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BOUNDARY VALUE PROBLEM OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS OF VARIABLE ORDER

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Abstract. In this work, we investigate the existence, uniqueness and the stability of solutions to the boundary value problem (BVP) of Caputo fractional differential equations of variable order by converting it into an equivalent standard Caputo BVP of the fractional constant order with the help of the generalized intervals and piecewise constant functions. The results obtained in this interesting study are novel and worthy based on the Krasnoselskii fixed point theorem and the Banach contraction principle. The Ulam-Hyers stability of the given variable-order Caputo fractional boundary value problem is established. A numerical examples is given at the end to support and validate the potentiality of our obtained results.

Keywords: Fractional differential equations of variable order, boundary value problem, fixed point theorem, Ulam-Hyers stability.

AMS Subject Classification: 26A33, 34K37.

1. Introduction

The main idea of fractional calculus is to constitute the natural numbers in the order of derivation operators with rational ones. Although it seems an elementary consideration, it has an exciting correspondence explaining some physical phenomena. The fractional calculus of variable fractional order is a generalization of constant order. While there are many studies on the existence of solutions to fractional constant-order problems, there are few research papers on the existence of solutions to the problems of fractional differential equations of variable order. we point out some of them to later. Therefore, investigating

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this special research topic makes all our results novel and worthwhile. In [5], Jiahui and Pengyu studied the uniqueness of solutions to initial value problem of Riemenn-Liouville fractional differential equations of variable-order. Zhang and Hu [22] established the existence of solutions and generalized Lyapunov-type inequalities of a variable-order Riemenn-Liouville boundary value problems.

Recently, Hristova et al. [6] and Refice et al. [9] turned to investigation of the boundary value problems of Hadamard fractional differential equations of variable order via Kuratowski measure of noncompactness Technique. Bouazza et al. [4] studied a variable-order multiterm boundary value problem and derived their results by two fixed point theorems. In 2021, Benkerrouche et al. [3] presented the existence results and Ulam-Hyers stability for implicit nonlinear fractional differential equations of variable order, for more studies we refer to ([1], [13], [14], [15], [17], [18]).

Some existence and Ulam-Hyers-Rassias stability properties for fractional differential equations are studied by many authors (see [6], [7], [9]).

Inspired by all previous studies, this work investigates the existence and the uniqueness of solutions to the following proposed boundary value problem (BVP) for Caputo fractional differential equations of variable order

$$\begin{cases} {}^cD_{0^+}^{\psi(t)}y(t) = f_1(t, y(t)), \;\; t \in J := [0, T] \\ y(0) = 0, \;\; y(T) = 0, \end{cases} \tag{1}$$

where $1 < \psi(t) \leq 2$, $f_1 : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD_{0^+}^{\psi(t)}$ is the Caputo fractional derivative of variable-order $\psi(t)$.

In this paper, we shall look for a solution of (1). Further, we study the stability of the obtained solution of (1) in the sense of Ulam-Hyers **(UH)**.

2. Preliminaries

This section introduces some important fundamental definitions that will be needed for obtaining our results in the next sections.
The symbol $C(J, \mathbb{R})$ represents the Banach space of continuous functions $\varpi : J \to \mathbb{R}$ with the norm

$$\|\varpi\| = Sup\{|\varpi(t)| : t \in J\},$$

For $-\infty < a_1 < a_2 < +\infty$, we consider the mappings $\psi(t) : [a_1, a_2] \to (0, +\infty)$ and $v(t) : [a_1, a_2] \to (n-1, n)$. Then, the left Riemenn-Liouville fractional integral (RLFI) of variable-order $\psi(t)$ for function $f_2(t)$ ([11], [12], [16]) is

$$I_{a_1^+}^{\psi(t)} f_2(t) = \int_{a_1}^{t} \frac{(t-s)^{\psi(t)-1}}{\Gamma(\psi(t))} f_2(s) ds, \;\; t > a_1, \tag{2}$$

and the left Caputo fractional derivative (CFD) of variable-order $v(t)$ for function $f_2(t)$ ([11], [12], [16]) is

$${}^cD_{a_1^+}^{v(t)} f_2(t) = \int_{a_1}^{t} \frac{(t-s)^{n-v(t)-1}}{\Gamma(n-v(t))} f_2^{(n)}(s) ds, \;\;\; t > a_1. \tag{3}$$

As anticipated, in case of $\psi(t)$ and $v(t)$ are constant, then (RLFI) and (CFD) are coincide with the standard Riemann-Liouville fractional integral and Caputo fractional derivative, respectively see e.g. [8, 11, 12].

Recall the following pivotal observation.

Lemma 2.1. *([8]) (page 96, 73) Let* $\alpha_1 > 0$, $a_1 > 0$, $f_2 \in L^1(a_1, a_2)$, ${}^cD^{\alpha_1}_{a_1^+} f_2 \in L^1(a_1, a_2)$. *Then,*

$$I^{\alpha_1}_{a_1^+}\, {}^cD^{\alpha_1}_{a_1^+} f_2(t) = f_2(t) + \lambda_0 + \lambda_1(t-a_1) + \lambda_2(t-a_1)^2 + ... + \lambda_{n-1}(t-a_1)^{n-1}$$

with $n-1 < \alpha_1 \le n$, $\lambda_j \in \mathbb{R}$, $j = 0, 1, ..., n-1$.
and

$${}^cD^{\alpha_1}_{a_1^+} I^{\alpha_1}_{a_1^+} f_2(t) = f_2(t).$$

Furthermore, for $\alpha_1,\ \alpha_2 > 0$, $a_1 > 0$, $f_2 \in L^1(a_1, a_2)$ *we have,*

$$I^{\alpha_1}_{a_1^+} I^{\alpha_2}_{a_1^+} f_2(t) = I^{\alpha_2}_{a_1^+} I^{\alpha_1}_{a_1^+} f_2(t) = I^{\alpha_1+\alpha_2}_{a_1^+} f_2(t).$$

Remark([19], [21], [23]) Note that the semigroup property is not fulfilled for general functions $\beta_1(t)$, $\beta_2(t)$, i.e.,

$$I^{\beta_1(t)}_{a_1^+} I^{\beta_2(t)}_{a_1^+} f_2(t) \neq I^{\beta_1(t)+\beta_2(t)}_{a_1^+} f_2(t).$$

