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COINCIDENCE POINT RESULTS FOR HYBRID PAIR OF MAPPINGS ON A DISLOCATED METRIC SPACE ENDOWED WITH AN ARBITRARY BINARY RELATION

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ABSTRACT. We introduce the notion of generalized S-contraction of Nadler type and prove some new coincidence point results for hybrid pair of mappings in 0-complete dislocated metric spaces endowed with an arbitrary binary relation. Our results generalize, extend and unify several well known comparable results including Nadler's fixed point theorem in the setting of dislocated metric spaces. As some applications of our main result, we can obtain several important fixed point results of multi-valued mappings satisfying some contractive type conditions of Kannan type, Fisher type etc. in this new framework. Moreover, we provide some examples to justify that the generalization is proper.

Keywords: dislocated metric, 0-completeness, generalized S-contractions, coincidence point.

AMS Subject Classification: 54H25, 47H10.

1. INTRODUCTION

Banach contraction principle [9] is one of the most impressing results in fixed point theory. Because of its simplicity and usefulness it has become a popular tool for solving various problems in nonlinear analysis. Several authors successfully extended this celebrated result in diverse ways. In 1994, Matthews [25] gave the concept of a partial metric space while studying denotational semantics of data flow networks and proved the well known Banach contraction principle in this setting. Complete partial metric space is a useful framework to model several complex problems in theory of computation. The works of [3, 5, 12, 18, 19, 21, 31] are viable and have opened new avenues for application in different fields of mathematics and applied sciences. In recent investigations, the study of fixed point theory for multi-valued mappings takes a vital role in many aspects. In this context, Nadler [28] proved that every multi-valued contraction on a complete metric space has a fixed point. Since then, many authors including Gordji [16], Berinde [10], Pathak [29] and

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others [13, 14, 27] studied lots of different types of fixed point theorems for single-valued and multi-valued contractions. In 2000, Hitzler and Seda [20] introduced the concept of dislocated metric space as a generalization of partial metric space and obtained an important characterization of the Banach contraction principle. After that, Amini-Harandi [17] initiated the notion of metric-like spaces. Karapinar *et al.*[22] noticed that the notions of dislocated metric spaces and metric-like spaces are exactly the same. Subsequently, E. Karapinar and others [1, 2, 15, 23, 24] considered Sehgal type contractions, F-contractive mappings, conditionally F-contractions and established some fixed point results in dislocated metric spaces. The study of fixed point theory combining a binary relation is a new development in the domain of contractive type multi-valued theory. The aim of this article is to introduce the notion of generalized S-contraction of Nadler type involving an arbitrary binary relation and obtain some new coincidence point results for such class of functions in dislocated metric spaces.

2. Preliminaries

Let (X, d) be a metric space, $CL(X)$ be the family of all nonempty closed subsets of X and $CB(X)$ be the family of all nonempty closed and bounded subsets of X. For A, $B \in CB(X)$, define $H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$, where $d(x, B) = \inf\{d(x, y) : y \in B\}$. Then H is called Pompeiu-Hausdorff metric on $CB(X)$. Let $T : X \to CB(X)$ be a multi-valued mapping. If there exists $\lambda \in (0, 1)$ such that $H(Tx, Ty) \leq \lambda\, d(x, y)$ for all x, $y \in X$, then T is called a multi-valued contraction. Detailed information about the Pompeiu-Hausdorff metric can be found in [11]. In this section, we recall some basic definitions, notations and crucial results in dislocated metric spaces.

Definition 2.1. [20] *Let X be a nonempty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a dislocated metric (or a metric-like) on X if for any x, y, $z \in X$, the following conditions hold:*

(σ_1) $\sigma(x, y) = 0 \Longrightarrow x = y$;
(σ_2) $\sigma(x, y) = \sigma(y, x)$;
(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is then called a dislocated metric (or metric-like) space.

It is valuable to note that a partial metric [25] is also a dislocated metric but the converse is not true, in general. A trivial example of a dislocated metric which is also a partial metric is given by $\sigma(x, y) = \max\{x, y\}$, for all x, $y \geq 0$.

