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FUZZY α-ψ^*-HOMOTOPY AND FUZZY α-ψ^*-COVERING SPACES

M. ROWTHRI[1*], B. AMUDHAMBIGAI[1], §

Abstract. In this paper, the concepts of fuzzy α-ψ^*-homotopies and fuzzy α-ψ^*-path homotopies are introduced. The intend of this article is to study the concepts of α-ψ^*-fundamental group in a fuzzy topological space and fuzzy α-ψ^*-covering spaces. Many properties concerning these concepts are provided.

Keywords: Fuzzy α-ψ^*-homotopies, Fuzzy α-ψ^*-paths, Fuzzy α-ψ^*-loops, Fuzzy α-ψ^*-path homotopy, and Fuzzy α-ψ^*-covering spaces.

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1. Introduction

Salleh and Tap [8] defined fuzzy topology on the unit interval. The homotopy theory in topology and its fundamental group were introduced and developed by Massey [5]. The concept of fuzzy homotopy theory in fuzzy topological spaces was introduced by Culvacioglu and Citil [2]. Salleh and Tap [8] introduced the concept of the fundamental group in fuzzy topological spaces based on the definition of fuzzy topology introduced by Chang [1]. Homotopy has many applications in engineering, image segmentation in medical field, Medical data structure and advanced sciences etc. Motivated by the application of *alpha*-open sets in medical field the concepts of fuzzy α-ψ^*-homotopy, fuzzy α-ψ^*-path homotopy, α-ψ^*-fundamental group in a fuzzy topological spaces are introduced and their properties are investigated. Defined that the set of all fuzzy α-ψ^*-path homotopy equivalence classes on the collection of fuzzy α-ψ^*-loops forms a α-ψ^*-fundamental group and it is shown that there exists a isomorphism between two α-ψ^*-fundamental groups. At last, the notion of fuzzy α-ψ^*-covering spaces is introduced and some of its properties are established via α-ψ^*-fundamental group.

Throughout this paper, $F\alpha O(X,\tau)$, $F\alpha C(X,\tau)$ and $\mathcal{FP}(X)$ denote the set of all fuzzy α-open sets in (X,τ), the set of all fuzzy α-closed sets in (X,τ) and the set of all fuzzy points x_t where $0 < t \leq 1$ over X respectively.

[1] PG and Research Department of Mathematics, Sri Sarada college for Women(Autonomous), Salem-16, Tamilnadu, India.
e-mail: rowth3.m@gmail.com; ORCID: https://orcid.org/0000-0003-2798-9667.
[*] Corresponding author.
e-mail: rbamudha@yahoo.co.in; ORCID: https://orcid.org/0000-0003-3836-7660.
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2. Preliminaries

In this section, some basic concepts necessary for this paper are recalled.

Definition 2.1. [9] *A function D from X to the unit interval $[0,1]$ is called a fuzzy set on X. The set $\{x \in X | D(x) > 0\}$ is called the support of D and is denoted by D_0.*

Definition 2.2. [9] *Let (X, T) be a (usual) topological space. The collection*

$$\widetilde{T} = \{G \mid G \text{ is a fuzzy set on } X \text{ and } G_0 \in T\}$$

is a fuzzy topology on X, called the fuzzy topology on X introduced by T. $(X, \widetilde{T})$ is called the fuzzy topological space introduced by (X, T).

Definition 2.3. [3] *Let $f, g : (X, \tau) \to (Y, \sigma)$ be two fuzzy continuous mappings. If there exists a fuzzy continuous mapping*

$$F : (X, \tau) \times (J, \tilde{\varepsilon}_J) \to (Y, \sigma)$$

such that $F(x_\lambda, 0) = f(x_\lambda)$ and $F(x_\lambda, 1) = g(x_\lambda)$ for every fuzzy point x_λ in (X, τ), then we say that f is fuzzy homotopic to g.

The mapping F is called a fuzzy homotopy between f and g, and write $f \simeq g$.

Definition 2.4. [4] *Let $(X, \mathscr{T})$, $(Y, \mathscr{V})$ be two fts's. The mapping $f : (X, \mathscr{T}) \to (Y, \mathscr{V})$ is fuzzy continuous at a point $x \in X$ iff for each open fuzzy set V in $\mathscr{V}$ containing the fuzzy point $y_\delta = (f(x))_\delta$, $0 < \delta \leq 1$, the inverse image $f^{-1}[V]$ is an open fuzzy set in $\mathscr{T}$ containing x_λ, $0 < \lambda \leq \delta$.*

Definition 2.5. [6] *Two paths f and f', mapping the interval $I = [0,1]$ into X, are said to be path homotopic if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that*

$$F(s, 0) = f(s) \text{ and } F(s, 1) = f'(s),$$
$$F(0, t) = x_0 \text{ and } F(1, t) = x_1,$$

for each $s \in I$ and each $t \in I$. We call F a path homotopy between f and f'.

Definition 2.6. [3] *Let $1_X : (X, \tau) \to (X, \tau)$ be an identity mapping. If 1_X is fuzzy homotopic to a constant, then (X, τ) is called a fuzzy contractible space.*

Definition 2.7. [7] *Let (X, τ) be a fuzzy topological space. A function*

$$\psi^* : F\alpha\text{O}(X, \tau) \to I^X$$

is called a fuzzy operator on $F\alpha\text{O}(X, \tau)$, if for each $\mu \in F\alpha\text{O}(X, \tau)$ with $\mu \neq 0_X$, $Fint(\mu) \leq \psi^*(\mu)$ and $\psi^*(0_X) = 0_X$.

Remark 2.1. [7] *It is easy to check that some examples of fuzzy operators on $F\alpha\text{O}(X, \tau)$ are the well known fuzzy operators viz. $Fint, Fint(Fcl), Fcl(Fint)$, $Fint(Fcl(Fint))$ and $Fcl(Fint(Fcl))$.*

Definition 2.8. [7] *Let (X, τ) be a fuzzy topological space and ψ^* be a fuzzy operator on $F\alpha\text{O}(X, \tau)$. Then any fuzzy α-open set $\mu \in I^X$ is called fuzzy α-ψ^*-open if $\mu \leq \psi^*(\mu)$. The complement of a fuzzy α-ψ^*-open set is said to be a fuzzy α-ψ^*-closed set.*

Definition 2.9. [7] *Let (X_1, τ_1) and (X_2, τ_2) be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha\text{O}(X_1, \tau_1)$ and $F\alpha\text{O}(X_2, \tau_2)$. Any function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is said to be a fuzzy α-ψ^*-continuous function if for every $\mu \in F\alpha$-ψ^*-$\text{O}(X_2, \tau_2)$, $f^{-1}(\mu) \in F\alpha$-ψ^*-$\text{O}(X_1, \tau_1)$.*

3. FUZZY α-ψ^*-HOMOTOPY

In this section, the concept of fuzzy α-ψ^*-homotopies is introduced. Then proved that the fuzzy α-ψ^*-homotopy is an equivalence relation. Some interesting properties of fuzzy α-ψ^*-homotopies are studied.

Definition 3.1. *Let (X, τ) be a topological space. Let $V \subseteq X$ and χ_V be the characteristic function of V. Then the fuzzy topology introduced by V is $V_\tau = \{\chi_V : V \in \tau\}$ and the pair (X, V_τ) is said to be a fuzzy topological space introduced by (X, τ).*

Notation 3.1. *Let I be the unit interval. Let ς be an Euclidean topology on I and (I, ς^*) be a fuzzy topological space introduced by the Euclidean space (I, ς).*

Proposition 3.1. *Let (X_1, τ_1) and (X_2, τ_2) be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha\mathrm{O}(X_1, \tau_1)$ and $F\alpha\mathrm{O}(X_2, \tau_2)$. Let $Y, Z \subseteq X_1$ and (Y, τ_Y), (Z, τ_Z) be two fuzzy topological subspaces of (X_1, τ_1), where τ_Y and τ_Z are fuzzy subspace topologies in (Y, τ_Y) and (Z, τ_Z) respectively. Let $1_{X_1} = (\chi_Y \vee \chi_Z)$, where χ_Y and χ_Z are fuzzy α-ψ^* closed sets in (X_1, τ_1). Let $\phi_1 : (Y, \tau_Y) \to (X_2, \tau_2)$ and $\phi_2 : (Z, \tau_Z) \to (X_2, \tau_2)$ be any two fuzzy α-ψ^*-continuous functions. If $\phi_1|_{Y\cap Z} = \phi_2|_{Y\cap Z}$, then $\varphi : (X_1, \tau_1) \to (X_2, \tau_2)$ defined by*

$$\varphi(x) = \begin{cases} \phi_1(x), & x \in Y, \\ \\ \phi_2(x), & x \in Z \end{cases}$$

is a fuzzy α-ψ^*-continuous function.

