1, w'_ts, and joining the vertex v_j^1 to v_j^is and joining the vertex u'_1 to $v \in V(G)$ where v be adjacent to w'_1. By using induction, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. Then we extend f to g on G by assigning 2 to two the vertices of $\{x, y, z\}$, 1 to one of the others, v'_1, w'_ts if $f(u'_1) \in \{0, 1\}$, $f(v) \in \{0, 1, 2\}$ or by assigning 2 to w'_1, two the vertices of $\{x, y, z\}$, 1 to one of the others, to $w'_2, \cdots, w'_{l'}$, 0 to v'_1 if $f(u'_1) = 2$, $f(v) \in \{0, 1, 2, 3\}$ or $f(u'_1) = 3$, $f(v) \in \{1, 2, 3\}$ or by assigning 2 to two the vertices of $\{x, y, z\}$, u'_1, 1 to one of the others, to $w'_2, \cdots, w'_{l'}$, 0 to v'_1, 3 to w'_1 if $f(u'_1) = 3$, $f(v) = 0$ or by assigning 2 to x, y, z, 1 to $w'_2, \cdots, w'_{l'}$, u'_1, 0 to v'_1, w'_1 if $f(u'_1) = 0$, $f(v) = 3$ or by assigning 2 to x, y, z, 1 to w'_ts, 0 to v'_1 if $f(u'_1) = 1$, $f(v) = 3$ and $g(t) = f(t)$ for otherwise. $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 5 + l' \leq \dfrac{4n}{3}$.

Subcase 8.3 Let $k''=1$ and $l'=k=k'=0$. If y is adjacent to v'_1, u'_1, then assume that G' is the graph obtained from G by removing the vertex y. By the induction hypothesis, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. We extend f to g on G by assigning 1 to y, and $g(t)=f(t)$ for otherwise. On the other hand let G' be the graph obtained from G by removing the vertices v, u, v'_1, where $v, u \in \{x, y, z\}$, u'_1 is not adjacent to u, v. By the induction hypothesis, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. We extend f to g on G by assigning 1 to u, v, 2 to v'_1 or 2 to one of v or u and 1 to one of the others, v'_1 and $g(t)=f(t)$ for otherwise. Thus $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + \dfrac{4(n-n')}{3} \leq \dfrac{4n}{3}$.

Subcase 8.4 Let $k''=1$ and $k \geq 2$, $k' \geq 1$. Let G' be the graph obtained from G by removing the vertices x, y, z, and joining the vertex v^1_j to v^i_js, and joining the vertex v'_1 to v^1_1. By using induction, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. Without loss of generality we first assume that $f(v^1_1)=1$, $f(u_1)=3$. Then we extend f to g on G by assigning 2 to y, 1 to x, z, and $g(t)=f(t)$ for otherwise.
If $f(v^1_1)=2$, $f(v^2_1)=1$, $f(u_1)=0$ then we extend f to g on G by assigning 2 to x, y, z, 1 to u_1, 0 to v^1_1, v^2_1, and $g(t)=f(t)$ for otherwise. If $f(v^1_1)=2$, $f(v^2_1)=0$, $f(u_1)=2$ then we extend f to g on G by assigning 2 to x, z, 1 to y, v^1_1, and $g(t)=f(t)$ for otherwise. If $f(v^1_1)=3$, $f(v^2_1)=0$, $f(u_1) \geq 1$ then we extend f to g on G by assigning 2 to u_1 if $f(u_1)=1$, also by assigning 2 to two vertices of $\{x, y, z\}$ where are adjacent to v'_1, 1 to one of the others, v^1_1, and $g(t)=f(t)$ for otherwise. It follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 \leq \dfrac{4n}{3}$.