The contrary situation can be illustrated by the following example.

Example 2.1. [7] *Let $X = \{1, 2, 3\}$ and consider the dislocated metric $\sigma : X\times X \to [0, \infty)$ given by*

$$\sigma(1, 1) = 0,\ \sigma(2, 2) = 1,\ \sigma(3, 3) = \frac{2}{3},\ \sigma(1, 2) = \sigma(2, 1) = \frac{9}{10},$$
$$\sigma(2, 3) = \sigma(3, 2) = \frac{4}{5},\ \sigma(1, 3) = \sigma(3, 1) = \frac{7}{10}.$$

Since $\sigma(2, 2) \neq 0$, σ is not a metric and since $\sigma(2, 2) > \sigma(1, 2)$, σ is not a partial metric.

In a dislocated metric space (X, σ), we define an open σ-ball $B_\sigma(x, r)$ for $x \in X$ and $r > 0$ as follows:

$$B_\sigma(x, r) = \{y \in X :\mid \sigma(x, y) - \sigma(x, x) \mid< r\}.$$

We now visualise the open σ-balls in a particular case.

Example 2.2. *Let* $X := [0,1]$ *and* $\sigma(x,y) = x + y$ *on* X*. Then* (X,σ) *is a dislocated metric space. In this case, for* $a \in X,\ r > 0$*, we have* $B_\sigma(a,r) = (a-r, a+r) \cap X$*.*

Definition 2.2. [22] *Let* (X,σ) *be a dislocated metric space. A subset* $A \subseteq X$ *is said to be* σ*-open if for any* $a \in A$*, there exists* $\epsilon > 0$ *such that* $B_\sigma(a,\epsilon) \subseteq A$*. Also,* $C \subseteq X$ *is a* σ*-closed subset of* X *if* $X \setminus C$ *is a* σ*-open subset of* X*. The family of all* σ*-open subsets of* X *will be denoted by* τ_σ*.*

Theorem 2.1. τ_σ *defines a topology on* (X,σ)*.*

Remark 2.1. *It is worth mentioning that the open* σ*-balls are not necessarily* σ*-open sets and the topology* τ_σ *is not even* T_0*. Also, the collection of open* σ*-balls may not form a base for a topology on* X*.*

We cite an example in support of the above remark.

Example 2.3. *Let* $X = \{a,b,c,d\}$ *and* $\sigma : X \times X \to [0,\infty)$ *be given by*

$$\sigma(a,a) = \sigma(b,b) = 5,\ \sigma(c,c) = \sigma(d,d) = 6,\ \sigma(a,b) = \sigma(b,a) = 5,\ \sigma(b,c) = \sigma(c,b) = 4,$$

$$\sigma(c,d) = \sigma(d,c) = 6,\ \sigma(a,d) = \sigma(d,a) = 7,\ \sigma(b,d) = \sigma(d,b) = 7,\ \sigma(a,c) = \sigma(c,a) = 4.$$

Then, (X,σ) *is a dislocated metric space.*

The open σ*-balls centred at* a *and radius* $\epsilon > 0$ *are as follows:*

$$B_\sigma(a,\epsilon) = \begin{cases} \{a,b\}, & if \quad 0 < \epsilon \leq 1, \\ \{a,b,c\}, & if \quad 1 < \epsilon \leq 2, \\ \{a,b,c,d\}, & if \quad \epsilon > 2. \end{cases}$$

The open σ*-balls centred at* b *and radius* $\epsilon > 0$ *are as follows:*

$$B_\sigma(b,\epsilon) = \begin{cases} \{a,b\}, & if \quad 0 < \epsilon \leq 1, \\ \{a,b,c\}, & if \quad 1 < \epsilon \leq 2, \\ \{a,b,c,d\}, & if \quad \epsilon > 2. \end{cases}$$

The open σ*-balls centred at* c *and radius* $\epsilon > 0$ *are as follows:*

$$B_\sigma(c,\epsilon) = \begin{cases} \{c,d\}, & if \quad 0 < \epsilon \leq 2, \\ \{a,b,c,d\}, & if \quad \epsilon > 2. \end{cases}$$

