TWMS J. App. and Eng. Math. V.13, N.3, 2023, pp. 1198-1207

ON THE SOLUTIONS OF SOME RECENT OPEN PROBLEMS IN METRIC FIXED POINT THEORY

N. TAŞ[1], §

Abstract. In this paper, our aim is to provide new solutions to some open problems given in [18] and [27] about the discontinuity at fixed point and related results. For this purpose, we prove new theorems and corollaries using simulation functions and known fixed-point techniques.

Keywords: Discontinuity, fixed point, simulation function, complete metric space.

AMS Subject Classification: 54H25, 54E50, 47H09, 47H10, 54C30, 46T99.

1. Introduction and background

Fixed-point theory is very important in the different areas of mathematics such as applied mathematics, topology, analysis etc. This theory was started with the Banach's contraction principle [1]. This principle has been generalized using the various techniques because there exist some examples of a self-mapping does not satisfy the Banach's contraction principle but has a fixed point. One of these techniques is to generalize the used contractive condition (for example, see [26] and the references therein). Another technique is to generalize the used metric spaces (for example, see [12] and the references therein). Recently, the geometric properties of fixed points have been investigated as a new direction in the fixed-point theory (see [18]). On the other hand, many contractive conditions used in the fixed-point theorems require that a self-mapping is continuous at fixed point. But there are some contractive conditions which do not require that the self-mapping to be continuous. In this context, the following open questions raised by Rhoades [27] and Özgür et al. [18] have been extensively studied, respectively:

Open Question 1: Does there exist a contractive condition which is strong enough to generate a fixed point but which does not force the self-mapping to be continuous at the fixed point?

Open Question 2: What are the geometric properties of fixed points in which case a self-mapping has more than one fixed point?

[1] Balıkesir University, Department of Mathematics, 10145 Balıkesir, Turkey.
e-mail: nihaltas@balikesir.edu.tr; ORCID: http://orcid.org/0000-0002-4535-4019.

§ Manuscript received: April 22, 2021; accepted: August 02, 2021.

Open Question 3: Does there exist a contractive condition that excludes the identity map from the obtained fixed-point (resp. fixed-circle, fixed-disc) results?

Many authors have investigated new solutions using various approaches to Open Question 1. For example, in [23], Pant obtained a solution using the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and the number defined as

$$m(a,b) = \max\{d(a,fa), d(b,fb)\}.$$

Theorem 1.1. [23] *Let (X,d) be a complete metric space and $f : X \to X$ be a self-mapping such that*

(i) $d(fa,fb) \leq \phi(m(a,b))$,

(ii) Given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varepsilon < m(a,b) < \varepsilon + \delta \Longrightarrow d(fa,fb) \leq \varepsilon.$$

Then f has a unique fixed point z in X and f is continuous at z if and only if $\lim\limits_{a\to z} m(a,z) = 0$.

After then, various solutions to Open Question 1 have been given (see [2, 3, 4, 5, 6, 7, 22, 24, 28, 30] for more details). These kind results are gained great importance because there exist some applications of the obtained solutions to some other areas such as biology, discontinuous neural networks, simulation functions etc. (see [10, 11, 24, 28, 29]).

More recently, the fixed-circle (or fixed-disc) problem has been studied using some classical fixed-point techniques related to Open Question 2. For example, Özgür and Taş obtained a solution to this question using Caristi's inequality (see [8]) as follows:

Theorem 1.2. [18] *Let (X,d) be a metric space and $C_{a_0,r} = \{a \in X : d(a,a_0) = r\}$ be any circle on X. Let us define the mapping $\varphi : X \to [0,\infty)$ such that $\varphi(a) = d(a,a_0)$ for all $a \in X$. If there exists a self-mapping $f : X \to X$ satisfying*

$(C1)$ $d(a,fa) \leq \varphi(a) - \varphi(fa)$,

$(C2)$ $d(fa,a_0) \geq r$ for each $a \in C_{a_0,r}$,

then the circle $C_{a_0,r}$ is a fixed circle of f.

Motivated by this fact, this question has been studied as a geometric approach to the generalization of fixed-point theory. For this purpose, some known techniques used in fixed-point theorems are adapted to the fixed-circle problem on metric spaces and some generalized metric spaces (see [18, 19, 20, 24]). The fixed-circle problem has brought a new light to the fixed-point theory and geometric thinking.