Subcase 8.5 Let $k'' \neq 1$ and $l \geq 1$ or $l' \geq 3$. Let G' be the graph obtained from G by removing the vertices x, y, z, w_ss, w'_ts, and joining the vertex v^1_j to v^i_js, and joining v'_1 to v'_rs. The induction hypothesis implies that, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. First assume that $f(v^1_1)=1$, $f(u_1)=3$. Then we extend f to g on G by assigning 0 to w_ss, 1 to w'_ts, 2 to x, y, z, v'_1, if $f(v'_1)=3$ or by assigning 0 to w_ss, 1 to w'_ts, $g(x)+g(y)+g(z)=5$ and $g(t)=f(t)$ for otherwise.
If $f(v^1_1)=2$, $f(v^2_1)=1$, $f(u_1)=0$, then we extend f to g on G by assigning 2 to x, y, z, 1 to u_1, w'_t, 0 to v^1_1, v^2_1, w_ss and $g(t)=f(t)$ for otherwise. It follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 5 + l' \leq \dfrac{4n}{3}$.

Subcase 8.6 Let $l'=2$ and $k''=0$. Let G' be the graph obtained from G by removing the vertices x, y, z, and joining the vertex v^1_j to v^i_js and joining vertex w'_1 to w'_2. By the induction, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. Without loss of generality we first assume that $f(v^1_1)=1$, $f(u_1)=3$. If $f(w'_1), f(w'_2) \in \{0, 1\}$, then we extend f to g on G by assigning $g(y)=2$, $g(x)=g(z)=1$ and $g(t)=f(t)$ for otherwise.
If $f(w'_1)=0, 1, 2, 3$, $f(w'_2)=2$, then extend f to g on G by assigning $g(v)=2$ for one vertex $v \in Q$ where $v \in N(w'_1)$, $g(x)=1$ for $x \in Q-\{v\}$ and $g(t)=f(t)$ for otherwise.
If $f(w'_1)=1, 2, 3$, $f(w'_2)=3$, then by assigning $g(v)=2$ for one vertex $v \in Q$ where $v \in N(w'_1)$, $g(x)=1$ for $x \in Q-\{v\}$ and $g(t)=f(t)$ for otherwise. If $f(w'_1)=0$, $f(w'_2)=3$, then by assigning $g(v)=2$ for one vertex $v \in Q$ where $v \in N(w'_1)$, $g(x)=1$ for $x \in Q-\{v\}$, $g(w'_1)=1$, $g(w'_2)=2$ and $g(t)=f(t)$ for otherwise.
Now let $f(v^1_1)=2$, $f(v^2_1)=1$, $f(u_1)=0$. We extend f to g on G by assigning 2 to x, y, z, 1 to u_1, 0 to v^1_1, v^2_1 and $g(t)=f(t)$ for otherwise. Then $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 \leq \dfrac{4n}{3}$.

Subcase 8.7 Let $l' = 1$ and $k'' = 0$ and w'_1 be adjacent to y. Let G' be the graph obtained from G by removing the vertices x, y, z, w'_1, and joining the vertex v_j^1 to v_j^is. By the induction hypothesis, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. We first assume that $f(v_1^1) = 1$, $f(u_1) = 3$. Then we extend f to g on G by assigning $g(y) = 2$, $g(x) = g(z) = g(w'_1) = 1$ and $g(t) = f(t)$ for otherwise.
If $f(v_1^1) = 2$, $f(v_1^2) = 1$, $f(u_1) = 0$, then we extend f to g on G by assigning 2 to x, y, z, 1 to u_1, w'_1, 0 to v_1^1, v_1^2 and $g(t) = f(t)$ for otherwise. It follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 5 \leq \dfrac{4n}{3}$.