The open σ*-balls centred at* d *and radius* $\epsilon > 0$ *are as follows:*

$$B_\sigma(d,\epsilon) = \begin{cases} \{c,d\}, & if \quad 0 < \epsilon \leq 1, \\ \{a,b,c,d\}, & if \quad \epsilon > 1. \end{cases}$$

Therefore, the family of all σ*-open subsets of* X *are*

$$\tau_\sigma = \{\emptyset, \{a,b\}, \{c,d\}, \{a,b,c,d\}\}.$$

Moreover, we see that the open σ*-balls are not* σ*-open sets as the open* σ*-ball* $B_\sigma(a,\frac{3}{2}) = \{a,b,c\}$ *is not a* σ*-open set. Also, for the pair* a*,* b*, there exists no* σ*-open set containing one of the points but not containing the other. This ensures that the topology* τ_σ *is not even* T_0*.*

Finally, we note that the collection of open σ*-balls may not form a base for a topology on* X*. For instance, we consider the open* σ*-balls* $B_\sigma(a,2) = \{a,\ b,\ c\},\ B_\sigma(b,3) = \{a,\ b,\ c,\ d\}$*.*

Then, $c \in B_\sigma(a,2) \cap B_\sigma(b,3)$ but there exists no $\epsilon > 0$ such that $B_\sigma(c,\epsilon) \subset B_\sigma(a,2) \cap B_\sigma(b,3)$.

Remark 2.2. *Let (X,σ) be a dislocated metric space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(w.r.t.) τ_σ if $\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x)$.*

Suppose that $\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x)$. We shall show that $x_n \to x$ w.r.t. τ_σ. Let $U \in \tau_\sigma$ and $x \in U$. Then there exists $\epsilon > 0$ such that $B_\sigma(x,\epsilon) \subseteq U$. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $\mid \sigma(x_n,x) - \sigma(x,x) \mid < \epsilon$ for all $n \geq n_0$. This ensures that $x_n \in B_\sigma(x,\epsilon)$ for all $n \geq n_0$ and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_σ on X.

Definition 2.3. [6] *Let (X,σ) be a dislocated metric space and let (x_n) be a sequence in X. Then*

- (i) *(x_n) converges to a point $x \in X$ if $\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x)$. This will be denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.*
- (ii) *(x_n) is called a σ-Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is finite.*
- (iii) *(X,σ) is said to be complete if for each σ-Cauchy sequence (x_n) in X, there is some $x \in X$ such that $\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n,x_m)$.*

Definition 2.4. *A sequence (x_n) in (X,σ) is called 0-Cauchy if*

$$\lim_{n,m\to\infty} \sigma(x_n,x_m) = 0.$$

The space (X,σ) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $\sigma(x,x) = 0$.

Lemma 2.1. *Let (X,σ) be a dislocated metric space.*

- (a) *(see [22]) If $\sigma(x_n,z) \to \sigma(z,z) = 0$ as $n \to \infty$, then $\sigma(x_n,y) \to \sigma(z,y)$ as $n \to \infty$ for each $y \in X$.*
- (b) *If (X,σ) is complete, then it is 0-complete.*

The converse assertion of (b) may not hold, in general. The following example supports the above remark.

Example 2.4. [30] *The space $X = [0,\infty) \cap \mathbb{Q}$ with the dislocated metric $\sigma(x,y) = max\,\{x,\, y\}$ is 0-complete, but it is not complete. Moreover, the sequence (x_n) with $x_n = 1$ for each $n \in \mathbb{N}$ is a σ-Cauchy sequence in (X,σ), but it is not a 0-Cauchy sequence.*

Definition 2.5. *Let (X,σ) be a dislocated metric space and $A \subseteq X$. The interior of A, denoted by A^0 or $Int(A)$ is the union of all σ-open sets contained in A. Clearly, $Int(A)$ is always a σ-open set. Moreover, A is σ-open if and only if $A = Int(A)$.*

Definition 2.6. *Let (X,σ) be a dislocated metric space and $A \subseteq X$. The closure of A, denoted by $\overline{A}$ or $cl(A)$ is the intersection of all σ-closed subsets of X which contains A. Clearly, $cl(A)$ is always a σ-closed set. Moreover, A is σ-closed if and only if $A = \overline{A}$.*

The following theorem can be obtained in a way similar to that in metric spaces.