Some contractive conditions have been introduced to exclude the identity map from the fixed-circle theorems related to Open Question 3 (see [18, 19]).

In this paper, our aim is to give new solutions to the above open questions. For this reason, we present a new technique using the family of simulation functions defined in [17]. The function $\zeta : [0,\infty)^2 \to \mathbb{R}$ is said to be a simulation function, if the followings hold:

(ζ_1) $\zeta(0,0) = 0$,

(ζ_2) $\zeta(t,s) < s - t$ for all $s,t > 0$,

(ζ_3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \zeta(t_n,s_n) < 0.$$

The set of all simulation functions is denoted by $\mathcal{Z}$ [17]. Using this set, many fixed-point results was obtained with various approaches (see [9, 13, 16, 17, 21, 25] and the references therein).

2. Main results

At first, using the set $\mathcal{Z}$, we present some new solutions to Open Question 1.

Let (X, d) be a complete metric space and $f : X \to X$ be a self-mapping in the whole paper unless otherwise stated.

Theorem 2.1. *If the following conditions hold*

(i) *Given* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$\varepsilon < m(a, b) < \varepsilon + \delta \Longrightarrow d(fa, fb) \leq \varepsilon,$$

(ii) $\zeta\left(d\left(fa, fb\right), m(a, b)\right) \geq 0,$

for all $a, b \in X$*, then* f *has a fixed point* x *and the sequence* $\{f^n a\}$ *for each* $a \in X$ *converges to the fixed point* x*. Also* f *is discontinuous at* x *if and only if* $\lim\limits_{a \to x} m(a, x) \neq 0$.

Proof. Let a_0 be any point in X. Let us define a sequence $\{a_n\}$ in X as

$$a_n = fa_{n-1} = f^n a_0.$$

If $a_n = a_{n+1}$ for some n then we have

$$a_n = a_{n+1} = a_{n+2} = a_{n+3} = \dots .$$

So $\{a_n\}$ is a Cauchy sequence and a_n is a fixed point of f. Hence we suppose $a_n \neq a_{n+1}$ for each n. Then we get

$$m(a_{n-1}, a_n) = \max\left\{d(a_{n-1}, fa_{n-1}), d(a_n, fa_n)\right\} = \max\left\{d(a_{n-1}, a_n), d(a_n, a_{n+1})\right\}$$

and so using the condition (ii), we have

$$0 \leq \zeta\left(d\left(fa_{n-1}, fa_n\right), m(a_{n-1}, a_n)\right). \tag{1}$$

Let us consider the following cases:

Case 1. Let $d(a_{n-1}, a_n) < d(a_n, a_{n+1})$. Using the inequality (1) and the condition (ζ_2), we obtain

$$0 \leq \zeta\left(d(a_n, a_{n+1}), d(a_n, a_{n+1})\right) < d(a_n, a_{n+1}) - d(a_n, a_{n+1}) = 0,$$

a contradiction.

Case 2. Let $d(a_{n-1}, a_n) > d(a_n, a_{n+1})$. Using the inequality (1) and the condition (ζ_2), we get

$$0 \leq \zeta\left(d(a_n, a_{n+1}), d(a_{n-1}, a_n)\right) < d(a_{n-1}, a_n) - d(a_n, a_{n+1})$$

and so

$$d(a_n, a_{n+1}) < d(a_{n-1}, a_n),$$

that is, $\{d(a_n, a_{n+1})\}$ is a strictly decreasing sequence of positive numbers and it tends to a limit $w \geq 0$. We assert $w = 0$. On the contrary, we assume $w > 0$. Therefore, there exists a positive integer n_0 with $n \geq n_0$ such that

$$w < d(a_n, a_{n+1}) < w + \delta(w), \tag{2}$$

or equivalently

$$w < m(a_n, a_{n+1}) < w + \delta(w).$$

From the condition (i), we get

$$d(fa_n, fa_{n+1}) \leq w,$$

contradicts with the inequality (2). Hence it should be $w = 0$, that is, $d(a_n, a_{n+1}) \to 0$ as $n \to \infty$. Now we prove that $\{a_n\}$ is a Cauchy sequence. To do this, let us fix an $\varepsilon > 0$. Without loss of generality, we suppose $\delta(\varepsilon) < \varepsilon$. Since $d(a_n, a_{n+1}) \to 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(a_n, a_{n+1}) < \delta,$$

for $n \geq n_0$. Using the mathematical induction and the Jachymski's technique (see [14, 15]), we show that