Exemple Let

$$\beta_1(t) = t,\ \ t \in [0,4], \quad \beta_2(t) = \begin{cases} 2, & t \in [0,1] \\ 3, & t \in]1,4]. \end{cases} \qquad f_2(t) = 2,\ \ t \in [0,4]$$

$$\begin{aligned} I^{\beta_1(t)}_{0^+} I^{\beta_2(t)}_{0^+} f_2(t) &= \int_0^t \frac{(t-s)^{\beta_1(t)-1}}{\Gamma(\beta_1(t))} \int_0^s \frac{(s-\tau)^{\beta_2(s)-1}}{\Gamma(\beta_2(s))} f_2(\tau) d\tau ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} [\int_0^1 \frac{(s-\tau)}{\Gamma(2)} 2d\tau + \int_1^s \frac{(s-\tau)^2}{\Gamma(3)} 2d\tau] ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} [2s - 1 + \frac{(s-1)^3}{3}] ds, \end{aligned}$$

and

$$I^{\beta_1(t)+\beta_2(t)}_{0^+} f_2(t) = \int_0^t \frac{(t-s)^{\beta_1(t)+\beta_2(t)-1}}{\Gamma(\beta_1(t)+\beta_2(t))} f_2(s) ds.$$

So, we get

$$\begin{aligned} I^{\beta_1(t)}_{0^+} I^{\beta_2(t)}_{0^+} f_2(t)|_{t=3} &= \int_0^3 \frac{(3-s)^2}{\Gamma(3)} [2s - 1 + \frac{(s-1)^3}{3}] ds \\ &= \frac{21}{10}, \end{aligned}$$

$$
\begin{aligned}
I_{0^+}^{\beta_1(t)+\beta_2(t)} f_2(t)|_{t=3} &= \int_0^3 \frac{(3-s)^{\beta_1(t)+\beta_2(t)-1}}{\Gamma(\beta_1(t)+\beta_2(t))} f_2(s)ds \\
&= \int_0^1 \frac{(3-s)^4}{\Gamma(5)} 2ds + \int_1^3 \frac{(3-s)^5}{\Gamma(6)} 2ds \\
&= \frac{1}{12}\int_0^1 (s^4 - 12s^3 + 54s^2 - 108s + 81)ds \\
&\quad + \frac{1}{60}\int_1^3 (-s^5 + 15s^4 - 90s^3 + 270s^2 - 405s + 243)ds \\
&= \frac{665}{180}.
\end{aligned}
$$

Therefore, we obtain

$$I_{0^+}^{\beta_1(t)} I_{0^+}^{\beta_2(t)} f_2(t)|_{t=3} \neq I_{0^+}^{\beta_1(t)+\beta_2(t)} f_2(t)|_{t=3}.$$

Lemma 2.2. *([24]) Let $u : J \to (1,2]$ be a continuous function, then for $f_2 \in C_\delta(J,\mathbb{R}) = \{f_2(t) \in C(J,\mathbb{R}),\ t^\delta f_2(t) \in C(J,\mathbb{R}), 0 \le \delta \le 1\}$, the variable order fractional integral $I_{0^+}^{\psi(t)} f_2(t)$ exists for any points on J.*

Lemma 2.3. *([24]) Let $u : J \to (1,2]$ be a continuous function, then $I_{0^+}^{\psi(t)} f_2(t) \in C(J,\mathbb{R})$ for $f_2 \in C(J,\mathbb{R})$.*

Definition 2.1. *([5], [20], [22])*
Let $I \subset \mathbb{R}$, I is called a generalized interval if it is either an interval, or $\{a_1\}$ or $\emptyset$.
A finite set $\mathcal{P}$ is called a partition of I if each x in I lies in exactly one of the generalized intervals E in $\mathcal{P}$.
A function $g : I \to \mathbb{R}$ is called piecewise constant with respect to partition $\mathcal{P}$ of I if for any $E \in \mathcal{P}$, g is constant on E.

Theorem 2.1. *(Krasnoselskii Fixed Point Theorem)([8]) Let S be a closed, bounded and convex subset of a real Banach space E and let W_1 and W_2 be operators on S satisfying the following conditions:*
(i) $W_1(S) + W_2(S) \subset S$,
(ii) W_1 is continuous on S and $W_1(S)$ is a relatively compact subset of E,
(iii) W_2 is a strict contraction on S, i.e., there exists $k \in [0,1)$, such that

$$\|W_2(\overline{y}) - W_2(y)\| \le k\|\overline{y} - y\|$$

for every $\overline{y}, y \in S$.
Then there exists $y \in S$ such that $W_1(y) + W_2(y) = y$.

Definition 2.2. *([2], [10]) The equation of (1) is **(UH)** stable if there exists $c_{f_1} > 0$, such that for any $\epsilon > 0$ and for every solution $z \in C(J,\mathbb{R})$ of the following inequality*

$$|{}^cD_{0^+}^{\psi(t)} z(t) - f_1(t, z(t))| \le \epsilon,\ t \in J \tag{4}$$

there exists a solution $y \in C(J,\mathbb{R})$ of equation (1) with

$$|z(t) - y(t)| \le c_{f_1}\epsilon,\ t \in J$$

3. Existence and uniqueness of solutions

Let us introduce the following assumption:

(H1): Let $n \in \mathbb{N}$ be an integer, $\mathcal{P} = \{J_1 := [0, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], ...J_n := (T_{n-1}, T]\}$ be a partition of the interval J, and let $\psi(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

$$\psi(t) = \sum_{\jmath=1}^{n} \psi_\jmath I_\jmath(t) = \begin{cases} \psi_1, & if\ t \in J_1, \\ \psi_2, & if\ t \in J_2, \\ \cdot \\ \cdot \\ \cdot \\ \psi_n, & if\ t \in J_n, \end{cases}$$

where $1 < \psi_\jmath \leq 2$ are constants, and $I_\jmath$ is the indicator of the interval $J_\jmath := (T_{\jmath-1}, T_\jmath],\ \jmath = 1, 2, ..., n$, (with $T_0 = 0,\ T_n = T$) such that

$$I_\jmath(t) = \begin{cases} 1, & for\ t \in J_\jmath, \\ 0, & for\ elsewhere. \end{cases}$$

For each $\jmath \in \{1, 2, ..., n\}$, the symbol $E_\jmath = C(J_\jmath, \mathbb{R})$, indicated the Banach space of continuous functions $y : J_\jmath \to \mathbb{R}$ equipped with the norm

$$\|y\|_{E_\jmath} = \sup_{t \in J_\jmath} |y(t)|.$$

Then, for any $t \in J_\jmath,\ \jmath = 1, 2, ..., n$ the left caputo fractional derivative of variable order $\psi(t)$ for function $y(t) \in C(J, \mathbb{R})$, defined by (3), could be presented as a sum of left caputo fractional derivatives of constant-orders $\psi_\jmath,\ \jmath = 1, 2, ..., n$

$$^cD_{0^+}^{\psi(t)} y(t) = \int_0^{T_1} \frac{(t-s)^{1-\psi_1}}{\Gamma(2-\psi_1)} y^{(2)}(s) ds + ... + \int_{T_{\jmath-1}}^{t} \frac{(t-s)^{1-\psi_\jmath}}{\Gamma(2-\psi_\jmath)} y^{(2)}(s) ds. \tag{5}$$

Thus, according to (5), (BVP)(1) can be written for any $t \in J_\jmath,\ \jmath = 1, 2, ..., n$ in the form

$$\int_0^{T_1} \frac{(t-s)^{1-\psi_1}}{\Gamma(2-\psi_1)} y^{(2)}(s) ds + ... + \int_{T_{\jmath-1}}^{t} \frac{(t-s)^{1-\psi_\jmath}}{\Gamma(2-\psi_\jmath)} y^{(2)}(s) ds = f_1(t, y(t)),\quad t \in J_\jmath. \tag{6}$$

In what follows we shall introduce the solution to the BVP (1).