Theorem 2.2. *Let (X,σ) be a dislocated metric space and A be any nonempty subset of X. Then, A is σ-closed if and only if for any sequence (x_n) in A which converges to x, we have $x \in A$.*

Theorem 2.3. *Let (X,σ) be a dislocated metric space, A be a σ-closed subset of X and $x \in X$. If $\sigma(x,A) = 0$, then $x \in A$, where $\sigma(x,A) = inf\{\sigma(x,y) : y \in A\}$.*

Proof. Let $\sigma(x,A) = 0$. Then for any $\epsilon > 0$, there exists $x_\epsilon \in A$ such that $\sigma(x,x_\epsilon) < \epsilon$. Therefore, for all $n \geq 1$, there exists $x_n \in A$ such that $\sigma(x,x_n) < \frac{1}{n}$. Hence, $\lim\limits_{n\to\infty} \sigma(x,x_n) = 0$. Again, $\sigma(x,x) \leq 2\sigma(x,x_n)$ ensures that $\sigma(x,x) = 0$. Thus, $\lim\limits_{n\to\infty} \sigma(x,x_n) = \sigma(x,x) = 0$. This means that (x_n) converges to x. By applying Theorem 2.2, it follows that $x \in A$. ☐

We now give an example to show that the converse of the above property is not true, in general.

Example 2.5. *Let $X = \{a, b\}$ and define $\sigma : X \times X \to [0,\infty)$ by*

$$\sigma(x,y) = \begin{cases} 2, & if\ \ x = y = a, \\ \\ 1, & otherwise. \end{cases}$$

Then (X,σ) is a dislocated metric space. We consider X as a σ-closed set and compute

$$\sigma(a,X) = \min\{\sigma(a,a),\, \sigma(a,b)\} = 1.$$

Thus, we find that $a \in X$ but $\sigma(a,X) \neq 0$.

Definition 2.7. [8] *Let (X,σ) be a dislocated metric space and A be a nonempty subset of X. The subset A is said to be bounded if there exist $x_0 \in X$ and $M > 0$ such that $a \in B_\sigma(x_0,M)$ for all $a \in A$.*

Let (X,σ) be a dislocated metric space and $CB^\sigma(X)$ be the set of all nonempty closed bounded subsets of X. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$, where 2^X denotes the collection of all nonempty subsets of X. For A, $B \in CB^\sigma(X)$, define

$$H_\sigma(A,B) = \max\{\sup_{x\in A} \sigma(x,B), \sup_{y\in B} \sigma(y,A)\},$$

where $\sigma(x,B) = \inf\{\sigma(x,y) : y \in B\}$. Such a map H_σ is called the Hausdorff dislocated metric induced by the dislocated metric σ.

Lemma 2.2. [8] *Let (X,σ) be a dislocated metric space. For all A, B, $C \in CB^\sigma(X)$, we have the following:*

(i) $H_\sigma(A,A) = \sup\{\sigma(a,A) : a \in A\}$;
(ii) $H_\sigma(A,B) = H_\sigma(B,A)$;
(iii) $H_\sigma(A,B) = 0 \Longrightarrow A = B$;
(iv) $H_\sigma(A,B) \leq H_\sigma(A,C) + H_\sigma(C,B)$.

Lemma 2.3. *Let (X,σ) be a dislocated metric space. For any A, $B \in CB^\sigma(X)$ and any x, $y \in X$, we have the following:*

(i) $\sigma(x,B) \leq \sigma(x,b)$ *for any* $b \in B$;
(ii) $\sigma(x,B) \leq H_\sigma(A,B)$ *for any* $x \in A$;
(iii) $\sigma(x,A) \leq \sigma(x,y) + \sigma(y,A)$.