$$d(a_k, a_{k+n}) < \varepsilon + \delta. \tag{3}$$

The inequality (3) holds for $n = 1$. We suppose that the inequality (3) is true for some n and we prove it for $n + 1$. Using the triangle inequality, we get

$$d(a_k, a_{k+n+1}) \leq d(a_k, a_{k+1}) + d(a_{k+1}, a_{k+n+1}).$$

It is enough to show that $d(a_{k+1}, a_{k+n+1}) \leq \varepsilon$. To do this, we prove that $m(a_k, a_{k+n}) \leq \varepsilon + \delta$, where

$$m(a_k, a_{k+n}) = \max\{d(a_k, fa_k), d(a_{k+n}, fa_{k+n})\}.$$

From the mathematical induction hypothesis, we get

$$d(a_k, a_{k+1}) < \varepsilon + \delta \text{ and } d(a_{k+n}, a_{k+n+1}) < \delta.$$

Hence $m(a_k, a_{k+n}) < \varepsilon + \delta$ and so by the condition (i), we have $d(a_{k+1}, a_{k+n+1}) \leq \varepsilon$. We have completed the induction. Therefore, $\{a_n\}$ is a Cauchy sequence. From the completeness hypothesis, there exists a point $x \in X$ such that $a_n \to x$ as $n \to \infty$ and also $fa_n \to x$ as $n \to \infty$. Now we claim that $fx = x$. If not, using the conditions (ii), (ζ_2) and (ζ_3), we get

$$m(a_n, x) = \max\{d(a_n, fa_n), d(x, fx)\} = \beta$$

and

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta\left(d(fa_n, fx), m(a_n, x)\right) = \limsup_{n\to\infty} \zeta\left(d(fa_n, fx), \beta\right) \\ &< \limsup_{n\to\infty} [\beta - d(a_{n+1}, fx)] = 0, \end{aligned}$$

a contradiction. So it should be $fx = x$, that is, x is a fixed point of f. To prove the last part of this theorem, we show that f is continuous at x if and only if $\lim_{a\to x} m(a, x) = 0$. Suppose that f is continuous at the fixed point x and $a_n \to x$ as $n \to \infty$. By the continuity of f, we have $fa_n \to fx = x$ and by the triangle inequality, we get

$$d(a_n, fa_n) \leq d(a_n, x) + d(x, fa_n) \to 0 \text{ as } n \to \infty.$$

Then we get $\lim_{n\to\infty} m(a_n, x) = 0$. Conversely, if $\lim_{a_n\to x} m(a_n, x) = 0$ then $d(a_n, fa_n) \to 0$ as $a_n \to x$. This implies $fa_n \to x = fx$, that is, f is continuous at the fixed point x. □

Using the self-mapping f defined as in Example 1 given in [23], we give the following example.

Example 2.1. *Let $X = [0, 2]$ be the complete metric space with the usual metric d defined as $d(a, b) = |a - b|$ for all $a, b \in X$. Let us define the self-mapping*

$$fa = \begin{cases} 1 & ; \quad a \leq 1 \\ 0 & ; \quad a > 1 \end{cases},$$

for all $a \in X$. Then f satisfies the conditions of Theorem 2.1 with

$$\delta(\varepsilon) = \begin{cases} 1 & ; \quad \varepsilon \geq 1 \\ 1 - \varepsilon & ; \quad \varepsilon < 1 \end{cases}$$

and

$$\zeta(t, s) = \begin{cases} 0 & ; \quad t, s \geq 1 \\ \frac{2}{3}s - t & ; \quad \text{otherwise} \end{cases}.$$

Consequently, f has a fixed point $a = 1$ and f is discontinuous at the point $a = 1$ if and only if $\lim_{a\to 1} m(a, 1) \neq 0$.

Assume that the self-mapping f satisfies the condition (i) of Theorem 2.1 in all of the following corollaries. Then we get the following results.

Corollary 2.1. *If the following condition holds*

$$d(fa, fb) \leq \lambda m(a, b),\ \lambda \in [0, 1) \tag{4}$$

for all $a, b \in X$, then f has a fixed point x and the sequence $\{f^n a\}$ for each $a \in X$ converges to the fixed point x. Also f is discontinuous at x if and only if $\lim_{a \to x} m(a, x) \neq 0$.