Definition 3.2. *Let (X_1, τ_1) and (X_2, τ_2) be any two fuzzy topological spaces. Let (I, ς^*) be a fuzzy topological space introduced by (I, ς) and ψ^* be a fuzzy operator on $F\alpha\mathrm{O}(X_1, \tau_1)$, $F\alpha\mathrm{O}(X_2, \tau_2)$ and $F\alpha\mathrm{O}(I, \varsigma^*)$. Let $\phi, \varphi : (X_1, \tau_1) \to (X_2, \tau_2)$ be any two fuzzy α-ψ^*-continuous functions. Then ϕ is said to be fuzzy α-ψ^*-homotopic to φ, if there exists a fuzzy α-ψ^*-continuous function $\mathbb{H} : (X_1, \tau_1) \times (I, \varsigma^*) \to (X_2, \tau_2)$ such that $\mathbb{H}(x_t, 0) = \phi(x_t)$ and $\mathbb{H}(x_t, 1) = \varphi(x_t)$ for each fuzzy point $x_t \in \mathcal{FP}(X_1)$. Moreover the function $\mathbb{H}$ is said to be a fuzzy α-ψ^*-homotopy between ϕ and φ, denoted by $\phi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \varphi$.*

Example 3.1. *Let $f, g : (X_1, \tau_1) \to (X_2, \tau_2)$ be any two fuzzy α-ψ^*-continuous functions. Let $\mathbb{H} : (X_1, \tau_1) \times (I, \varsigma^*) \to (X_2, \tau_2)$ be defined as $H(x_t, t) = (1-t)f(x_t) + tg(x_t)$ for all $x_t \in \mathcal{FP}(X)$. Then $H(x_t, 0) = f(x_t)$ and $H(x_t, 1) = g(x_t)$. Thus f is fuzzy α-ψ^*-homotopic to g.*

Proposition 3.2. *Let (X_1, τ_1) and (X_2, τ_2) be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha\mathrm{O}(X_1, \tau_1)$ and $F\alpha\mathrm{O}(X_2, \tau_2)$. Let $\mathbb{H} : (X_1, \tau_1) \times (I, \varsigma^*) \to (X_2, \tau_2)$ be a fuzzy α-ψ^*-continuous function and $\phi_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \phi_2$. Then "$\simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}}$" is an equivalence relation.*

Proof. Let (X_1, τ_1) and (X_2, τ_2) be any two fuzzy topological spaces. Let (I, ς^*) be a fuzzy topological space introduced by (I, ς) and ψ^* be a fuzzy operator on $F\alpha O(I, \varsigma^*)$.
(i) Reflexive : Let $\varphi : (X_1, \tau_1) \to (X_2, \tau_2)$ be any fuzzy α-ψ^*-continuous function. Assume $\mathbb{H} : (X_1, \tau_1) \times (I, \varsigma^*) \to (X_2, \tau_2)$ be such that $\mathbb{H}(x_t, r) = \varphi(x_t)$, for each fuzzy point $x_t \in \mathcal{FP}(X_1)$ and $r \in I$. Then $\mathbb{H}$ is a fuzzy α-ψ^*-continuous function. Also, $\mathbb{H}(x_t, 0) = \varphi(x_t)$ and $\mathbb{H}(x_t, 1) = \varphi(x_t)$. Therefore, $\varphi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \varphi$.
(ii) Symmetric : Suppose that $\varphi, \rho : (X_1, \tau_1) \to (X_2, \tau_2)$ are two fuzzy α-ψ^*-continuous functions. Let $\varphi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \rho$. Then there exists a fuzzy α-ψ^*-continuous function $\mathbb{H} : (X_1, \tau_1) \times (I, \varsigma^*) \to (X_2, \tau_2)$ such that $\mathbb{H}(x_t, 0) = \varphi(x_t)$ and $\mathbb{H}(x_t, 1) = \rho(x_t)$ for

each fuzzy point $x_t \in \mathcal{FP}(X_1)$. Let $\mathbb{G} : (X_1, \tau_1) \times (I, \varsigma^*) \to (X_2, \tau_2)$ be such that $\mathbb{G}(x_t, r) = \mathbb{H}(x_t, 1-r)$, for all $r \in I$. Since $\mathbb{H}$ is fuzzy α-ψ^*-continuous, $\mathbb{G}$ is a fuzzy α-ψ^*-continuous function. Also, $\mathbb{G}(x_t, 0) = \mathbb{H}(x_t, 1) = \rho(x_t)$ and $\mathbb{G}(x_t, 1) = \mathbb{H}(x_t, 0) = \varphi(x_t)$, for each fuzzy point $x_t \in \mathcal{FP}(X_1)$. Therefore, $\rho \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \varphi$.

(iii) Transitive : Suppose that $\varphi, \rho, \phi : (X_1, \tau_1) \to (X_2, \tau_2)$ are any three fuzzy α-ψ^*-continuous functions. Let $\varphi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \rho$ and $\rho \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \phi$. Since $\varphi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \rho$, there exists a fuzzy α-ψ^*-continuous function $\mathbb{H} : (X_1, \tau_1)\times(I, \varsigma^*) \to (X_2, \tau_2)$ such that $\mathbb{H}(x_t, 0) = \varphi(x_t)$ and $\mathbb{H}(x_t, 1) = \rho(x_t)$, for each fuzzy point $x_t \in \mathcal{FP}(X_1)$. Similarly, since $\rho \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \phi$, there exists a fuzzy α-ψ^*-continuous function $\mathbb{G} : (X_1, \tau_1)\times(I, \varsigma^*) \to (X_2, \tau_2)$ such that $\mathbb{G}(x_t, 0) = \rho(x_t)$ and $\mathbb{G}(x_t, 1) = \phi(x_t)$, for each fuzzy point $x_t \in \mathcal{FP}(X_1)$.

Let $\mathbb{P} : (X_1, \tau_1) \times (I, \varsigma^*) \to (X_2, \tau_2)$ be defined by

$$\mathbb{P}(x_t, r) = \begin{cases} \mathbb{H}(x_t, 2r), & \text{if } 0 \le r \le \frac{1}{2} \\ \\ \mathbb{G}(x_t, 2r-1), & \text{if } \frac{1}{2} \le r \le 1 \end{cases}$$

for each fuzzy point $x_t \in \mathcal{FP}(X_1)$ and $r \in I$. Since $\mathbb{H}$ and $\mathbb{G}$ are fuzzy α-ψ^*-continuous functions and by Proposition 3.1, $\mathbb{P}$ is a fuzzy α-ψ^*-continuous function. Further $\mathbb{P}(x_t, 0) = \mathbb{H}(x_t, 0) = \varphi(x_t)$ and $\mathbb{P}(x_t, 1) = \mathbb{G}(x_t, 1) = \phi(x_t)$. Therefore, $\varphi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \phi$.

Hence "$\simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}}$" is an equivalence relation. □

Proposition 3.3. *Let (X_1, τ_1), (X_2, τ_2) and (X_3, τ_3) be any three fuzzy topological spaces. Let ψ^* be a fuzzy operator on $F\alpha O(X_1, \tau_1)$, $F\alpha O(X_2, \tau_2)$ and $F\alpha O(X_3, \tau_3)$. If $\varphi : (X_1, \tau_1) \to (X_2, \tau_2)$ and $\phi : (X_2, \tau_2) \to (X_3, \tau_3)$ are fuzzy α-ψ^*-continuous functions, then $\phi \circ \varphi : (X_1, \tau_1) \to (X_3, \tau_3)$ is also a fuzzy α-ψ^*-continuous function.*

Proof. Let $\lambda \in F\alpha$-ψ^*-$O(X_3, \tau_3)$. As ϕ is a fuzzy α-ψ^*-continuous function, $\phi^{-1}(\lambda) \in F\alpha$-$\psi^*$-$O(X_2, \tau_2)$. Since φ is a fuzzy α-ψ^*-continuous function and $\phi^{-1}(\lambda) \in F\alpha$-$\psi^*$-$O(X_2, \tau_2)$, $\varphi^{-1}(\phi^{-1}(\lambda)) \in F\alpha$-$\psi^*$-$O(X_1, \tau_1)$. Thus

$$\varphi^{-1}(\phi^{-1}(\lambda)) = (\phi \circ \varphi)^{-1}(\lambda)$$

is a fuzzy α-ψ^*-open set in (X_1, τ_1). Hence $\phi \circ \varphi$ is a fuzzy α-ψ^*-continuous function. □

Proposition 3.4. *Let (X_1, τ_1), (X_2, τ_2) and (X_3, τ_3) be any three fuzzy topological spaces. Let ψ^* be a fuzzy operator on $F\alpha O(X_1, \tau_1)$, $F\alpha O(X_2, \tau_2)$ and $F\alpha O(X_3, \tau_3)$. If ϕ_1 and ϕ_2 are the fuzzy α-ψ^*-continuous functions from (X_1, τ_1) to (X_2, τ_2) and that φ_1 and φ_2 are the fuzzy α-ψ^*-continuous functions from (X_2, τ_2) to (X_3, τ_3), then, $\varphi_1 \circ \phi_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \varphi_2 \circ \phi_2$.*

Proof. The proof is apparent from the following steps:

(i) $\varphi_1 \circ \phi_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \varphi_1 \circ \phi_2$.