Definition 3.1. *BVP (1) has a solution, if there are functions $y_\jmath, \jmath = 1, 2, ..., n$, so that, $y_\jmath \in C([0, T_\jmath], \mathbb{R})$ fulfilling equation (6) and $y_\jmath(0) = 0 = y_\jmath(T_\jmath)$.*

Let the function $y \in C(J, \mathbb{R})$ be such that $y(t) \equiv 0$ on $t \in [0, T_{\jmath-1}]$ and it solves integral equation (6). Then (6) is reduced to

$$^cD_{T_{\jmath-1}^+}^{\psi_\jmath} y(t) = f_1(t, y(t)),\quad t \in J_\jmath.$$

We shall deal with following BVP

$$\begin{cases} ^cD_{T_{\jmath-1}^+}^{\psi_\jmath} y(t) = f_1(t, y(t)),\quad t \in J_\jmath \\ y(T_{\jmath-1}) = 0, y(T_\jmath) = 0. \end{cases} \tag{7}$$

For our purpose, the upcoming lemma will be a corner stone of the solution of BVP (7).

Lemma 3.1. *Let $\jmath \in \{1, 2, ..., n\}$ be a natural number, $f_1 \in C(J_\jmath \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0, 1)$ such that $t^\delta f_1 \in C(J_\jmath \times \mathbb{R}, \mathbb{R})$.*
Then, the function $y \in E_\jmath$ is a solution of BVP (7) if and only if x solves the integral equation

$$y(t) = -(T_\jmath - T_{\jmath-1})^{-1}(t - T_{\jmath-1}) I_{T_{\jmath-1}^+}^{\psi_\jmath} f_1(T_\jmath, y(T_\jmath)) + I_{T_{\jmath-1}^+}^{\psi_\jmath} f_1(t, y(t)). \tag{8}$$

Proof. We presume that $y \in E_{\jmath}$ is solution of BVP (7). Employing the operator $I^{\psi_{\jmath}}_{T^+_{\jmath-1}}$ to both sides of (7) and regarding Lemma 2.1, we find

$$y(t) = \lambda_1 + \lambda_2(t - T_{\jmath-1}) + \frac{1}{\Gamma(\psi_{\jmath})} \int_{T_{\jmath-1}}^{t} (t-s)^{\psi_{\jmath}-1} f_1(s, y(s))ds, \quad t \in J_{\jmath}.$$

By $y(T_{\jmath-1}) = 0$, we get $\lambda_1 = 0$.
Let $y(t)$ satisfy $y(T_{\jmath}) = 0$. So, we observe that

$$\lambda_2 = -(T_{\jmath} - T_{\jmath-1})^{-1} I^{\psi_{\jmath}}_{T^+_{\jmath-1}} f_1(T_{\jmath}, y(T_{\jmath}))$$

Then, we find

$$\begin{aligned} y(t) &= -(T_{\jmath} - T_{\jmath-1})^{-1}(t - T_{\jmath-1}) I^{\psi_{\jmath}}_{T^+_{\jmath-1}} f_1(T_{\jmath}, y(T_{\jmath})) \\ &\quad + I^{\psi_{\jmath}}_{T^+_{\jmath-1}} f_1(t, y(t)), \quad t \in J_{\jmath}. \end{aligned}$$

Conversely, let $y \in E_{\jmath}$ be solution of integral equation (8). Regarding the continuity of function $t^{\delta} f_1$ and Lemma 2.1, we deduce that y is the solution of BVP (7).

□

We will prove the existence and uniqueness of solutions for the BVP (7). This result is based on Theorem 2.1 and the Banach contraction principle.

Theorem 3.1. *Let the conditions of Lemma 3.1 be satisfied and there exist a constant* $K > 0$*, such that,*
$t^{\delta}|f_1(t, x_1) - f_1(t, x_2)| \leq K|x_1 - x_2|$*, for any* $x_1,\ x_2 \in \mathbb{R}$*,* $t \in J_{\jmath}$*. and the inequality*

$$\frac{2K(T_{\jmath} - T_{\jmath-1})^{\psi_{\jmath}-1}(T_{\jmath}^{1-\delta} - T_{\jmath-1}^{1-\delta})}{(1-\delta)\Gamma(\psi_{\jmath})} < 1. \tag{9}$$

holds. Then, BVP (7) has a unique solution in $E_{\jmath}$*.*

Proof. We construct the operators

$$W_1, W_2 : E_{\jmath} \to E_{\jmath}$$

as follow:

$$W_1 y(t) = -(T_{\jmath} - T_{\jmath-1})^{-1}(t - T_{\jmath-1}) I^{\psi_{\jmath}}_{T^+_{\jmath-1}} f_1(T_{\jmath}, y(T_{\jmath})), \quad t \in J_{\jmath}. \tag{10}$$

$$W_2 y(t) = I^{\psi_{\jmath}}_{T^+_{\jmath-1}} f_1(T_{\jmath}, y(T_{\jmath})), \quad t \in J_{\jmath}. \tag{11}$$

It follows from the properties of fractional integrals and from the continuity of function $t^{\delta} f_1$ that the operators $W_1, W_2 : E_{\jmath} \to E_{\jmath}$ defined in (10), (11) are well defined.
Let

$$R_{\jmath} \geq \frac{\frac{2f^{\star}(T_{\jmath} - T_{\jmath-1})^{\psi_{\jmath}}}{\Gamma(\psi_{\jmath})}}{1 - \frac{2K(T_{\jmath} - T_{\jmath-1})^{\psi_{\jmath}-1}(T_{\jmath}^{1-\delta} - T_{\jmath-1}^{1-\delta})}{(1-\delta)\Gamma(\psi_{\jmath})}},$$

with

$$f^{\star} = \sup_{t \in J_{\jmath}} |f_1(t, 0)|.$$

We consider the set

$$B_{R_{\jmath}} = \{y \in E_{\jmath},\ \|y\|_{E_{\jmath}} \leq R_{\jmath}\}.$$

Clearly $B_{R_\jmath}$ is nonempty, closed, convex and bounded. Now, we demonstrate that W_1, W_2 satisfies the assumption of the Theorem 2.1. We shall prove it in four phases.
STEP 1: Claim: $W_1(B_{R_\jmath}) + W_2(B_{R_\jmath}) \subseteq (B_{R_\jmath})$.
For $y \in B_{R_\jmath}$ and by (H2), we get