Proof. Let us define the mapping $\zeta_1 : [0, \infty)^2 \to \mathbb{R}$ by

$$\zeta_1(t, s) = \lambda s - t,$$

for all $s, t \in [0, \infty)$ (see Corollary 2.10 given in [17]). Then the self-mapping f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_1 \in \mathcal{Z}$. Therefore, the proof can be easily obtained by taking $\zeta = \zeta_1$ in Theorem 2.1. □

Corollary 2.2. *If the following condition holds*

$$d(fa, fb) \leq m(a, b) - \varphi(m(a, b)),$$

for all $a, b \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is lower semi continuous function and $\varphi^{-1}(0) = \{0\}$, then f has a fixed point x and the sequence $\{f^n a\}$ for each $a \in X$ converges to the fixed point x. Also f is discontinuous at x if and only if $\lim_{a \to x} m(a, x) \neq 0$.

Proof. Let us define the mapping $\zeta_2 : [0, \infty)^2 \to \mathbb{R}$ by

$$\zeta_2(t, s) = s - \varphi(s) - t,$$

for all $s, t \in [0, \infty)$ (see Corollary 2.11 given in [17]). Then the self-mapping f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_2 \in \mathcal{Z}$. Therefore, the proof can be easily seen by taking $\zeta = \zeta_2$ in Theorem 2.1. □

Corollary 2.3. *If the following condition holds*

$$d(fa, fb) \leq \varphi(m(a, b))m(a, b),$$

for all $a, b \in X$, where $\varphi : [0, \infty) \to [0, 1)$ is a mapping such that $\limsup_{t \to r^+} \varphi(t) < 1$ for all $r > 0$, then f has a fixed point x and the sequence $\{f^n a\}$ for each $a \in X$ converges to the fixed point x. Also f is discontinuous at x if and only if $\lim_{a \to x} m(a, x) \neq 0$.

Proof. Let us define the mapping $\zeta_3 : [0, \infty)^2 \to \mathbb{R}$ by

$$\zeta_3(t, s) = s\varphi(s) - t,$$

for all $s, t \in [0, \infty)$ (see Corollary 2.13 given in [17]). Then the self-mapping f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_3 \in \mathcal{Z}$. Therefore, the proof can be easily seen by taking $\zeta = \zeta_3$ in Theorem 2.1. □

Corollary 2.4. *If the following condition holds*

$$d(fa, fb) \leq \eta(m(a, b)),$$

for all $a, b \in X$, where $\eta : [0, \infty) \to [0, \infty)$ is an upper semi continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, then f has a fixed point x and the sequence $\{f^n a\}$ for each $a \in X$ converges to the fixed point x. Also f is discontinuous at x if and only if $\lim_{a \to x} m(a, x) \neq 0$.

Proof. Let us define the mapping $\zeta_4 : [0,\infty)^2 \to \mathbb{R}$ by

$$\zeta_4(t,s) = \eta(s) - t,$$

for all $s,t \in [0,\infty)$ (see Corollary 2.14 given in [17]). Then the self-mapping f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_4 \in \mathcal{Z}$. Therefore, the proof can be easily seen by taking $\zeta = \zeta_4$ in Theorem 2.1. □

Corollary 2.5. *If the following condition holds*

$$\int_0^{d(fa,fb)} \phi(u)du \leq m(a,b),$$

for all $a,b \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^{\varepsilon} \phi(u)du$ exists and $\int_0^{\varepsilon} \phi(u)du > \varepsilon$, for each $\varepsilon > 0$, then f has a fixed point x and the sequence $\{f^n a\}$ for each $a \in X$ converges to the fixed point x. Also f is discontinuous at x if and only if $\lim_{a\to x} m(a,x) \neq 0$.

Proof. Let us define the mapping $\zeta_5 : [0,\infty)^2 \to \mathbb{R}$ by

$$\zeta_5(t,s) = s - \int_0^{t} \phi(u)du,$$

for all $s,t \in [0,\infty)$ (see Corollary 2.15 given in [17]). Then the self-mapping f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_5 \in \mathcal{Z}$. Therefore, the proof can be easily seen by taking $\zeta = \zeta_5$ in Theorem 2.1. □

Remark 2.1. 1) *If a self-mapping f satisfies the inequality (4) of Corollary 2.1, then f satisfies the condition (iii) of Theorem* 1 *given in* [23].

2) *Notice that Corollary 2.4 can be considered same as Theorem* 2.6 *given in* [22].

3) *The above results are the new solutions of Open Question 1.*

Following Corollary 2.3 given in [2], we get the following corollary.