(ii) $\varphi_1 \circ \phi_2 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \varphi_2 \circ \phi_2$.

(iii) Transitivity of (i) and (ii) implies that $\varphi_1 \circ \phi_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \varphi_2 \circ \phi_2$. □

Proposition 3.5. *Let (X_1, τ_1), (X_2, τ_2) and (X_3, τ_3) be any three fuzzy topological spaces. Let ψ^* be a fuzzy operator on $F\alpha O(X_1, \tau_1)$, $F\alpha O(X_2, \tau_2)$ and $F\alpha O(X_3, \tau_3)$. Let $\phi, \varphi : (X_1, \tau_1) \to (X_2, \tau_2)$ be any two fuzzy α-ψ^*-continuous functions such that $\phi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \varphi$. If $\sigma : (X_2, \tau_2) \to (X_3, \tau_3)$ is a fuzzy α-ψ^*-continuous function, then $\sigma \circ \phi, \sigma \circ \varphi : (X_1, \tau_1) \to (X_3, \tau_3)$ are fuzzy α-ψ^*-continuous functions and $\sigma \circ \phi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*\mathscr{H}}} \sigma \circ \varphi$.*

Proof. The proof is apparent. □

Proposition 3.6. *Let (X_1, τ_1), (X_2, τ_2) and (X_3, τ_3) be any three fuzzy topological spaces. Let ψ^* be a fuzzy operator on $F\alpha O(X_1, \tau_1)$, $F\alpha O(X_2, \tau_2)$ and $F\alpha O(X_3, \tau_3)$. Let $\phi, \varphi : (X_1, \tau_1) \to (X_2, \tau_2)$ be any two fuzzy α-ψ^*-continuous functions such that $\phi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{H}} \varphi$. Also let $\sigma, \wp : (X_2, \tau_2) \to (X_3, \tau_3)$ be any two fuzzy α-ψ^*-continuous functions such that $\sigma \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{H}} \wp$. Then $\sigma \circ \phi, \wp \circ \varphi : (X_1, \tau_1) \to (X_3, \tau_3)$ are fuzzy α-ψ^*-continuous function and $\sigma \circ \phi \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{H}} \wp \circ \varphi$.*

Proof. The proof is apparent. □

4. Fuzzy α-ψ^*-Path Homotopy

In this section, the concepts of fuzzy α-ψ^*-paths, fuzzy α-ψ^*-loops and fuzzy α-ψ^*-path homotopy in fuzzy topological spaces are introduced and the properties related with these concepts are discussed. Also, a characterization of fuzzy α-ψ^*-contractible space is studied.

Definition 4.1. *Let (X, τ) be any fuzzy topological space and (I, ς^*) be a fuzzy topological space introduced by (I, ς). Let ψ^* be a fuzzy operator on both $F\alpha O(X, \tau)$ and $F\alpha O(I, \varsigma^*)$. Let $x_{t_1}, y_{t_2} \in \mathcal{FP}(X)$ be any two fuzzy points. A fuzzy α-ψ^*-path $\gamma : (I, \varsigma^*) \to (X, \tau)$ from x_{t_1} to y_{t_2} is a fuzzy α-ψ^*-continuous function such that $\gamma(0) = x_{t_1}$ and $\gamma(1) = y_{t_2}$, $0 < t_i \leq 1, i = 1, 2$. Then the fuzzy points x_{t_1} and y_{t_2} are called the initial and terminal points of γ.*

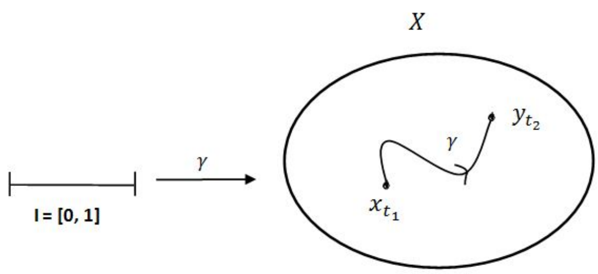


Figure 1. Fuzzy α-ψ^*-path

Definition 4.2. *Let (X, τ) be any fuzzy topological space and (I, ς^*) be a fuzzy topological space introduced by (I, ς). Let ψ^* be a fuzzy operator on both $F\alpha O(X, \tau)$ and $F\alpha O(I, \varsigma^*)$. Let γ be a fuzzy α-ψ^*-path in (X, τ) from x_{t_1} to y_{t_2}, where $x_{t_1}, y_{t_2} \in \mathcal{FP}(X)$. The inverse of γ is the fuzzy α-ψ^*-path in (X, τ) from y_{t_2} to x_{t_1} defined by $\gamma^i(t) = \gamma(1 - t)$ for all $t \in I$.*

Proposition 4.1. *Let (X, τ) be any fuzzy topological space and ψ^* be a fuzzy operator on $F\alpha O(X, \tau)$. Let $x_{t_1}, y_{t_2} \in \mathcal{FP}(X)$ be any two fuzzy points and if there is a fuzzy α-ψ^*-path in (X, τ) with initial point and terminal points x_{t_1}, y_{t_2} respectively, then there exists a fuzzy α-ψ^*-path in (X, τ) with initial and terminal points y_{t_2}, x_{t_1} respectively.*

Proof. Let (I, ς^*) be a fuzzy topological space introduced by (I, ς) and ψ^* be a fuzzy operator on $F\alpha O(I, \varsigma^*)$. Let γ be a fuzzy α-ψ^*-path in (X, τ) with‘ initial and terminal points x_{t_1}, y_{t_2} respectively where $x_{t_1}, y_{t_2} \in \mathcal{FP}(X)$. Then $\gamma : (I, \varsigma^*) \to (X, \tau)$ is a fuzzy α-ψ^*-continuous function with $\gamma(0) = x_{t_1}$ and $\gamma(1) = y_{t_2}$. Let $\beta : (I, \varsigma^*) \to (X, \tau)$ be such that $\beta(t) = \gamma(1-t)$ for every $t \in I$. Then $\beta(0) = \gamma(1-0) = \gamma(1) = y_{t_2}$ and $\beta(1) = \gamma(1-1) = \gamma(0) = x_{t_1}$. Since γ is a fuzzy α-ψ^*-continuous function, β is also a fuzzy α-ψ^*-continuous function. Therefore β is a fuzzy α-ψ^*-path in (X,τ) with initial and terminal points y_{t_2}, x_{t_1} respectively. □

Definition 4.3. *Let (X,τ) be any fuzzy topological space and (I,ς^*) be a fuzzy topological space introduced by (I,ς). Let ψ^* be a fuzzy operator on both $F\alpha\mathrm{O}(X,\tau)$ and $F\alpha\mathrm{O}(I,\varsigma^*)$. Let $x_{t_1}, y_{t_2}, z_{t_3} \in \mathcal{FP}(X)$ and let γ and δ be any two fuzzy α-ψ^*-paths in (X,τ) from x_{t_1} to y_{t_2} and y_{t_2} to z_{t_3} respectively. Then the product of γ and δ is the fuzzy α-ψ^*-path $\gamma * \delta$ in (X,τ) from x_{t_1} to z_{t_3} which is defined by*

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ \delta(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

for all $t \in I$.