$$\begin{aligned}
|W_1 y(t) + W_2 y(t)| &\leq \frac{(T_\jmath - T_{\jmath-1})^{-1}(t - T_{\jmath-1})}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y(s))| ds \\
&\quad + \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t} (t - s)^{\psi_\jmath - 1} |f_1(s, y(s))| ds \\
&\leq \frac{2}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y(s))| ds \\
&\leq \frac{2}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y(s)) - f_1(s, 0)| ds \\
&\quad + \frac{2}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, 0)| ds \\
&\leq \frac{2}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} s^{-\delta} (K|y(s)|) ds + \frac{2f^\star (T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath)} \\
&\leq \frac{2K(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}(T_\jmath^{1-\delta} - T_{\jmath-1}^{1-\delta})}{(1-\delta)\Gamma(\psi_\jmath)} R_\jmath + \frac{2f^\star (T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath)} \\
&\leq R_\jmath,
\end{aligned}$$

which means that $W_1(B_{R_\jmath}) + W_2(B_{R_\jmath}) \subseteq (B_{R_\jmath})$.

STEP 2: Claim: W_1 is continuous.
We presume that the sequence (y_n) converges to y in $E_\jmath$ and $t \in J_\jmath$. Then,

$$\begin{aligned}
|(W_1 y_n)(t) - (W_1 y)(t)| &\leq \frac{(T_\jmath - T_{\jmath-1})^{-1}(t - T_{\jmath-1})}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y_n(s)) \\
&\quad - f_1(s, y(s))| ds \\
&\leq \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y_n(s)) - f_1(s, y(s))| ds \\
&\leq \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} s^{-\delta} (T_\jmath - s)^{\psi_\jmath - 1} (K|y_n(s) - y(s)|) ds \\
&\leq \frac{K}{\Gamma(\psi_\jmath)} \|y_n - y\|_{E_\jmath} \int_{T_{\jmath-1}}^{T_\jmath} s^{-\delta} (T_\jmath - s)^{\psi_\jmath - 1} ds \\
&\leq \frac{K(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}({T_\jmath}^{1-\delta} - {T_{\jmath-1}}^{1-\delta})}{(1-\delta)\Gamma(\psi_\jmath)} \|y_n - y\|_{E_\jmath}
\end{aligned}$$

i.e., we obtain

$$\|(W_1 y_n) - (W_1 y)\|_{E_\jmath} \to 0 \ \ as \ \ n \to \infty.$$

Ergo, the operator W_1 is a continuous on $E_\jmath$.
Step 3: Claim: W_1 is compact
Now, we will show that $W_1(B_{R_\jmath})$ is relatively compact, meaning that W_1 is compact. Clearly $W_1(B_{R_\jmath})$ is uniformly bounded because by Step 1, we have $W_1(B_{R_\jmath}) = \{W_1(y) : y \in B_{R_\jmath}\} \subset W_1(B_{R_\jmath}) + W_2(B_{R_\jmath}) \subseteq (B_{R_\jmath})$ thus for each $y \in B_{R_\jmath}$ we have $\|W_1(y)\|_{E_\jmath} \leq R_\jmath$ which means that $W_1(B_{R_\jmath})$is bounded. It remains to indicate that $W_1(B_{R_\jmath})$ is equicontinuous.