Corollary 2.6. *If the following conditions hold*

(i) *Given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < t < \varepsilon + \delta \Longrightarrow \zeta(t,\varepsilon) \geq 0,$$

for any $t > 0$,

(ii) $\zeta\left(d\left(fa,fb\right), d(a,b)\right) \geq 0$ *for all $a,b \in X$,*

then f has a fixed point x and the sequence $\{f^n a\}$ for each $a \in X$ converges to the fixed point x.

Proof. By the similar arguments used in the proof of Theorem 2.1 and the properties of a simulation function, the proof can be easily seen. □

Following Theorem 2.9 given in [22], we give the following theorem using the set $\mathcal{Z}$:

Theorem 2.2. *Let f be a continuous self-mapping of X and the following conditions hold*

(i) *Given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < m(a,b) < \varepsilon + \delta \Longrightarrow d(fa,fb) \leq \varepsilon,$$

(ii) $\zeta\left(d\left(fa,fb\right), m(a,b)\right) \geq 0$ *for all $a,b \in X$,*

(iii) $\zeta\left(d\left(fa,f^2a\right),m(a,fa)\right)\geq 0$ *for all* $a\in X$,
(iv) $\inf\{m(a,b):a,b\in X\}<M$*, where* $M=\sup\{m(a,b):a,b\in X\}$*, that is,* $m(a,b)$ *is not constant on* $X\times X$.
If $\delta(\varepsilon)$ *is continuous in* $(0,M)$ *then* f *has a fixed point* x.

Proof. Let $b\in X$. By the conditions (ii), (iii) and (ζ_2), we get

$$\begin{aligned} 0 &\leq \zeta(d(f^nb,f^{n+1}b),m(f^{n-1}b,f^nb))=\zeta(d(f^nb,f^{n+1}b),d(f^{n-1}b,f^nb)) \\ &< d(f^{n-1}b,f^nb)-d(f^nb,f^{n+1}b), \end{aligned}$$

whence

$$d(f^nb,f^{n+1}b)<d(f^{n-1}b,f^nb)<d(f^{n-2}b,f^{n-1}b)<\ldots<d(b,fb), \tag{5}$$

that is, $\{d(f^nb,f^{n+1}b)\}$ is a strictly decreasing sequence of positive numbers. By the condition (iv), there exist $x,y\in X$ such that $m(x,y)<M$, that is, $d(x,fx)<M$ and $d(y,fy)<M$. Using the inequality (5), we get

$$d(f^nx,f^{n+1}x)<d(f^{n-1}x,f^nx)<\ldots<d(x,fx)<M.$$

If $f^nx=f^{n+1}x$ for some n then f^nx is a fixed point of f. Therefore we can suppose that $f^nx\neq f^{n+1}x$ for each n. Since $\{d(f^nx,f^{n+1}x)\}$ is a strictly decreasing sequence of positive real numbers, it converges to a real number w such that $0\leq w<M$. By the similar arguments used in the proof of Theorem 2.1, we have $w=0$ and also $\{f^nx\}$ is Cauchy. From the completeness hypothesis, there exists a point $u\in X$ such that $\lim\limits_{n\to\infty}f^nx=u$. By the continuity of f, we have $\lim\limits_{n\to\infty}f(f^nx)=fu$, that is, $\lim\limits_{n\to\infty}f^nx=fu$. So we obtain $fu=u$, that is, u is a fixed point of f. □

Remark 2.2. *It can be also investigated new results on discontinuity at the fixed point using the continuity of* f^2 *(resp. the continuity of* f^p *or the orbitally continuity of* f*) instead of the continuity condition given in Theorem 2.2.*

Let the number $m^*(a,b)$ be defined as follows:

$$m^*(a,b)=\max\left\{d(a,b),d(a,fa),d(b,fb),\frac{d(a,fb)+d(b,fa)}{2}\right\}.$$

Using this number and simulation functions, various types of fixed-point theorems were obtained (for example, see [21]). Also, using this number, a solution was given to Open Question 1 in [2]. Now we get a new solution to Open Question 1 using this number and the set of simulation functions.