Subcase 8.8 Let $l' = 1$ and $k'' = 0$ and w'_1 do not be adjacent to y and let y be adjacent to at least two vertices other than x, z and v_j^is in G. Let G' be the graph obtained from G by removing the vertices x, z, w'_1, and joining the vertices v_j^1s to v_j^is. By the induction hypothesis, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. First assume that $f(v_1^1) = 1$, $f(u_1) = 3$. Then we extend f to g on G by assigning $g(w'_1) = 2$, $g(x) = g(z) = 1$ and $g(t) = f(t)$ for otherwise.
If $f(v_1^1) = 2$, $f(v_1^2) = 1$, $f(u_1) = 0$, then we extend f to g on G by assigning 2 to the vertices of x, u_1, z, 1 to w'_1, 0 to v_1^1, v_1^2 and $g(t) = f(t)$ for otherwise. It follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 4 \leq \frac{4n}{3}$.

Subcase 8.9 Let $l' = 1$ and $k'' = 0$ and w'_1 is not adjacent to y and let y be adjacent to at most one vertex other than x, z and v_j^is in G. First assume that y is not adjacent to any vertex other than x, z and v_j^is in G. Let G' be the graph obtained from G by removing the vertices x, y, z, w'_1, and joining the vertex v_j^1 to v_j^is. By the induction hypothesis, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. Then we extend f to g on G by assigning $g(w'_1) = 1$, $g(x) = g(z) = 2$, $g(y) = 0$ and $g(t) = f(t)$ for otherwise.
Now assume that y is adjacent to one vertex other than x, z and v_j^is in G, say v. Assume that $N_H(w'_1) = u$. Let G' be the graph obtained from G by removing the vertices x, y, z, w'_1, and joining the vertex v_j^1 to v_j^is and joining v to u. Using induction hypothesis, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. Assume that $f(v_1^1) = 1$, $f(u_1) = 3$. We extend f to g on G by assigning 2 to x, z, 1 to y, 0 to w'_1 or 2 to x, z, 1 to w'_1, 0 to y if $f(v) \in \{0, 1\}$, $f(u) \in \{0, 1\}$, and by assigning 1 to x, y, z, 2 to w'_1 or 0 to y, 1 to x, z, 3 to w'_1 if $f(u) \in \{0, 1, 2, 3\}$ and $f(v) \in \{2, 3\}$, and by assigning 1 to x, z, w'_1, and 2 to y if $f(u) = 2$, $f(v) \in \{0, 1\}$ and by assigning 3 to y and 0 to w'_1, 1 to x, z or 2 to y, 1 to x, z, w'_1 if $f(u) = 3$, $f(v) \in \{0, 1\}$ and $g(t) = f(t)$ for otherwise.
If $f(v_1^1) = 2$, $f(v_1^2) = 1$, $f(u_1) = 0$, then we extend f to g on G by assigning 2 to x, y, z, 1 to u_1, w'_1, 0 to v_1^1, v_1^2 and $g(t) = f(t)$ for otherwise. Thus $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + 5 \leq \dfrac{4(n-4)}{3} + 5 \leq \dfrac{4n}{3}$.

Subcase 8.10 Let $1 \leq l' \leq 2$ and $k'' \geq 2$. Let G' be the graph obtained from G by removing the vertices x, y, z,w'_ts, and joining the vertex v_j^1 to v_j^is, and joining v'_1 to v'_rs. By the induction hypothesis, there exists a $CTDRD$ function f of G' such that $\omega(f) \leq \frac{4(n')}{3}$. First assume that $f(v_1^1) = 1$, $f(u_1) = 3$. Then we extend f to g on G by assigning 2 to x, y, u'_1, 1 to z, w'_ts If $f(v'_1) = 0$, $f(u'_1) = 3$ and by assigning 2 to x, y, z, 1 to w'_ts, 0 to v'_2 If $f(v'_1) = 0$, $f(u'_1) = 2$, $f(v'_2) = 2$, by assigning 2 to x, y, z, 1 to w'_ts, 0 to v'_2, v'_3 If $f(v'_1) = 0$, $f(u'_2), f(u'_3) \geq 1$, $f(v'_2) = f(v'_3) = 2$, by assigning 2 to x, y, z, 1 to w'_ts, 0 to v'_2 If $f(v'_1) = 0$, $f(u'_2) \geq 1$, $f(v'_2) = 3$, and by assigning 2 to x, y, z, 1 to w'_ts, 0 to v'_2 if $f(v'_1) = f(u'_2) = 1$, $f(v'_2) = 2$ and 2 to x, y, 1 to z, w'_ts, 0 to v'_1 if $f(v'_1) = 1$, $f(u'_1) = 2$ and by assigning 2 to x, y, z, 1 to u'_1, w'_ts, 0 to v'_1 or 2 to x, y, z, 1 to w'_ts, 0 to v'_1 If $f(v'_1) = 3$ and by assigning