For $t_1, t_2 \in J_\jmath,\ t_1 < t_2$ and $y \in B_{R_\jmath}$, we have

$$\begin{aligned}
&\Big|(W_1y)(t_2) - (W_1y)(t_1)\Big| \\
= &\Big| - \frac{(T_\jmath - T_{\jmath-1})^{-1}(t_2 - T_{\jmath-1})}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} f_1(s, y(s))ds \\
&+ \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t_2} (t_2 - s)^{\psi_\jmath - 1} f_1(s, y(s))ds + \frac{(T_\jmath - T_{\jmath-1})^{-1}(t_1 - T_{\jmath-1})}{\Gamma(\psi_\jmath)} \\
&\int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} f_1(s, y(s))ds - \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t_1} (t_1 - s)^{\psi_\jmath - 1} f_1(s, y(s))ds\Big| \\
\leq &\frac{(T_\jmath - T_{\jmath-1})^{-1}}{\Gamma(\psi_\jmath)} \Big((t_2 - T_{\jmath-1}) - (t_1 - T_{\jmath-1})\Big) \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y(s))|ds \\
&+ \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t_1} \Big((t_2 - s)^{\psi_\jmath - 1} - (t_1 - s)^{\psi_\jmath - 1}\Big)|f_1(s, y(s))|ds \\
&+ \frac{1}{\Gamma(\psi_\jmath)} \int_{t_1}^{t_2} (t_2 - s)^{\psi_\jmath - 1} |f_1(s, y(s))|ds \\
\leq &\frac{(T_\jmath - T_{\jmath-1})^{-1}}{\Gamma(\psi_\jmath)} \Big((t_2 - T_{\jmath-1}) - (t_1 - T_{\jmath-1})\Big) \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y(s)) - f_1(s, 0)|ds \\
&+ \frac{(T_\jmath - T_{\jmath-1})^{-1}}{\Gamma(\psi_\jmath)} \Big((t_2 - T_{\jmath-1}) - (t_1 - T_{\jmath-1})\Big) \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, 0)|ds \\
&+ \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t_1} \Big((t_2 - s)^{\psi_\jmath - 1} - (t_1 - s)^{\psi_\jmath - 1}\Big)|f_1(s, y(s)) - f_1(s, 0)|ds \\
&+ \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t_1} \Big((t_2 - s)^{\psi_\jmath - 1} - (t_1 - s)^{\psi_\jmath - 1}\Big)|f_1(s, 0)|ds \\
&+ \frac{1}{\Gamma(\psi_\jmath)} \int_{t_1}^{t_2} (t_2 - s)^{\psi_\jmath - 1} |f_1(s, y(s)) - f_1(s, 0)|ds + \frac{1}{\Gamma(\psi_\jmath)} \int_{t_1}^{t_2} (t_2 - s)^{\psi_\jmath - 1} |f_1(s, 0)|ds \\
\leq &\frac{(T_\jmath - T_{\jmath-1})^{-1}}{\Gamma(\psi_\jmath)} \Big((t_2 - T_{\jmath-1}) - (t_1 - T_{\jmath-1})\Big) \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} s^{-\delta}(K|y(s)|)ds \\
&+ \frac{f^\star (T_\jmath - T_{\jmath-1})^{-1}}{\Gamma(\psi_\jmath)} \Big((t_2 - T_{\jmath-1}) - (t_1 - T_{\jmath-1})\Big) \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} ds \\
&+ \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t_1} s^{-\delta}\Big((t_2 - s)^{\psi_\jmath - 1} - (t_1 - s)^{\psi_\jmath - 1}\Big)(K|y(s)|)ds \\
&+ \frac{f^\star}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t_1} \Big((t_2 - s)^{\psi_\jmath - 1} - (t_1 - s)^{\psi_\jmath - 1}\Big)ds \\
&+ \frac{1}{\Gamma(\psi_\jmath)} \int_{t_1}^{t_2} s^{-\delta}(t_2 - s)^{\psi_\jmath - 1}(K|y(s)|)ds + \frac{f^\star}{\Gamma(\psi_\jmath)} \int_{t_1}^{t_2} (t_2 - s)^{\psi_\jmath - 1} ds \\
\leq &\frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 2}}{\Gamma(\psi_\jmath)} \Big((t_2 - T_{\jmath-1}) - (t_1 - T_{\jmath-1})\Big)(K\|y\|_{E_\jmath}) \int_{T_{\jmath-1}}^{T_\jmath} s^{-\delta} ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{f^{\star}(T_{\jmath}-T_{\jmath-1})^{\psi_{\jmath}-1}}{\Gamma(\psi_{\jmath}+1)}\Big((t_2-T_{\jmath-1})-(t_1-T_{\jmath-1})\Big)\\
&+\frac{1}{\Gamma(\psi_{\jmath})}(K\|y\|_{E_{\jmath}})\int_{T_{\jmath-1}}^{t_1}s^{-\delta}((t_2-t_1)^{\psi_{\jmath}-1})ds\\
&+\frac{f^{\star}}{\Gamma(\psi_{\jmath})}\Big(\frac{(t_2-T_{\jmath-1})^{\psi_{\jmath}}}{\psi_{\jmath}}-\frac{(t_2-t_1)^{\psi_{\jmath}}}{\psi_{\jmath}}-\frac{(t_1-T_{\jmath-1})^{\psi_{\jmath}}}{\psi_{\jmath}}\Big)\\
&+\frac{(t_2-t_1)^{\psi_{\jmath}-1}}{\Gamma(\psi_{\jmath})}(K\|y\|_{E_{\jmath}})\int_{t_1}^{t_2}s^{-\delta}ds+\frac{f^{\star}}{\Gamma(\psi_{\jmath})}\frac{(t_2-t_1)^{\psi_{\jmath}}}{\psi_{\jmath}}\\
\leq\;&\frac{K(T_{\jmath}-T_{\jmath-1})^{\psi_{\jmath}-2}(T_{\jmath}{}^{1-\delta}-T_{\jmath-1}{}^{1-\delta})}{(1-\delta)\Gamma(\psi_{\jmath})}\|y\|_{E_{\jmath}}\Big((t_2-T_{\jmath-1})-(t_1-T_{\jmath-1})\Big)\\
&+\frac{f^{\star}(T_{\jmath}-T_{\jmath-1})^{\psi_{\jmath}-1}}{\Gamma(\psi_{\jmath}+1)}\Big((t_2-T_{\jmath-1})-(t_1-T_{\jmath-1})\Big)\\
&+\Big(\frac{K(t_1{}^{1-\delta}-T_{\jmath-1}{}^{1-\delta})(t_2-t_1)^{\psi_{\jmath}-1}}{(1-\delta)\Gamma(\psi_{\jmath})}\Big)\|y\|_{E_{\jmath}}\\
&+\frac{f^{\star}}{\Gamma(\psi_{\jmath}+1)}\Big((t_2-T_{\jmath-1})^{\psi_{\jmath}}-(t_2-t_1)^{\psi_{\jmath}}-(t_1-T_{\jmath-1})^{\psi_{\jmath}}\Big)\\
&+\frac{K(t_2{}^{1-\delta}-t_1{}^{1-\delta})(t_2-t_1)^{\psi_{\jmath}-1}}{(1-\delta)\Gamma(\psi_{\jmath})}\|y\|_{E_{\jmath}}+\frac{f^{\star}(t_2-t_1)^{\psi_{\jmath}}}{\Gamma(\psi_{\jmath}+1)}\\
\leq\;&\Big(\frac{K(T_{\jmath}-T_{\jmath-1})^{\psi_{\jmath}-2}(T_{\jmath}{}^{1-\delta}-T_{\jmath-1}{}^{1-\delta})}{(1-\delta)\Gamma(\psi_{\jmath})}\|y\|_{E_{\jmath}}+\frac{f^{\star}(T_{\jmath}-T_{\jmath-1})^{\psi_{\jmath}-1}}{\Gamma(\psi_{\jmath}+1)}\Big)\\
&\Big((t_2-T_{\jmath-1})-(t_1-T_{\jmath-1})\Big)\\
&+\Big(\frac{K(t_2{}^{1-\delta}-T_{\jmath-1}{}^{1-\delta})}{(1-\delta)\Gamma(\psi_{\jmath})}\|y\|_{E_{\jmath}}\Big)(t_2-t_1)^{\psi_{\jmath}-1}\\
&+\frac{f^{\star}}{\Gamma(\psi_{\jmath}+1)}\Big((t_2-T_{\jmath-1})^{\psi_{\jmath}}-(t_1-T_{\jmath-1})^{\psi_{\jmath}}\Big)
\end{aligned}
$$

Hence $\|(W_1y)(t_2)-(W_1y)(t_1)\|_{E_{\jmath}}\to 0$ as $|t_2-t_1|\to 0$. It implies that $W_1(B_{R_{\jmath}})$ is equicontinuous.

STEP 4: Claim: W_2 is a strict contraction on $B_{R_{\jmath}}$.

For $\overline{y}(t), y(t)\in E_{\jmath}$, we obtain that

$$
\begin{aligned}
|(W_2y)(t)-(W_2\overline{y})(t)| &\leq \frac{1}{\Gamma(\psi_{\jmath})}\int_{T_{\jmath-1}}^{t}(t-s)^{\psi_{\jmath}-1}|f_1(s,y(s))-f_1(s,\overline{y}(s))|ds\\
&\leq \frac{1}{\Gamma(\psi_{\jmath})}\int_{T_{\jmath-1}}^{T_{\jmath}}(T_{\jmath}-s)^{\psi_{\jmath}-1}|f_1(s,y(s))-f_1(s,\overline{y}(s))|ds\\
&\leq \frac{1}{\Gamma(\psi_{\jmath})}\int_{T_{\jmath-1}}^{T_{\jmath}}(T_{\jmath}-s)^{\psi_{\jmath}-1}s^{-\delta}(K|y(s)-\overline{y}(s)|)ds\\
&\leq \frac{K}{\Gamma(\psi_{\jmath})}\|y-\overline{y}\|_{E_{\jmath}}\int_{T_{\jmath-1}}^{T_{\jmath}}s^{-\delta}(T_{\jmath}-s)^{\psi_{\jmath}-1}ds
\end{aligned}
$$

$$\leq \quad \frac{K({T_\jmath}^{1-\delta} - {T_{\jmath-1}}^{1-\delta})(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}}{(1-\delta)\Gamma(\psi_\jmath)} \|y - \overline{y}\|_{E_\jmath}$$

Consequently by (9), the operator W_2 is strict contraction.
Therefore, all conditions of Theorem 2.1 are fulfilled and thus, there exists $\widetilde{y_\jmath} \in B_{R_\jmath}$, such that $W_1\widetilde{y_\jmath} + W_2\widetilde{y_\jmath} = \widetilde{y_\jmath}$, which is a solution of the BVP (7). Since $B_{R_\jmath} \subset E_\jmath$, then $\widetilde{y_\jmath} \in E_\jmath$.