Theorem 2.3. *If the following conditions hold*
(i) Given $\varepsilon>0$ *there exists a* $\delta(\varepsilon)>0$ *such that*

$$\varepsilon<m^*(a,b)<\varepsilon+\delta\Longrightarrow d(fa,fb)\leq\varepsilon,$$

(ii) $\zeta\left(d\left(fa,fb\right),m^*(a,b)\right)\geq 0$,
for all $a,b\in X$*, then* f *has a unique fixed point* x *and the sequence* $\{f^na\}$ *for each* $a\in X$ *converges to the fixed point* x*. Also* f *is discontinuous at* x *if and only if* $\lim\limits_{a\to x}m^*(a,x)\neq 0$.

Proof. By the similar arguments used in the proof of Theorem 2.1, it is easy to see that f has a fixed point x. Now we show that the fixed point x is unique. To do this, we suppose that y is another fixed point of f such that $x\neq y$. From the conditions (ii), (ζ_2) and the symmetry property, we have

$$\begin{aligned} m^*(x,y) &= \max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\} \\ &= d(x,y) \end{aligned}$$

and

$$0 \leq \zeta\left(d\left(fx, fy\right), m^*(x, y)\right) = \zeta\left(d\left(x, y\right), d(x, y)\right) < d(x, y) - d(x, y) = 0,$$

a contradiction. So we get $x = y$, that is, x is a unique fixed point of f. The last part of this theorem is clear. □

Example 2.2. *If we consider the metric space (X, d) and the self-mapping f given in Example 2.1, then f satisfies the conditions Theorem 2.3. Consequently, f has a fixed point $a = 1$ and f is discontinuous at the point $a = 1$ if and only if $\lim_{a \to 1} m^*(a, 1) \neq 0$.*

In [20], a new solution was given using the number $m^*(a, b)$ and the set of simulation functions on metric spaces to Open Question 2. In [20], Open Question 3 has been left for the obtained fixed-disc results. To obtain a solution to this question, we use the number $m(a, b)$ and the set of simulation functions. Therefore, we exclude the identity map I_X ($I_X : X \to X$ is a function defined by $I_X(x) = x$ for all $x \in X$) from the obtained fixed-disc results in [20].

Theorem 2.4. *Let (X, d) be a metric space and $f : X \to X$ be a self-mapping. There exist $a_0 \in X$ and $\zeta \in \mathcal{Z}$ such that*

$$d(a, fa) > 0 \Longrightarrow \zeta(d(a, fa), m(a, a_0)) \geq 0, \tag{6}$$

for all $a \in X$ if and only if $f = I_X$.

Proof. At first, we show that a_0 is a fixed point of f. Let $a_0 \neq fa_0$. Then we have $d(a_0, fa_0) > 0$. By the conditions (6) and (ζ_2), we get

$$\begin{aligned} 0 &\leq \zeta(d(a_0, fa_0), m(a_0, a_0)) = \zeta(d(a_0, fa_0), d(a_0, fa_0)) \\ &< d(a_0, fa_0) - d(a_0, fa_0) = 0, \end{aligned}$$

a contradiction. It should be

$$a_0 = fa_0. \tag{7}$$

Let us take $a \in X - \{a_0\}$ and $a \neq fa$. Hence we obtain $d(a, fa) > 0$ and so using the conditions (6), (ζ_2) and the equality (7), we find

$$\begin{aligned} 0 &\leq \zeta(d(a, fa), m(a, a_0)) = \zeta(d(a, fa), d(a, fa)) \\ &< d(a, fa) - d(a, fa) = 0, \end{aligned}$$

a contradiction. Therefore, we have $a = fa$. Consequently, we get $f = I_X$. The converse statement of this theorem can be easily proved. □

3. Conclusion

In the present paper, some solutions are given to the above mentioned open questions using the set of simulation functions. By similar approaches, new solutions can be derived to these open questions on metric and some generalized metric spaces. Also, some applications of the obtained results can be investigated in other research areas such as discontinuous activation functions, integral equations etc.

4. Acknowledgment

The author would like to thank the anonymous referees for their comments that helped us improve this article.

References

[1] Banach, S., (1922), Sur les operations dans les ensembles abstrait et leur application aux equations integrals, Fundam. Math., 3, 133-181.

[2] Bisht, R. K., Pant, R. P., (2017), A remark on discontinuity at fixed point, J. Math. Anal. Appl., 445(2), 1239-1242.

[3] Bisht, R. K., Pant, R.P., (2017), Contractive definitions and discontinuity at fixed point, Appl. Gen. Topol. 18(1), 173-182.

[4] Bisht, R. K., Hussain, N., (2017), A note on convex contraction mappings and discontinuity at fixed point, J. Math. Anal., 8(4), 90-96.