Definition 4.4. *Let (X,τ) be any fuzzy topological space and ψ^* be a fuzzy operator on $F\alpha\mathrm{O}(X,\tau)$. Let $I_X : (X,\tau) \to (X,\tau)$ be any fuzzy α-ψ^*-continuous function with $I_X(x_t) = x_t$ for all $x_t \in \mathcal{FP}(X)$. Let $y_t \in \mathcal{FP}(X)$. If I_X is fuzzy α-ψ^*-homotopic to a fuzzy α-ψ^*-continuous function $C_X : (X,\tau) \to (X,\tau)$ with $C_X(x_t) = y_t$ for all $x_t \in \mathcal{FP}(X)$, then (X,τ) is said to be a fuzzy α-ψ^*-contractible space.*

Example 4.1. *Let (X,τ) be any fuzzy topological space, $y_t \in \mathcal{FP}(X)$ and ψ^* be a fuzzy operator on $F\alpha\mathrm{O}(X,\tau)$. Define the functions $I_X : (X,\tau) \to (X,\tau)$ and $C_X : (X,\tau) \to (X,\tau)$ as $I_X(x_t) = x_t$ and $C_X(x_t) = y_t$ for all $x_t \in \mathcal{FP}(X)$. Clearly, I_X and C_X are fuzzy α-ψ^*-continuous functions. Define the function $H(x_t,t) = (1-t)I_X(x_t) + tC_X(x_t)$ for all $x_t \in \mathcal{FP}(X)$. Then $H(x_t,0) = I_X(x_t)$ and $H(x_t,1) = C_X(x_t)$. Thus I_X is fuzzy α-ψ^*-homotopic to C_X. Hence (X,τ) is a fuzzy α-ψ^*-contractible space.*

Definition 4.5. *Let (X,τ) be a fuzzy topological space (I,ς^*) be a fuzzy topological space introduced by (I,ς). Let ψ^* be a fuzzy operator on both $F\alpha\mathrm{O}(X,\tau)$ and $F\alpha\mathrm{O}(I,\varsigma^*)$. Let $x_{t_1}, y_{t_2} \in \mathcal{FP}(X)$. Then (X,τ) is said to be a fuzzy α-ψ^*-path connected space if for each pair of fuzzy points $x_{t_1}, y_{t_2} \in \mathcal{FP}(X)$, there exists a fuzzy α-ψ^*-path $\gamma : (I,\varsigma^*) \to (X,\tau)$ such that $\gamma(0) = x_{t_1}$ and $\gamma(1) = y_{t_2}$.*

Proposition 4.2. *Let (X,τ) and (Y,σ) be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha\mathrm{O}(X,\tau)$ and $F\alpha\mathrm{O}(Y,\sigma)$. Let $x_{t_1} \in \mathcal{FP}(X)$. Then (X,τ) is fuzzy α-ψ^*-contractible if and only if any fuzzy α-ψ^*-continuous function $f : (Y,\sigma) \to (X,\tau)$ is fuzzy α-ψ^*-homotopic to a function $C_X : (X,\tau) \to (X,\tau)$ such that $C_X(x_t) = x_{t_1}$ for all $x_t \in \mathcal{FP}(X)$.*

Proof. Let (X,τ) be a fuzzy α-ψ^*-contractible space. Then there exists a fuzzy α-ψ^*-homotopy $\mathbb{H} : (X,\tau) \times (I,\varsigma^*) \to (X,\tau)$ between the fuzzy α-ψ^*-continuous function $I_X : (X,\tau) \to (X,\tau)$ and the fuzzy α-ψ^*-continuous function $C_X : (X,\tau) \to (X,\tau)$ such that $I_X(x_t) = x_t$ and $C_X(x_t) = x_{t_1}$ for all $x_t \in \mathcal{FP}(X)$. Let $f : (Y,\sigma) \to (X,\tau)$ be a fuzzy α-ψ^*-continuous function. By Proposition 3.5, $I_X \circ f$ is fuzzy α-ψ^*-homotopic to $C_X \circ f$. Also $(I_X \circ f)(x_t) = I_X(f(x_t)) = f(x_t)$ and $C_X \circ f : (Y,\sigma) \to (X,\tau)$ is such that $(C_X \circ f)(x_t) = C_X(f(x_t)) = x_{t_1} = C_X(x_t)$ for all $x_t \in \mathcal{FP}(X)$. Thus $(I_X \circ f) = f$ and $(C_X \circ f) = C_X$. Hence f is fuzzy α-ψ^*-homotopic to a function C_X.

Conversely, suppose that $Y = X$ and $\sigma = \tau$. Assume that $f : (X,\tau) \to (X,\tau)$ is such that $f(x_t) = x_t$. Thus $f = I_X$. Since f is fuzzy α-ψ^*-homotopic to C_X, I_X is fuzzy α-ψ^*-homotopic to C_X. Hence (X,τ) is fuzzy α-ψ^*-contractible. □

Definition 4.6. *Let (X,τ) be a fuzzy topological space. Let (I,ς_1^*) and (I,ς_2^*) be any two fuzzy topological spaces introduced by (I,ς_1) and (I,ς_2) respectively. Let ψ^* be a fuzzy operator on $F\alpha O(X,\tau)$, $F\alpha O(I,\varsigma_1^*)$ and $F\alpha O(I,\varsigma_2^*)$. Any two fuzzy α-ψ^*-paths γ_1 and γ_2 in (X,τ) from x_{t_1} to y_{t_2}, where $x_{t_1}, y_{t_2} \in \mathcal{FP}(X)$ are said to be a fuzzy α-ψ^*-path homotopy (denoted by, $\gamma_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{PH}} \gamma_2$) if there exists a fuzzy α-ψ^*-continuous function $\mathbb{H} : (I,\varsigma_1^*) \times (I,\varsigma_2^*) \to (X,\tau)$ such that*

$$\mathbb{H}(0,s_t) = x_{t_1} \text{ and } \mathbb{H}(1,s_t) = y_{t_2}, \text{ for all } s_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_2^*),$$
$$\mathbb{H}(r_t,0) = \gamma_1(r_t) \text{ and } \mathbb{H}(r_t,1) = \gamma_2(r_t), \text{ for all } r_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_1^*).$$

Proposition 4.3. *Let (X,τ) be a fuzzy topological space. Let (I,ς_1^*) and (I,ς_2^*) be any two fuzzy topological spaces introduced by (I,ς_1) and (I,ς_2) respectively. Let ψ^* be a fuzzy operator on $F\alpha O(X,\tau)$, $F\alpha O(I,\varsigma_1^*)$ and $F\alpha O(I,\varsigma_2^*)$. If γ_1 and γ_2 are any two fuzzy α-ψ^*-paths having same initial point as well as the same terminal point and $\gamma_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{PH}} \gamma_2$, then " $\simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{PH}}$ " is an equivalence relation.*

Proof. Let $x_{t_1}, y_{t_2} \in \mathcal{FP}(X)$. Let $r_t \in \mathcal{FP}(I)$ in (I,ς_1^*) and $s_t \in \mathcal{FP}(I)$ in (I,ς_2^*).
(i) Reflexive : Let $\gamma : (I,\varsigma^*) \to (X,\tau)$ be any fuzzy α-ψ^*-path with $\gamma(0) = x_{t_1}$, $\gamma(1) = y_{t_2}$. Let $\mathbb{H} : (I,\varsigma_1^*) \times (I,\varsigma_2^*) \to (X,\tau)$ be a fuzzy α-ψ^*-continuous function such that $\mathbb{H}(r_t,s_t) = \gamma(r_t)$. Thus

$$\mathbb{H}(0,s_t) = x_{t_1} \text{ and } \mathbb{H}(1,s_t) = y_{t_2}, \text{ for all } s_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_2^*),$$
$$\mathbb{H}(r_t,0) = \gamma(r_t) \text{ and } \mathbb{H}(r_t,1) = \gamma(r_t), \text{ for all } r_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_1^*).$$

Therefore $\mathbb{H}$ is a fuzzy α-ψ^*-path-homotopy from γ to itself. Hence the relation is reflexive.
(ii) Symmetric : Suppose that, $\gamma_1, \gamma_2 : (I,\varsigma^*) \to (X,\tau)$ are any two fuzzy α-ψ^*-paths with $\gamma_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{PH}} \gamma_2$. Then there exists a fuzzy α-ψ^*-continuous function $\mathbb{H} : (I,\varsigma_1^*) \times (I,\varsigma_2^*) \to (X,\tau)$ such that

$$\mathbb{H}(0,s_t) = x_{t_1} \text{ and } \mathbb{H}(1,s_t) = y_{t_2}, \text{ for all } s_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_2^*),$$
$$\mathbb{H}(r_t,0) = \gamma_1(r_t) \text{ and } \mathbb{H}(r_t,1) = \gamma_2(r_t), \text{ for all } r_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_1^*).$$