2 to two vertices of x, y, z, 1 to one of the others, u_1', w_t's, 0 to v_1' or 2 to x, y, z, 1 to w_t's, 0 to v_1' If $f(v_1') = 2$.
If $f(v_1^1) = 2$, $f(v_1^2) = 1$, $f(u_1) = 0$, then we extend f to g on G by assigning 2 to x, y, z, 1 to u_1, w_t's, 0 to v_1^1, v_1^2 and $g(t) = f(t)$ for otherwise. It follows that $\gamma_{ctdR}(G) \leq \gamma_{ctdR}(G') + \frac{4(n-n')}{3} \leq \frac{4n}{3}$. □

As an immediate consequence from Theorem 4.1 it follows that:

Corollary 4.1. *Let G be a simple graph of order n with $\delta(G) \geq 2$. Then*

$$\gamma_{ctdR}(G) \leq \lfloor \frac{4n}{3} \rfloor.$$

5. Conclusions

We already discuss on the covering total double Roman domination of trees, 2-edge connected graphs and graph with minimum degree 2. Therefore, there exist some problems as follows.
1. Let G be a graph of order n with minimum degree 2. Characterize the graph G for which $\gamma_{ctdR}(G) \leq \lfloor \frac{4n}{3} \rfloor$.
2. The complexity the CTDRD of graph G.
3. What is the relationship between CTDRD with other domination parameters?
4. The CTDRD of (Cartesian, lexicographic, strong) products of graphs may be studied.

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References

[1] Beeler, R. A., Haynes, T. W. and Hedetniemi, S. T., (2016), Double Roman domination, Discrete Applied Mathematics, 211, pp. 23-29.
[2] Bondy, J. A. and Murty, U. S. R., (1982), Graph Theory with Applications, The Macmillan Press Ltd. London and Basingstoke.
[3] Hao, G., Volkmann, L. and Mojdeh, D. A., (2020), Total double Roman domination in graphs, Commun. Comb. Optim., 5 no. 1, pp. 27-39.
[4] Haynes, T. W., Hedetniemi, S. T. and Henning, M. A., (2020), Topics in Domination in Graphs, https://doi.org/10.1007/978-3-030-51117-3 Springer Nature Switzerland, AG.
[5] Haynes, T. W. and Hedetniemi, S. T. and Slater, P. J., (1998), Fundamentals of Domination in graphs, New York: Marcel Dekker.
[6] Jafari Rad, N. and Rahbani, H., (2019), Some progress on the double Roman domination in graphs, Discuss. Math. Graph Theory, 39, pp. 41-53.
[7] Khodkar, A., Mojdeh, D. A., Samadi, B. and Yero I. G., Covering Italian domination in graphs, To appear in Discrete Applied Mathematic
[8] Mojdeh, D. A. and Mansouri, Zh., (2020), On the independent double Roman domination in graphs, Bulletin of the Iranian Mathematical Society, 46, pp. 905-915, DOI: 10.1007/s41980-019-00300-9.
[9] Mojdeh, D. A. and Parsian, A. and Masoumi, I., Characterization of double Roman trees, to appear in Ars Combinatoria.
[10] West, D. B., (2001), Introduction to Graph theory, Second edition, Prentice Hall.

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