[5] Bisht, R. K., (2017), On power contractions and discontinuity at the fixed point, Math. Stud., 86(1-2), 31-37.

[6] Bisht, R. K., Rakočević, V., (2018), Generalized Meir-Keeler type contractions and discontinuity at fixed point, Fixed Point Theory, 19(1), 57-64.

[7] Bisht, R. K., Rakočević, V., (2018), Fixed points of convex and generalized convex contractions, Rend. Circ. Mat. Palermo, II. Ser.

[8] Caristi, J., (1976), Fixed point theorems for mappings satisfying inwardness conditions, Trans. Am. Math. Soc., 215, 241-251.

[9] Chanda, A., Dey, L.K., Radenović, S., (2018), Simulation functions: a survey of recent results. RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.

[10] Cromme, L.J., Diener, I., (1991), Fixed point theorems for discontinuous mapping, Math. Program., 51(1-3), 257-267.

[11] Cromme, L,J., (1997), Fixed point theorems for discontinuous functions and applications, Nonlinear Anal., Theory Methods Appl., 30(3), 1527-1534.

[12] Dosenovic, T., Radenovic, S., Sedghi, S., (2018), Generalized metric spaces: survey, TWMS J. Pure Appl. Math., 9(1), 3-17.

[13] Felhi, A., Aydi, H., Zhang, D., (2016), Fixed points for α-admissible contractive mappings via simulation functions, J. Nonlinear Sci. Appl., 9, 5544-5560.

[14] Jachymski, J., (1994), Common fixed point theorems for some families of maps, Indian J. Pure Appl. Math., 25(9), 925-937.

[15] Jachymski, J., (1995), Equivalent conditions and Meir-Keeler type theorems, J. Math. Anal. Appl., 194(1), 293-303.

[16] Karapınar, E., (2016), Fixed points results via simulation functions, Filomat, 30(8), 2343-2350.

[17] Khojasteh, F., Shukla, S., Radenović, S., (2015), A new approach to the study of fixed point theory for simulation functions, Filomat, 29(6), 1189-1194.

[18] Özgür, N. Y., Taş, N., (2019), Some fixed-circle theorems on metric spaces, Bull. Malays. Math. Sci. Soc., 42, 1433-1449.

[19] Özgür, N. Y., Taş, N., (2019), Fixed-circle problem on S-metric spaces with a geometric viewpoint, Facta Univ., Ser. Math. Inf., 34(3), 459-472.

[20] Özgür, N. Y., (2019), Fixed-disc results via simulation functions, Turk. J. Math., 43, 2794-2805.

[21] Padcharoen, A., Kumam, P., Saipara, P., Chaipunya, P., (2018), Generalized Suzuki type $\mathcal{Z}$-contraction in complete metric spaces, Kragujevac J. Math., 42(3), 419-430.

[22] Pant, A., Pant, R. P., (2017), Fixed points and continuity of contractive maps, Filomat, 31(11), 3501-3506.

[23] Pant, R. P., (1999), Discontinuity and fixed points, J. Math. Anal. Appl., 240(1), 284-289.

[24] Pant, R. P., Özgür, N. Y., Taş, N., (2020), On discontinuity problem at fixed point, Bull. Malays. Math. Sci. Soc., 43, 499-517.

[25] Radenovic, S., Vetro, F., Vujaković, J., (2017), An alternative and easy approach to fixed point results via simulation functions, Demonstr. Math., 50, 223-230.

[26] Rhoades, B. E., (1977), A comparison of various definitions of contractive mappings, Trans. Am. Math. Soc., 226, 257-290.

[27] Rhoades, B. E., (1988), Contractive definitions and continuity, Contemp. Math., 72, 233-245.

[28] Taş, N., Özgür, N. Y., (2019), A new contribution to discontinuity at fixed point, Fixed Point Theory, 20(2), 715-728.

[29] Todd, M. J., (1976), The computation of fixed points and applications, Berlin, Heidelberg, New York: Springer-Verlag.

[30] Zheng, D., Wang, P., (2017), Weak θ-ϕ-contraction and discontinuity, J. Nonlinear Sci. Appl. 10(5), 2318-2323.

Nihal Taş graduated from Ege University. She is currently working as an associate professor at Balıkesir University in Turkey. Her research interests include topological spaces, soft set theory, fixed point theory and fixed circle problem.