Define a map $\mathbb{H}' : (I,\varsigma_1^*) \times (I,\varsigma_2^*) \to (X,\tau)$ by $\mathbb{H}'(r_t,s_t) = \mathbb{H}(r_t, 1-s_t)$.
Then $\mathbb{H}'$ is a fuzzy α-ψ^*-continuous function and

$$\mathbb{H}'(0,s_t) = x_{t_1} \text{ and } \mathbb{H}'(1,s_t) = y_{t_2}, \text{ for all } s_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_2^*),$$
$$\mathbb{H}'(r_t,0) = \mathbb{H}(r_t,1) = \gamma_2(r_t) \text{ and } \mathbb{H}'(r_t,1) = \mathbb{H}(r_t,0) = \gamma_1(r_t),$$
$$\text{for all } r_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_1^*).$$

Thus $\gamma_2 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{PH}} \gamma_1$. Hence the relation is symmetric.
(iii) Transitive : Suppose $\gamma_1, \gamma_2, \gamma_3 : (I,\varsigma^*) \to (X,\tau)$ are any three fuzzy α-ψ^*-paths and $\gamma_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{PH}} \gamma_2$ and $\gamma_2 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{PH}} \gamma_3$. Let

$$\mathbb{H}_1 : (I,\varsigma_1^*) \times (I,\varsigma_2^*) \to (X,\tau) \text{ and } \mathbb{H}_2 : (I,\varsigma_1^*) \times (I,\varsigma_2^*) \to (X,\tau)$$

be two fuzzy α-ψ^*-homotopies such that

$$\mathbb{H}_1(0,s_t) = x_{t_1} \text{ and } \mathbb{H}_1(1,s_t) = y_{t_2}, \text{ for all } s_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_2^*),$$
$$\mathbb{H}_1(r_t,0) = \gamma_1(r_t) \text{ and } \mathbb{H}_1(r_t,1) = \gamma_2(r_t), \text{ for all } r_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_1^*).$$

and

$$\mathbb{H}_2(0,s_t) = x_{t_1} \text{ and } \mathbb{H}_2(1,s_t) = y_{t_2}, \text{ for all } s_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_2^*),$$

$$\mathbb{H}_2(r_t,0) = \gamma_2(r_t) \text{ and } \mathbb{H}_2(r_t,1) = \gamma_3(r_t), \text{ for all } r_t \in \mathcal{FP}(I) \text{ in } (I,\varsigma_1^*).$$

Define a map $\mathbb{H}_3 : (I, \varsigma_1^*) \times (I, \varsigma_2^*) \to (X, \tau)$ by

$$\mathbb{H}_3(r_t, s_t) = \begin{cases} \mathbb{H}_1(r_t, 2s_t), & \text{if } 0 \le s_t \le \frac{1}{2} \\ \\ \mathbb{H}_2(r_t, 2s_t - 1), & \text{if } \frac{1}{2} \le s_t \le 1. \end{cases}$$

Now,

$$\mathbb{H}_3(0, s_t) = x_{t_1} \text{ and } \mathbb{H}_3(1, s_t) = y_{t_2}, \text{ for all } s_t \in \mathcal{FP}(I) \text{ in } (I, \varsigma_2^*),$$
$$\mathbb{H}_3(r_t, 0) = \mathbb{H}_1(r_t, 0) = \gamma_1(r_t) \text{ and } \mathbb{H}_3(r_t, 1) = \mathbb{H}_2(r_t, 0) = \gamma_3(r_t),$$
$$\text{for all } r_t \in \mathcal{FP}(I) \text{ in } (I, \varsigma_1^*).$$

Then $\mathbb{H}_3$ is fuzzy a α-ψ^*-continuous function by Proposition 3.1, Thus $\gamma_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{P}\mathscr{H}} \gamma_3$. Hence the relation is transitive.

Therefore " $\simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{P}\mathscr{H}}$ " is an equivalence relation. □

Definition 4.7. *Let (X, τ) be any fuzzy topological space and (I, ς^*) be a fuzzy topological space introduced by (I, ς). Let ψ^* be a fuzzy operator on both $F\alpha O(X, \tau)$ and $F\alpha O(I, \varsigma^*)$. Let $\gamma : (I, \varsigma^*) \to (X, \tau)$ be a fuzzy α-ψ^*-path and $x_t \in \mathcal{FP}(X)$. If the initial point and the terminal point of γ are equal, that is $\gamma(0) = x_t = \gamma(1)$, then the fuzzy α-ψ^*-path γ is called as the fuzzy α-ψ^*-loop based on x_t. The collection of all fuzzy α-ψ^*-loops associated with x_t in (X, τ) is denoted by $\Upsilon((X, \tau), x_t)$.*

Definition 4.8. *Let (X, τ) be a fuzzy topological space. Let (I, ς_1^*) and (I, ς_2^*) be any two fuzzy topological spaces introduced by (I, ς_1) and (I, ς_2) respectively. Let ψ^* be a fuzzy operator on $F\alpha O(X, \tau)$, $F\alpha O(I, \varsigma_1^*)$ and $F\alpha O(I, \varsigma_2^*)$. Let $x_t \in \mathcal{FP}(X)$. Any two fuzzy α-ψ^*-loops l_1 and l_2 in (X, τ) at x_t are said to be fuzzy α-ψ^*-loop homotopic at x_t (denoted by, $\mathrm{l}_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{L}\mathscr{H}} \mathrm{l}_2$) if there exists a fuzzy α-ψ^*-continuous function $\mathscr{G} : (I, \varsigma_1^*) \times (I, \varsigma_2^*) \to (X, \tau)$ such that*

$$\mathscr{G}(0, p_{t'}) = \mathscr{G}(1, p_{t'}) = x_t, \textit{ for all } p_{t'} \in \mathcal{FP}(I) \textit{ in } (I, \varsigma_2^*),$$
$$\mathscr{G}(s_t, 0) = \mathrm{l}_1(s_t) \textit{ and } \mathscr{G}(s_t, 1) = \mathrm{l}_2(s_t), \textit{ for all } s_t \in \mathcal{FP}(I) \textit{ in } (I, \varsigma_1^*).$$

Proposition 4.4. *Let (X, τ) be a fuzzy topological space. Let (I, ς_1^*) and (I, ς_2^*) be any two fuzzy topological spaces introduced by (I, ς_1) and (I, ς_2). Let ψ^* be a fuzzy operator on $F\alpha O(X, \tau)$, $F\alpha O(I, \varsigma_1^*)$ and $F\alpha O(I, \varsigma_2^*)$. Let $\gamma_1, \gamma_2 : (I, \varsigma_1^*) \to (X, \tau)$ be any two fuzzy α-ψ^*-paths. If $\mathbb{H} : (I, \varsigma_1^*) \times (I, \varsigma_2^*) \to (X, \tau)$ is fuzzy α-ψ^*-loop homotopy between γ_1 and γ_2, that is $\gamma_1 \simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{L}\mathscr{H}} \gamma_2$, then "$\simeq_{\mathscr{F}_{\alpha\text{-}\psi^*}\mathscr{L}\mathscr{H}}$" is an equivalence relation on $\Upsilon((X, \tau), x_t)$.*

Proof. The proof is obvious by taking $x_{t_1} = y_{t_2}$ in the Proposition 4.3. □

Notation 4.1. *Let (X, τ) be any fuzzy topological space and (I, ς^*) be a fuzzy topological space introduced by (I, ς). Let ψ^* be a fuzzy operator on both $F\alpha O(X, \tau)$ and $F\alpha O(I, \varsigma^*)$. If $\gamma \in \Upsilon((X, \tau), x_t)$, then $[\gamma]$ denotes the fuzzy α-ψ^*-path homotopy equivalence class that contains γ and $\pi_1((X, \tau), x_t)$ denotes the set of all fuzzy α-ψ^*-path homotopy equivalence classes on*
$\Upsilon((X, \tau), x_t)$, that is,

$$\pi_1((X, \tau), x_t) = \{[\gamma] : \gamma \textit{ is a fuzzy } \alpha\text{-}\psi^*\textit{-loop in } X \textit{ based on } x_t\}.$$

Definition 4.9. *An operation "$\circ$" is defined on $\pi_1((X, \tau), x_t)$ by*

$$[\gamma_1] \circ [\gamma_2] = [\gamma_1 * \gamma_2]$$

where $[\gamma_1], [\gamma_2] \in \pi_1((X, \tau), x_t)$ and $\gamma_1 * \gamma_2$ is defined as in Definition 4.3.