Now, we shall show the uniqueness of solutions for the BVP (7) based on the Banach contraction principle. Consider the operator

$$W : E_\jmath \to E_\jmath$$

defined by :

$$(Wy)(t) = (W_1y)(t) + (W_2y)(t), \quad for \;\; y(t) \in E_\jmath.$$

STEP 5: Claim: W is a contraction.
For $\overline{y}(t), y(t) \in E_\jmath$, we obtain that

$$\begin{aligned}
|(Wy)(t) - (W\overline{y})(t)| &\leq \tfrac{(T_\jmath - T_{\jmath-1})^{-1}(t-T_{\jmath-1})}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y(s)) - f_1(s, \overline{y}(s))| ds \\
&\quad + \tfrac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t} (t-s)^{\psi_\jmath - 1} |f_1(s, y(s)) - f_1(s, \overline{y}(s))| ds \\
&\leq \tfrac{2}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, y(s)) - f_1(s, \overline{y}(s))| ds \\
&\leq \tfrac{2}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} s^{-\delta} (K|y(s) - \overline{y}(s)|) ds \\
&\leq \tfrac{2K(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}(T_\jmath^{1-\delta} - T_{\jmath-1}^{1-\delta})}{(1-\delta)\Gamma(\psi_\jmath)} \|y - \overline{y}\|_{E_\jmath}
\end{aligned}$$

Ergo, by (9), we deduce that the operator W forms a contraction. Hence, by the Banach's contraction principal, W has a unique fixed point $\widetilde{y_\jmath} \in E_\jmath$, which is a unique solution of the BVP (7). the claim of Theorem 3.1 is proved. □

Now, we will prove the existence result for BVP (1). Introduce the following assumption:

(H2): Let $f_1 \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0, 1)$ such that $t^\delta f_1 \in C(J \times \mathbb{R}, \mathbb{R})$ and there exist a constant $K > 0$, such that,
$t^\delta |f_1(t, x_1) - f_1(t, x_2)| \leq K|x_1 - x_2|$, for any $x_1, \; x_2 \in \mathbb{R}$ and $t \in J$.

Theorem 3.2. *Let the conditions (H1), (H2) and inequality (9) be satisfied for all $\jmath \in \{1, 2, ..., n\}$.*
Then, the problem (1) possesses a unique solution in $C(J, \mathbb{R})$.

Proof. For any $\jmath \in \{1, 2, ..., n\}$ according to Theorem 3.1 the BVP (7) possesses a unique solution $\widetilde{y_\jmath} \in E_\jmath$.
For any $\jmath \in \{1, 2, ..., n\}$ we define the function

$$y_\jmath = \begin{cases} 0, & t \in [0, T_{\jmath-1}], \\ \widetilde{y_\jmath}, & t \in J_\jmath, \end{cases}$$

Thus, the function $y_\jmath \in C([0, T_\jmath], \mathbb{R})$ solves the integral equation (6) for $t \in J_\jmath$ with $y_\jmath(0) = 0, \; y_\jmath(T_\jmath) = \widetilde{y_\jmath}(T_\jmath) = 0$.

Then, the function

$$y(t)=\begin{cases} y_1(t), \quad t\in J_1,\\ y_2(t)=\begin{cases} 0, \quad t\in J_1,\\ \widetilde{y}_2, \quad t\in J_2\end{cases} .\\ \cdot\\ \cdot\\ \cdot\\ y_n(t)=\begin{cases} 0, \quad t\in[0,T_{\jmath-1}],\\ \widetilde{y}_\jmath, \quad t\in J_\jmath\end{cases}\end{cases} \tag{12}$$

is a unique solution of the BVP (1) in $C(J,\mathbb{R})$. □

4. Ulam-Hyers stability

Theorem 4.1. *Let the conditions (H1), (H2) and inequality (9) be satisfied. Then, BVP (1) is **(UH)** stable.*

Proof. Let $\epsilon>0$ an arbitrary number and the function $z(t)$ from $z\in C(J_\jmath,\mathbb{R})$ satisfy inequality (4). For any $\jmath\in\{1,2,...,n\}$ we define the functions $z_1(t)\equiv z(t), t\in[1,T_1]$ and for $\jmath=2,3,...,n:$

$$z_\jmath(t)=\begin{cases} 0, \quad t\in[0,T_{\jmath-1}],\\ z(t), \quad t\in J_\jmath,\end{cases}$$

For any $\jmath\in\{1,2,...,n\}$ according to equality (5) for $t\in J$ we get

$$^{c}D^{\psi(t)}_{T_{\jmath-1}^+}z_\jmath(t)=\int_{T_{\jmath-1}}^{t}\frac{(t-s)^{1-\psi_\jmath}}{\Gamma(2-\psi_\jmath)}z^{(2)}(s)ds.$$

Taking the (RLFI) $I^{\psi_\jmath}_{T^+_{\jmath-1}}$ of both sides of the inequality (4), we obtain

$$\begin{aligned}\Big|z_\jmath(t) \quad &+ \quad \frac{(T_\jmath-T_{\jmath-1})^{-1}(t-T_{\jmath-1})}{\Gamma(\psi_\jmath)}\int_{T_{\jmath-1}}^{T_\jmath}(T_\jmath-s)^{\psi_{\jmath-1}}f_1(s,z_\jmath(s))ds\\ &-\frac{1}{\Gamma(\psi_\jmath)}\int_{T_{\jmath-1}}^{t}(t-s)^{\psi_{\jmath-1}}f_1(s,z_\jmath(s))ds\Big|\\ &\le \quad \epsilon\int_{T_{\jmath-1}}^{t}\frac{(t-s)^{\psi_\jmath-1}}{\Gamma(\psi_\jmath)}ds\\ &\le \quad \epsilon\frac{(T_\jmath-T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath+1)}\end{aligned}$$

According to Theorem 3.2, BVP (1) has a unique solution $y\in C(J,\mathbb{R})$ defined by $y(t)=y_\jmath(t)$ for $t\in J_\jmath,\ \jmath=1,2,...,n$, where

$$y_\jmath=\begin{cases} 0, \quad t\in[0,T_{\jmath-1}],\\ \widetilde{y}_\jmath, \quad t\in J_\jmath,\end{cases} \tag{13}$$

and $\widetilde{y}_\jmath\in E_\jmath$ is a solution of BVP (7). According to Lemma 3.1 the integral equation

$$\begin{aligned}\widetilde{y}_\jmath(t)=-&\frac{(T_\jmath-T_{\jmath-1})^{-1}(t-T_{\jmath-1})}{\Gamma(\psi_\jmath)}\int_{T_{\jmath-1}}^{T_\jmath}(T_\jmath-s)^{\psi_{\jmath-1}}f_1(s,\widetilde{y}_\jmath(s))ds\\ &+\frac{1}{\Gamma(\psi_\jmath)}\int_{T_{\jmath-1}}^{t}(t-s)^{\psi_{\jmath-1}}f_1(s,\widetilde{y}_\jmath(s))ds \end{aligned}\tag{14}$$

holds.