Proof. (i) and (ii) are obvious. We now prove part (iii). For any x, $y \in X$ and $a \in A$, we have

$$\sigma(x,a) \leq \sigma(x,y) + \sigma(y,a).$$

This implies that,

$$\sigma(x,A) \leq \sigma(x,a) \leq \sigma(x,y) + \sigma(y,a)$$

and hence $\sigma(x, A) \leq \sigma(x, y) + \sigma(y, A)$. $\square$

Lemma 2.4. [8] *Let A, $B \in CB^\sigma(X)$ and $a \in A$. Then for all $\epsilon > 0$, there exists a point $b \in B$ such that $\sigma(a, b) \leq H_\sigma(A, B) + \epsilon$.*

Let (X, σ) be a dislocated metric space, ρ be a binary relation over X and $S = \rho \cup \rho^{-1}$. Then for x, $y \in X$, $xSy \Leftrightarrow (x\rho y \text{ or } y\rho x)$. In fact, $xSy \Rightarrow ySx$ for x, $y \in X$.

Definition 2.8. *(X, σ, S) is called regular if the following condition holds:*

If the sequence (u_n) in X and the point $u \in X$ are such that $u_n S u_{n+1}$ for all $n \geq 1$ and $\lim_{n\to\infty} \sigma(u_n, u) = \sigma(u, u) = 0$, then there exists a subsequence (u_{n_i}) of (u_n) such that $u_{n_i} S u$ for all $i \geq 1$.

Definition 2.9. *Let (X, σ) be a dislocated metric space and ρ be a binary relation over X. Then the mapping $f : X \to X$ is called S-preserving if f maps comparable elements into comparable elements, that is,*

$$x, y \in X,\ xSy \Rightarrow (fx)\, S\, (fy).$$

For subsets A, B of X, we use the following notation:

$$A\, S\, B \Leftrightarrow aSb \ for\ all\ a \in A,\ b \in B.$$

Definition 2.10. *Let (X, σ) be a dislocated metric space and ρ be a binary relation over X. Then the mapping $T : X \to CB^\sigma(X)$ is called S-preserving if*

$$\forall x, y \in X,\ xSy \Rightarrow (Tx)\, S\, (Ty).$$

Definition 2.11. *Let (X, σ) be a dislocated metric space and ρ be a binary relation over X. Let $T : X \to CB^\sigma(X)$ be a multi-valued mapping and $g : X \to X$ be a single-valued mapping. Then T is called S-preserving w.r.t. g if*

$$\forall x, y \in X,\ (gx)S(gy) \Rightarrow (Tx)\, S\, (Ty).$$

Definition 2.12. *Let (X, σ) be a dislocated metric space and $T : X \to CB^\sigma(X)$ and $g : X \to X$ be two mappings. If $y = gx \in Tx$ for some x in X, then x is called a coincidence point of T and g and y is called a point of coincidence of T and g.*

Theorem 2.4. [26] *Let (X, d) be a metric space and let $T : X \to CL(X)$ and $f : X \to X$ be a hybrid pair of mappings such that $T(X) \subseteq f(X)$ and $f(X)$ a complete subspace of X. Assume that there exists $r \in (0, 1)$ such that*

$$H(Tx, Ty) \leq r\, d(fx, fy) \tag{1}$$

for all x, $y \in X$. Then f and T have a point of coincidence in $f(X)$.

3. Main Results

We begin with the following definition.

Definition 3.1. *Let (X, σ) be a dislocated metric space, ρ be a binary relation over X and let $S = \rho \cup \rho^{-1}$. Then the pair (T, f) of mappings $T : X \to CB^\sigma(X)$ and $f : X \to X$ is called a generalized S-contraction of Nadler type if there exists $\beta \in (0, 1)$ such that*

$$H_\sigma(Tx, Ty) \leq \beta\, M_\sigma(fx, fy), \tag{2}$$

for all x, $y \in X$ with $(fx)S(fy)$ where

$$M_\sigma(fx, fy) = max\{\sigma(fx, fy),\ \sigma(fx, Tx),\ \sigma(fy, Ty),\ \frac{\sigma(fx, Ty) + \sigma(fy, Tx)}{4}\}.$$

We now present our main result.