Definition 4.10. *Let (X,τ) be any fuzzy topological space and (I,ς^*) be a fuzzy topological space introduced by (I,ς). Let ψ^* be a fuzzy operator on both $F\alpha O(X,\tau)$ and $F\alpha O(I,\varsigma^*)$. Let $\mathscr{I}:(I,\varsigma^*)\to(X,\tau)$ be the fuzzy α-ψ^*-loop defined by $\mathscr{I}(s_t)=x_t$ for each $s_t\in\mathcal{FP}(I)$ in (I,ς^*) and $x_t\in\mathcal{FP}(X)$. Then $\pi_1((X,\tau),x_t)$ is said to be α-ψ^*-fundamental group of (X,τ) at x_t if the following conditions are satisfied:*

(i) *Identity : If $[\gamma],[\mathscr{I}]\in\pi_1((X,\tau),x_t)$, then $[\mathscr{I}]\circ[\gamma]=[\gamma]\circ[\mathscr{I}]=[\gamma]$;*
(ii) *Inverse : If $[\gamma],[\gamma^i]\in\pi_1((X,\tau),x_t)$, then $[\gamma]\circ[\gamma^i]=[\gamma^i]\circ[\gamma]=[\mathscr{I}]$;*
(iii) *Associative : If $[\gamma_1],[\gamma_2],[\gamma_3]\in\pi_1((X,\tau),x_t)$, then*

$$([\gamma_1]\circ[\gamma_2])\circ[\gamma_3]=[\gamma_1]\circ([\gamma_2]\circ[\gamma_3]).$$

Definition 4.11. *Let (X_1,τ_1) and (X_2,τ_2) be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha O(X_1,\tau_1)$ and $F\alpha O(X_2,\tau_2)$. Let $x_{t_1}\in\mathcal{FP}(X_1)$, $y_{t_2}\in\mathcal{FP}(X_2)$. Let $\pi_1((X_1,\tau_1),x_{t_1})$ and $\pi_1((X_2,\tau_2),y_{t_2})$ be any two α-ψ^*-fundamental groups of (X_1,τ_1) at x_{t_1} and (X_2,τ_2) at y_{t_2} respectively. Any function $f:\pi_1((X_1,\tau_1),x_{t_1})\to\pi_1((X_2,\tau_2),y_{t_2})$ is said to be a fuzzy α-ψ^*-homomorphism if $f([\gamma_1]\circ[\gamma_2])=f([\gamma_1])\circ f([\gamma_2])$, for all $[\gamma_1],[\gamma_2]\in\pi_1((X_1,\tau_1),x_{t_1})$.*

Moreover, the fuzzy α-ψ^*-homomorphism is said to be a fuzzy α-ψ^*-isomorphism if it is bijective.

Proposition 4.5. *Let (X,τ) be a fuzzy α-ψ^*-path connected space where ψ^* is a fuzzy operator on $F\alpha O(X,\tau)$. Let $x_{t_1},y_{t_2}\in\mathcal{FP}(X)$ be any two fuzzy points and $\pi_1((X,\tau),x_{t_1})$ and $\pi_1((X,\tau),y_{t_2})$ are two α-ψ^*-fundamental groups of (X,τ) at x_{t_1} and y_{t_2} respectively. Then there exists a fuzzy α-ψ^*-isomorphism from $\pi_1((X,\tau),x_{t_1})$ onto $\pi_1((X,\tau),y_{t_2})$.*

Proof. Let γ be a fuzzy α-ψ^*-path from x_{t_1} to y_{t_2} in (X,τ) and γ^i be the inverse fuzzy α-ψ^*-path of γ such that $\gamma^i(t)=\gamma(1-t)$. Let $\gamma_\diamond:\pi_1((X,\tau),x_{t_1})\to\pi_1((X,\tau),y_{t_2})$ be defined by $\gamma_\diamond([\sigma])=[\gamma^i]\circ[\sigma]\circ[\gamma]$ for each $[\sigma]\in\pi_1((X,\tau),x_{t_1})$. Now for all $[\sigma],[\rho]\in\pi_1((X,\tau),x_{t_1})$,

$$\begin{aligned}\gamma_\diamond([\sigma]\circ[\rho])&=\gamma_\diamond[\sigma*\rho],\ \text{as in Definition 4.9}\\&=[\gamma^i]\circ[\sigma*\rho]\circ[\gamma]\\&=[\gamma^i*\sigma*\rho*\gamma],\ \text{as in Definition 4.9}\\&=[\gamma^i*\sigma*\rho*\gamma]\\&=[\gamma^i*\sigma]\circ[\rho*\gamma],\ \text{as in Definition 4.9}\\&=[\gamma^i*\sigma]\circ[\mathscr{I}]\circ[\rho*\gamma],\ \text{ by (i) of Definition 4.10}\\&=[\gamma^i*\sigma]\circ[\mathscr{I}*\rho*\gamma]\\&=[\gamma^i*\sigma]\circ[\gamma*\gamma^i*\rho*\gamma]\\&=[\gamma^i*\sigma*\gamma*\gamma^i*\rho*\gamma],\ \text{as in Definition 4.9}\\&=[\gamma^i*\sigma*\gamma*\gamma^i*\rho*\gamma]\\&=[\gamma^i*\sigma*\gamma]\circ[\gamma^i*\rho*\gamma]\\&=\gamma_\diamond([\sigma])\circ\gamma_\diamond([\rho]).\end{aligned}$$

Thus $\gamma_\diamond([\sigma]\circ[\rho])=\gamma_\diamond([\sigma])\circ\gamma_\diamond([\rho])$. Hence, $\gamma_\diamond$ is a fuzzy α-ψ^*-homomorphism. Similarly, if $\gamma^i_\diamond:\pi_1((X,\tau),y_{t_2})\to\pi_1((X,\tau),x_{t_1})$ is defined by $\gamma^i_\diamond([\sigma])=[\gamma]\circ[\sigma]\circ[\gamma^i]$ for each $[\sigma]\in\pi_1((X,\tau),x_{t_1})$, then $\gamma^i_\diamond:\pi_1((X,\tau),y_{t_2})\to\pi_1((X,\tau),x_{t_1})$ is also a fuzzy α-ψ^*-homomorphism.

Now for each $[\sigma] \in \pi_1((X,\tau), x_{t_1})$,

$$\begin{aligned}(\gamma^i_\diamond \circ \gamma_\diamond)([\sigma]) &= \gamma^i_\diamond(\gamma_\diamond([\sigma]))\\ &= \gamma^i_\diamond[\gamma^i * \sigma \ * \gamma]\\ &= [\gamma * (\gamma^i * \sigma \ * \gamma) * \gamma^i]\\ &= [(\gamma * \gamma^i) * \sigma \ * (\gamma * \gamma^i)], \quad \text{by (iii) of Definition 4.10}\\ &= [\mathscr{I} * \sigma \ * \mathscr{I}], \quad \text{by (i) of Definition 4.10}\\ &= [\sigma].\end{aligned}$$

Thus $(\gamma^i_\diamond \circ \gamma_\diamond)([\sigma]) = [\sigma]$. Hence $\gamma^i_\diamond \circ \gamma_\diamond$ is an identity function on $\pi_1((X,\tau), x_{t_1})$. Similarly, $(\gamma_\diamond \circ \gamma^i_\diamond)([\sigma]) = [\sigma]$. Hence $\gamma_\diamond \circ {\gamma^i}_\diamond$ is also an identity function on $\pi_1((X,\tau), x_{t_1})$. Therefore, $\gamma_\diamond$ is a fuzzy α-ψ^*-isomorphism. Hence $\gamma_\diamond$ is a fuzzy α-ψ^*-isomorphism between $\pi_1((X,\tau), x_{t_1})$ and $\pi_1((X,\tau), y_{t_2})$. □

5. FUZZY α-ψ^*-COVERING SPACES

In this section, the concepts of fuzzy α-ψ^*-open functions, fuzzy α-ψ^*-homeomorphisms and fuzzy α-ψ^*-covering spaces are introduced and some interesting properties are discussed.