Let $t \in J_\jmath,\ \jmath = 1, 2, ..., n$. Then by Eq (13) and (14) we get

$$\begin{aligned}
& |z(t) - y(t)| = |z(t) - y_\jmath(t)| = |z_\jmath(t) - \widetilde{y}_\jmath(t)| \\
= & \Big| z_\jmath(t) + \frac{(T_\jmath - T_{\jmath-1})^{-1}(t - T_{\jmath-1})}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} f_1(s, \widetilde{y}_\jmath(s)) ds \\
& - \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t} (t - s)^{\psi_\jmath - 1} f_1(s, \widetilde{y}_\jmath(s)) ds \Big| \\
= & \Big| z_\jmath(t) + \frac{(T_\jmath - T_{\jmath-1})^{-1}(t - T_{\jmath-1})}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} f_1(s, z_\jmath(s)) ds \\
& - \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t} (t - s)^{\psi_\jmath - 1} f_1(s, z_\jmath(s)) ds \Big| \\
& + \frac{(T_\jmath - T_{\jmath-1})^{-1}(t - T_{\jmath-1})}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} |f_1(s, z_\jmath(s)) - f_1(s, \widetilde{y}_\jmath(s))| ds \\
& + \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t} (t - s)^{\psi_\jmath - 1} |f_1(s, z_\jmath(s)) - f_1(s, \widetilde{y}_\jmath(s))| ds \\
\leq & \ \epsilon \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath + 1)} + \frac{(T_\jmath - T_{\jmath-1})^{-1}(t - T_{\jmath-1})}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{T_\jmath} (T_\jmath - s)^{\psi_\jmath - 1} s^{-\delta} (K|z_\jmath(s) - \widetilde{y}_\jmath(s)|) ds \\
& + \frac{1}{\Gamma(\psi_\jmath)} \int_{T_{\jmath-1}}^{t} (t - s)^{\psi_\jmath - 1} s^{-\delta} (K|z_\jmath(s) - \widetilde{y}_\jmath(s)|) ds \\
\leq & \ \epsilon \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath + 1)} + \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}}{\Gamma(\psi_\jmath)} (K\|z_\jmath - \widetilde{y}_\jmath\|_{E_\jmath}) \int_{T_{\jmath-1}}^{T_\jmath} s^{-\delta} ds \\
& + \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}}{\Gamma(\psi_\jmath)} (K\|z_\jmath - \widetilde{y}_\jmath\|_{E_\jmath}) \int_{T_{\jmath-1}}^{t} s^{-\delta} ds \\
\leq & \ \epsilon \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath + 1)} + \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}({T_\jmath}^{1-\delta} - {T_{\jmath-1}}^{1-\delta})}{(1-\delta)\Gamma(\psi_\jmath)} (K\|z_\jmath - \widetilde{y}_\jmath\|_{E_\jmath}) \\
& + \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}(t^{1-\delta} - {T_{\jmath-1}}^{1-\delta})}{(1-\delta)\Gamma(\psi_\jmath)} (K\|z_\jmath - \widetilde{y}_\jmath\|_{E_\jmath}) \\
\leq & \ \epsilon \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath + 1)} + \frac{2K(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}({T_\jmath}^{1-\delta} - {T_{\jmath-1}}^{1-\delta})}{(1-\delta)\Gamma(\psi_\jmath)} \|z_\jmath - \widetilde{y}_\jmath\|_{E_\jmath} \\
\leq & \ \epsilon \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath + 1)} + \mu \|z - y\|,
\end{aligned}$$

where

$$\mu = \max_{\jmath=1,2,\ldots,n} \frac{2K(T_\jmath - T_{\jmath-1})^{\psi_\jmath - 1}({T_\jmath}^{1-\delta} - {T_{\jmath-1}}^{1-\delta})}{(1-\delta)\Gamma(\psi_\jmath)}.$$

Then,

$$\|z - y\|(1 - \mu) \leq \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{\Gamma(\psi_\jmath + 1)} \epsilon.$$

We obtain, for each $t \in J_\jmath$

$$|z(t) - y(t)| \leq \|z - y\| \leq \frac{(T_\jmath - T_{\jmath-1})^{\psi_\jmath}}{(1-\mu)\Gamma(\psi_\jmath + 1)}\epsilon := c_{f_1}\epsilon.$$

Therefore, the BVP (1) is **(UH)** stable. □

5. Example

Let us consider the following fractional boundary value problem,

$$\begin{cases} {}^cD_{0^+}^{\psi(t)} y(t) = \frac{t^{-\frac{1}{3}} e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |y(t)|)}, & t \in J := [0, 2], \\ y(0) = 0, \ \ y(2) = 0. \end{cases} \tag{15}$$

Let

$$f_1(t, x) = \frac{t^{-\frac{1}{3}} e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + x)}, \ (t, x) \in [0, 2] \times [0, +\infty).$$

$$\psi(t) = \begin{cases} \frac{3}{2}, & t \in J_1 := [0, 1], \\ \frac{9}{5}, & t \in J_2 :=]1, 2]. \end{cases} \tag{16}$$

Then, we have

$$\begin{aligned} t^{\frac{1}{3}}|f_1(t, x_1) - f_1(t, x_2)| &= \left| \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)} \left(\frac{1}{1 + x_1} - \frac{1}{1 + x_2} \right) \right| \\ &\leq \frac{e^{-t}|x_1 - x_2|}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + x_1)(1 + x_2)} \\ &\leq \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)}|x_1 - x_2| \\ &\leq \frac{1}{(e + 5)}|x_1 - x_2|. \end{aligned}$$

Hence the condition (H2) holds with $\delta = \frac{1}{3}$ and $K = \frac{1}{e+5}$.
By (16), according to BVP (7) we consider two auxiliary BVP for Caputo fractional differential equations of constant order

$$\begin{cases} {}^cD_{0^+}^{\frac{3}{2}} y(t) = \frac{t^{-\frac{1}{3}} e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |y(t)|)}, & t \in J_1, \\ y(0) = 0, \ \ y(1) = 0. \end{cases} \tag{17}$$

and

$$\begin{cases} {}^cD_{1^+}^{\frac{9}{5}} y(t) = \frac{t^{-\frac{1}{3}} e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |y(t)|)}, & t \in J_2, \\ y(1) = 0, \ \ y(2) = 0. \end{cases} \tag{18}$$

Next, we prove that the condition (9) is fulfilled for $\jmath = 1$. Indeed,

$$\frac{2K({T_1}^{1-\delta} - {T_0}^{1-\delta})(T_1 - T_0)^{\psi_1 - 1}}{(1 - \delta)\Gamma(\psi_1)} = \frac{2}{\frac{2}{3}(e + 5)\Gamma(\frac{3}{2})} \simeq 0.4435 < 1$$

Accordingly the condition (9) is achieved. By Theorem 3.1, BVP (17) has a unique solution $\widetilde{y}_1 \in E_1$.
We prove that the condition (9) is fulfilled for $\jmath = 2$. Indeed,

$$\frac{2K({T_2}^{1-\delta} - {T_1}^{1-\delta})(T_2 - T_1)^{\psi_2 - 1}}{(1-\delta)\Gamma(\psi_2)} = \frac{2}{e+5}\frac{2^{\frac{2}{3}} - 1}{\frac{2}{3}\Gamma(\frac{9}{5})} \simeq 0.2447 < 1$$

Thus, the condition (9) is satisfied.
According to Theorem 3.1, the BVP (18) possesses a unique solution $\widetilde{y}_2 \in E_2$.
Then, by Theorem 3.2, the BVP (15) has a unique solution

$$y(t) = \begin{cases} \widetilde{y}_1(t), & t \in J_1, \\ y_2(t), & t \in J_2. \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{y}_2(t), & t \in J_2. \end{cases}$$

According to Theorem 4.1, BVP (15) is **(UH)** stable.

6. Conclusions

In this paper, we proposed the boundary value problem for fractional differential equations of variable order involving Caputo derivative of variable order, which is a piecewise constant function. Based the essential difference about the variable order fractional calculus (derivative and integral) and the integer order and the constant fractional order calculus (derivative and integral), we carry on essential analysis to (BVP) (1). According to our analysis, we give the definition of solution to the (BVP) (1), using the Krasnoselskii fixed point theorem and the Banach contraction principle, we examined the existence and uniqueness of the solutions to our problem (Theorem 3.2) and we studied the (**UH**) stability of the solutions (Theorem 4.1), Finally, as applications, an example is presented to illustrate our result

References

[1] Aguilar, G. J. F., (2018), Analytical and numerical solutions of nonlinear alcoholism model via variable-order fractional differential equations, Phys. A., 494, 52-57.

[2] Benchohra, M., Lazreg, J. E., (2017), Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative, Stud. Univ. Babes-Bolyai Math., 62(1), 27-38.

[3] Benkerrouche, A., Souid, M. S., Sitthithakerngkiet, K., Hakem, A., (2021), Implicit nonlinear fractional differential equations of variable order, Boundary Value Problems, 2021:64, 1-16.

[4] Bouazza, Z., Etemad, S., Souid, M.S., Rezapour, Sh., Martinez, F., Kaabar, M.K.A., (2021), A study on the solutions of a multiterm FBVP of variable order, J. Funct. Spaces., 9939147, 1-9.

[5] Jiahui, A., Pengyu, C., (2019), Uniqueness of solutions to initial value problem of fractional differential equations of variable-order, Dynamic Systems and Applications, 28(3), 607-623.

[6] Hristova, S., Benkerrouche, A., Souid, M.S., Hakem, A., (2021), Boundary value problems of Hadamard fractional differential equations of variable order, Symmetry, 13(5), 896, 1-16.

[7] Kaabar, M. K. A., Refice, A., Souid, M. S., Martinez, F., Etemad, S., Siri, Z., Rezapour S., (2021), Existence and U-H-R stability of solutions to the implicit nonlinear FBVP in the variable order settings Mathematics, 9, 1693, 1-17.

[8] Kilbas, A.A., Srivastava, H. M., Trujillo, J. J., (2006), Theory and applications of fractional differenatial equations, North-Holland Mathematics Studies, 204, Elsevier Science B V, Amsterdam.

[9] Refice, A., Souid, M. S., Stamova, I., (2021), On the boundary value problems of Hadamard fractional differential equations of variable order via Kuratowski MNC technique, Mathematics, 9, 1134, 1-16.

[10] Rus, I. A., (2010), Ulam stabilities of ordinary differential equations in a Banach space, Carpathian J. Math., 26, 103-107.

[11] Samko,S. G., (1995), Fractional integration and differentiation of variable order, Anal. Math., 21, 213-236.

[12] Samko,S. G., Boss, B., (1993), Integration and differentiation to a variable fractional order, Integr.Transforms Spec. Funct., 1, 277-300.

[13] Sun, H., Chen, W., Wei, H.,Chen, Y., (2011), A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems, Eur. Phys. J. Special Topics., 193, 185-192.

[14] Sun, H., Chen, W., Chen, Y.Q., (2009), Variable-order fractional differential operators in anomalous diffusion modeling, Phys. A., 388, 4586-4592.

[15] Tavares,D., Almeida, R., Torres, D. F. M., (2016), Caputo derivatives of fractional variable order Numerical approximations, Commun. Nonlinear Sci. Numer. Simul., 35 , 69-87.

[16] Valerio, D., Costa, J.S., (2011), Variable-order fractional derivatives and their numerical approximations, Signal Process, 91, 470-483.

[17] Vanterler da C. S. J., Capelas de Oliverira, E., (2018), Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation, Comput. Appl. Math., 37, 5375-5394.

[18] Yang, J., Yao, H., Wu, B., (2018), An efficient numberical method for variable order fractional functional differential equation, Appl. Math. Lett., 76, 221-226.

[19] Zhang, S., (2013), Existence of solutions for two point boundary value problems with singular differential equations of variable order, Electronic Journal of Differential Equations, 245, 1-16.

[20] Zhang, S., (2018), The uniqueness result of solutions to initial value problems of differential equations of variable-order, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math., 112, 407-423.

[21] Zhang, S., Hu, L., (2019), Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis, Mathematics, 7(286), 1-23.

[22] Zhang, S., Hu, L., (2020), The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order, Aims Mathematics, 5(4), 2923-2943.

[23] Zhang, S., Hu, L., (2019), The existeness and uniqueness result of solutions to initial value problems of nonlinear difiusion equations involving with the conformable variable, Azerbaijan Journal of Mathematics, 9(1), 22-45.

[24] Zhang, S., Sun, S., Hu, L., (2018), Approximate solutions to initial value problem for differential equation of variable order, Journal of Fractional Calculus and Applications, 9(2), 93-112.

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