Definition 5.1. *Let (X,τ) and (Y,σ) be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha\mathrm{O}(X,\tau)$ and $F\alpha\mathrm{O}(Y,\sigma)$. Any function $f : (X,\tau) \to (Y,\sigma)$ is said to be a fuzzy α-ψ^*-open function if for each $\lambda \in F\alpha$-ψ^*-$\mathrm{O}(X,\tau)$ the image $f(\lambda) \in F\alpha$-ψ^*-$\mathrm{O}(Y,\sigma)$.*

Definition 5.2. *Let (X,τ) and (Y,σ) be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha\mathrm{O}(X,\tau)$ and $F\alpha\mathrm{O}(Y,\sigma)$. If the bijective function $f : (X,\tau) \to (Y,\sigma)$ and its inverse function are fuzzy α-ψ^*-continuous functions, then the function f is said to be a fuzzy α-ψ^*-homeomorphism. Moreover, (X,τ) and (Y,σ) are said to be fuzzy α-ψ^*-homeomorphic spaces.*

Definition 5.3. *Let (X,τ) be a fuzzy topological space and ψ^* be a fuzzy operator on $F\alpha\mathrm{O}(X,\tau)$. A collection $\mathscr{S} = \{\lambda_i \in F\alpha$-$\psi^*$-$\mathrm{O}(X,\tau), i \in J, J$ is an indexed set$\}$ is called a fuzzy α-ψ^*-open cover of (X,τ) if $\vee_{i\in J}\lambda_i = 1_X$.*

Definition 5.4. *Let (X,τ) and $(\tilde{X},\tilde{\tau})$ be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha\mathrm{O}(X,\tau)$ and $F\alpha\mathrm{O}(\tilde{X},\tilde{\tau})$. Let $X_i \subseteq X$, $i \in J$, where J is an indexed set and $\{\chi_{X_i} \in F\alpha$-ψ^*-$\mathrm{O}(X,\tau)\}$ be a fuzzy α-ψ^*-open cover of (X,τ), where χ_{X_i} is a characteristic function of X_i, for each $i \in J$ respectively. Let $\phi : (\tilde{X},\tilde{\tau}) \to (X,\tau)$ be a fuzzy α-ψ^*-continuous function.*

Then any fuzzy α-ψ^*-open subspace (X_i, τ_{X_i}) of (X,τ) is said to be fuzzy α-ψ^*-evenly covered by the function ϕ if

$$\phi^{-1}(\chi_{X_i}) = \vee_{j=1}^{n}\{\chi_{S_j} \in F\alpha\text{-}\psi^*\text{-}\mathrm{O}(\tilde{X},\tilde{\tau})\},$$

where $S_j \subseteq \tilde{X}$, χ_{S_j} is a characteristic function of S_j and $\{\chi_{S_j}\}_{j=1}^n$ is a non-overlapping family and also each $\phi|_{S_j} : (S_j, \tilde{\tau}_{S_j}) \to (X_i, \tau_{X_i})$ is an onto fuzzy α-ψ^*-homeomorphism. Then ϕ is said to be a fuzzy α-ψ^*-covering function and $(\tilde{X},\tilde{\tau})$ is said to be a fuzzy α-ψ^* covering space of (X,τ). Also for each $j \in J$, χ_{S_j} is called a fuzzy α-ψ^*-path component of $\phi^{-1}(\chi_{X_i})$ and each member in $\{\chi_{X_i}\}$ of a fuzzy α-ψ^*-open cover of (X,τ) is called a fuzzy α-ψ^*-admissible open set in (X,τ).

Proposition 5.1. *Let (X,τ) and $(\tilde{X},\tilde{\tau})$ be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha O(X,\tau)$ and $F\alpha O(\tilde{X},\tilde{\tau})$. Then the fuzzy α-ψ^*-covering function $\phi:(\tilde{X},\tilde{\tau})\to(X,\tau)$ is always a fuzzy α-ψ^*-open function.*

Proof. Let $\lambda\in F\alpha$-ψ^*-$O(\tilde{X},\tilde{\tau})$ and $x_t\leq\phi(\lambda)$ where $x_t\in\mathcal{FP}(X)$. Assume that $\tilde{x}_t\leq\lambda$ where $\tilde{x}_t\in\mathcal{FP}(\tilde{X})$, such that $\phi(\tilde{x}_t)=x_t$. Since ϕ is a fuzzy α-ψ^*-covering function, there exists a fuzzy α-ψ^*-evenly covered subspace (X_1,τ_{X_1}) of (X,τ) such that $x_t\leq\chi_{X_1}$ and $\phi^{-1}(\chi_{X_1})=\vee_{j=1}^{n}\{\chi_{S_j}\in F\alpha$-$\psi^*$-$O(\tilde{X},\tilde{\tau})\}$, where $S_j\subseteq\tilde{X}$ and $\{\chi_{S_j}\}_{i=1}^{n}$ is a non-overlapping family and $\phi|_{S_j}:(S_j,\tilde{\tau}_{S_j})\to(X_1,\tau_{X_1})$ for each $j\in J$, J is an indexed set, is an onto fuzzy α-ψ^*-homeomorphism.

Let $\tilde{x}_t\leq\chi_{S_1}$. Since $\lambda,\chi_{S_1}\in F\alpha$-$\psi^*$-$O(\tilde{X},\tilde{\tau})$, $(\lambda\wedge\chi_{S_1})\in F\alpha$-$\psi^*$-$O(\tilde{X},\tilde{\tau})$. As $\phi|_{S_1}:(S_1,\tilde{\tau}_{S_1})\to(X_1,\tau_{X_1})$ is an onto fuzzy α-ψ^*-homeomorphism,

$$\phi|_{S_1}(\lambda\wedge\chi_{S_1})\in F\alpha\text{-}\psi^*\text{-}O(X_1,\tau_{X_1}).$$

Thus $\phi(\lambda\wedge\chi_{S_1})\in F\alpha$-$\psi^*$-$O(X_1,\tau_{X_1})$. Then $\phi(\lambda\wedge\chi_{S_1})\in F\alpha$-$\psi^*$-$O(X,\tau)$. Since $\tilde{x}_t\leq\lambda$ and $\tilde{x}_t\leq\chi_{S_1}$, $\tilde{x}_t\leq(\lambda\wedge\chi_{S_1})$. Thus, $\phi(\tilde{x}_t)\leq\phi(\lambda\wedge\chi_{S_1})$. Clearly, $x_t\leq\phi(\lambda\wedge\chi_{S_1})$.

Since $\phi(\lambda\wedge\chi_{S_1})\leq\phi(\lambda)$ and $x_t\leq\phi(\lambda\wedge\chi_{S_1})\leq\phi(\lambda)$, $\phi(\lambda)\in F\alpha$-ψ^*-$O(X,\tau)$. Hence ϕ is a fuzzy α-ψ^*-open function. □

Definition 5.5. *Let (X,τ) be a fuzzy topological space and ψ^* be a fuzzy operator on $F\alpha O(X,\tau)$. Then (X,τ) is said to be fuzzy α-ψ^*-locally path connected if for any $x_t\in\mathcal{FP}(X)$ and for any $\lambda\in F\alpha$-$\psi^*O(X,\tau)$ with $x_t\leq\lambda$, there exist some fuzzy α-ψ^*-path connected open subspace (Y,τ_Y) of (X,τ) such that $x_t\leq\chi_Y\leq\lambda$, where χ_Y is a characteristic function of Y.*

Proposition 5.2. *Let (X,τ) and $(\tilde{X},\tilde{\tau})$ be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha O(X,\tau)$ and $F\alpha O(\tilde{X},\tilde{\tau})$. Let $A\subseteq X$ and $\phi:(\tilde{X},\tilde{\tau})\to(X,\tau)$ be a fuzzy α-ψ^*-covering function. Let (A,τ_A) be a fuzzy α-ψ^*-locally path connected and fuzzy α-ψ^*-connected subspace of (X,τ). If $\tilde{A}\subseteq\tilde{X}$ and the characteristic function $\chi_{\tilde{A}}$ of $\tilde{A}$ is a fuzzy α-ψ^*-component of $\phi^{-1}(\chi_{\tilde{A}})$, then $\phi|_{\tilde{A}}:(\tilde{A},\tilde{\tau}_{\tilde{A}})\to(A,\tau_A)$ is a fuzzy α-ψ^*-covering function.*

Proof. Let $x_t\in\mathcal{FP}(A)$ and choose a fuzzy α-ψ^*-admissible open set χ_U such that $x_t\leq\chi_U$ where $A,U\subseteq X$ and χ_U is a characteristic function of U is such that $\chi_U\in F\alpha$-$\psi_1^*O(X,\tau)$. Let $\tilde{U}_i\subseteq\tilde{X}, i=1,2,...n$ and $\{\chi_{\tilde{U}_i}\}$ be the collection of fuzzy α-ψ^*-path components of $\phi^{-1}(\chi_U)$. Since ϕ is a fuzzy α-ψ^*-covering function, $\phi|_{\tilde{U}_i}:(\tilde{U}_i,\tilde{\tau}_{\tilde{U}_i})\to(U,\tau_U)$ is an onto fuzzy α-ψ^*-homeomorphism. Clearly, $((U\cap A),\tau_{U\cap A})$ is fuzzy α-ψ^*-evenly covered by $\{\chi_{\tilde{U}_i}\wedge\phi^{-1}(\chi_A)\}_{i=1}^{n}$. Since (A,τ_A) is fuzzy α-ψ^*-locally path connected, there exists a fuzzy α-ψ^*-path connected open subspace (V,τ_{A_V}) of (A,τ_A) where $V\subseteq A$ such that $x_t\leq\chi_V$ and $\chi_V\leq(\chi_U\wedge\chi_A)$. Then (V,τ_{A_V}) is fuzzy α-ψ^*-evenly covered by ϕ. Thus any fuzzy α-ψ^*-component $\chi_{\tilde{V}_i}$ of $\phi^{-1}(\chi_V)$ is such that $\chi_{\tilde{V}_i}q\chi_{\tilde{A}}$, then $\chi_{\tilde{V}_i}\leq\chi_{\tilde{A}}$. Thus $\phi|_{\tilde{A}}:(\tilde{A},\tilde{\tau}_{\tilde{A}})\to(A,\tau_A)$ is a fuzzy α-ψ^*-covering function. □

Definition 5.6. *Let (X,τ) and (Y,σ) be any two fuzzy topological spaces. Let ψ^* be a fuzzy operator on both $F\alpha O(X,\tau)$ and $F\alpha O(Y,\sigma)$. Let $\phi:(X,\tau)\to(Y,\sigma)$ be a fuzzy α-ψ^*-continuous function and $[\gamma]\in\pi_1((X,\tau),x_t)$ where γ is a fuzzy α-ψ^*-loop in X based at x_t. Then the fuzzy α-ψ^*-induced homomorphism of p is denoted by $\phi_*:\pi_1((X,\tau),x_t)\to\pi_1((Y,\sigma),\gamma(x_t))$ and it is defined by $\phi_*([\gamma])=[\phi\circ\gamma]$ for all $[\gamma]\in\pi_1((X,\tau),x_t)$.*

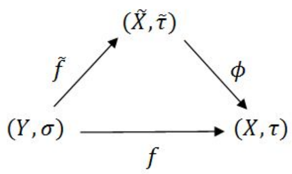
Definition 5.7. *Let (X,τ), $(\tilde{X},\tilde{\tau})$ and (Y,σ) be any three fuzzy topological spaces. Let ψ^* be a fuzzy operator on $F\alpha O(X,\tau)$, $F\alpha O(\tilde{X},\tilde{\tau})$ and $F\alpha O(Y,\sigma)$. Let $\phi:(\tilde{X},\tilde{\tau})\to(X,\tau)$ be a fuzzy α-ψ^*-covering function and $f:(Y,\sigma)\to(X,\tau)$ be any function. Then a lift of f is a fuzzy α-ψ^*-continuous function $\tilde{f}:(Y,\sigma)\to(\tilde{X},\tilde{\tau})$ such that $\phi\circ\tilde{f}=f$. In otherwords, $\tilde{f}$ lifts f.*

Proposition 5.3. *Let (X,τ), $(\tilde{X},\tilde{\tau})$ and (Y,σ) be any three fuzzy topological spaces. Let ψ^* be a fuzzy operator on $F\alpha O(X,\tau)$, $F\alpha O(\tilde{X},\tilde{\tau})$ and $F\alpha O(Y,\sigma)$. Let $\phi:(\tilde{X},\tilde{\tau})\to(X,\tau)$ be a fuzzy α-ψ^*-covering function and $f:(Y,\sigma)\to(X,\tau)$ be a fuzzy α-ψ^*-continuous function. If a lift of f exists, then*

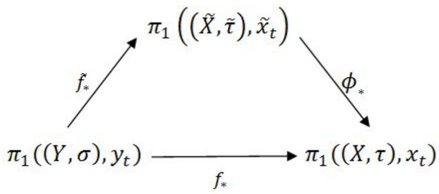
$$f_*(\pi_1((Y,\sigma),y_t))\leq\phi_*(\pi_1((\tilde{X},\tilde{\tau}),\tilde{x}_t))$$

where $\tilde{x}_t\in\mathcal{FP}(\tilde{X})$ and $y_t\in\mathcal{FP}(Y)$.

Proof. Let $\tilde{f}:(Y,\sigma)\to(\tilde{X},\tilde{\tau})$ be a lift of f. Then by Definition 5.7, $f=\phi\circ\tilde{f}$.



This implies that $f_*=(\phi\circ\tilde{f})_*$.



Let us choose $y_t\in\mathcal{FP}(Y)$ such that $\tilde{f}(y_t)=\tilde{x}_t$ where $\tilde{x}_t\in\mathcal{FP}(\tilde{X})$.
Then for $[\gamma]\in\pi_1((Y,\sigma),y_t)$,

$$\begin{aligned}f_*([\gamma])&=(\phi\circ\tilde{f})_*([\gamma])\\&=[\phi\circ\tilde{f}\circ\gamma],\quad\text{by Definition 5.6}\\&=[\phi\circ(\tilde{f}\circ\gamma)]\\&=\phi_*([\tilde{f}\circ\gamma]).\end{aligned}$$

Since $\tilde{f}\circ\gamma$ is a fuzzy α-ψ^*-loop at $\tilde{x}_t$, $\tilde{f}\circ\gamma\in\pi_1((\tilde{X},\tilde{\tau}),\tilde{x}_t)$. This implies that $f_*([\gamma])\in\pi_1((\tilde{X},\tilde{\tau}),\tilde{x}_t)$. Hence $f_*(\pi_1((Y,\sigma),y_t))\leq\phi_*(\pi_1((\tilde{X},\tilde{\tau}),\tilde{x}_t))$

□

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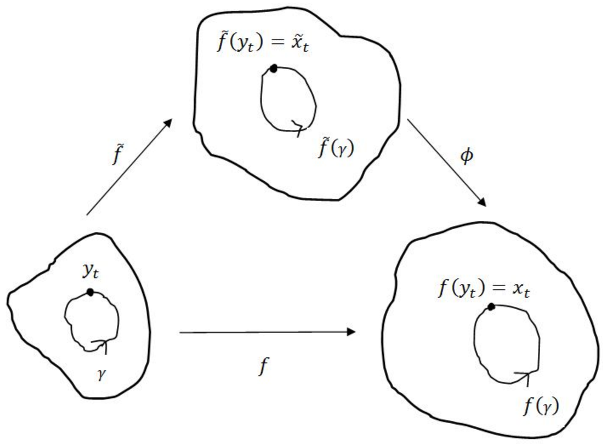

7. CONCLUSION

In this paper, the concepts of fuzzy α-ψ^*-homotopies and fuzzy α-ψ^*-path homotopies are introduced and some of their interesting properties are studied. Also, the concept of α-ψ^*-fundamental group in a fuzzy topological space is established and its role on fuzzy α-ψ^*-homotopy is also discussed. Finally, the notion of fuzzy α-ψ^*-covering spaces is introduced and some of its properties are studied.

References

[1] Chang, C. L., (1968), Fuzzy topological spaces,Journal of Mathematical Analysis and Applications, 24, pp. 182-190.

[2] Culvacioglu, G., and Citil, M., (2006), On fuzzy homotopy theory, Adv. Stud. Contemp. Math. (Kyungshang), 12, 1, pp. 163-166.

[3] Erdal Guner, (2007) Fuzzy contractibilty, Commun. Fac. Sci. Univ. Ank. Series A1, 56, 2, pp. 11-16.

[4] Mario Ferraro and David H. Foster, (1987), Differentiation of fuzzy continuous mappings on fuzzy topological vector spaces, J. Math. Anal. Appl., 121, pp. 589-601.

[5] Massey, W. S., (1967), Algebraic Topology : An Introduction, Harcourt Brace and World, New York.

[6] Munkres, J., (2000), Topology, Second edition, Prentice Hall, New Jersey, U.S.A.

[7] Rowthri, M. and Amudhambigai, B., (2020), Fuzzy α-ψ^*-Irreducible Spaces, International Journal of Recent Technology and Engineering, 9, 1, pp. 208-211.

[8] Salleh, A. R. and Md. Tap, A. O., (1987), The fundamental groupoid of fuzzy topological spaces, Sains Malaysina, 16, 4, pp. 447-454.

[9] Zheng Chong-you, (1984), Fuzzy path and fuzzy connectedness, Fuzzy Sets and Systems, 14, pp. 273-280.

M. Rowthri (M.Sc., M.Phil.), is a PhD research scholar in PG and Research Department of Mathematics in Sri Sarada College for Women(Autonomous), Salem-16, Tamilnadu, India. She is doing research in fuzzy operator on fuzzy topological spaces.

Dr. B. Amudhambigai (M.Sc., MPhil, Ph.D) is working as an Assistant Professor in PG and Research Department of Mathematics at Sri Sarada College for Women(Autonomous), Salem-16, Tamilnadu, India. Her research area includes fuzzy topology, fuzzy Cryptography and Mathematical Modeling.