Theorem 3.1. *Let (X, σ) be a dislocated metric space, ρ be a binary relation over X and let $S = \rho \cup \rho^{-1}$. Let $T : X \to CB^{\sigma}(X)$ and $f : X \to X$ be such that $T(X) \subseteq f(X)$ and $f(X)$ a 0-complete subspace of X. Assume that T is S-preserving w.r.t. f and (T, f) is generalized S-contraction of Nadler type. Suppose also that the following conditions hold:*

(i) *(X, σ, S) is regular;*
(ii) *there exists $x_0 \in X$ such that $(fx_0)Sz$ for some $z \in Tx_0$.*

Then f and T have a point of coincidence u(say) in $f(X)$ with $\sigma(u, u) = 0$.

Proof. Suppose there exists $x_0 \in X$ such that $(fx_0)Sz$ for some $z \in Tx_0$. If $fx_0 \in Tx_0$, then there is nothing to prove. So, we assume that $fx_0 \notin Tx_0$. This ensures that $\sigma(fx_0, Tx_0) > 0$, since Tx_0 is σ-closed. Therefore, $\sigma(fx_0, y) > 0$ for all $y \in Tx_0$. As $Tx_0 \subseteq f(X)$ is nonempty, there exists $x_1 \in X$ such that $z = fx_1 \in Tx_0$, $\sigma(fx_0, fx_1) > 0$ and $(fx_0)S(fx_1)$. If $fx_1 \in Tx_1$, then f and T have a point of coincidence in $f(X)$. So, we assume that $fx_1 \notin Tx_1$. Since Tx_0, $Tx_1 \in CB^{\sigma}(X)$ and $fx_1 \in Tx_0$, by Lemma 2.4, there exists $fx_2 \in Tx_1$ for some $x_2 \in X$ such that

$$\sigma(fx_1, fx_2) \leq H_{\sigma}(Tx_0, Tx_1) + \frac{1-\beta}{2} M_{\sigma}(fx_0, fx_1).$$

Since $fx_1 \notin Tx_1$, we have $\sigma(fx_1, Tx_1) > 0$ and consequently, $\sigma(fx_1, fx_2) > 0$. As T is S-preserving w.r.t. f and $(fx_0)S(fx_1)$, $fx_1 \in Tx_0$, $fx_2 \in Tx_1$, it follows that $(fx_1)S(fx_2)$. If $fx_2 \in Tx_2$, then the theorem is proved. So, we assume that $fx_2 \notin Tx_2$. Proceeding similarly to that of the above, there exists $fx_3 \in Tx_2$ for some $x_3 \in X$ and $\sigma(fx_2, fx_3) > 0$ such that

$$\sigma(fx_2, fx_3) \leq H_{\sigma}(Tx_1, Tx_2) + \frac{1-\beta}{2} M_{\sigma}(fx_1, fx_2).$$

As T is S-preserving w.r.t. f and $(fx_1)S(fx_2)$, $fx_2 \in Tx_1$, $fx_3 \in Tx_2$, it follows that $(fx_2)S(fx_3)$. Continuing in this way, we can obtain a sequence (fx_n) in $f(X)$ such that $fx_n \in Tx_{n-1}$, $fx_n \notin Tx_n$, $\sigma(fx_n, fx_{n+1}) > 0$, $(fx_n)S(fx_{n+1})$ for $n = 0, 1, 2, \cdots$ and

$$\sigma(fx_n, fx_{n+1}) \leq H_{\sigma}(Tx_{n-1}, Tx_n) + \frac{1-\beta}{2} M_{\sigma}(fx_{n-1}, fx_n), \ for\ all\ n \in \mathbb{N}. \tag{3}$$

By using condition (2), we obtain from condition (3) that

$$\begin{aligned} \sigma(fx_n, fx_{n+1}) &\leq \beta M_{\sigma}(fx_{n-1}, fx_n) + \frac{1-\beta}{2} M_{\sigma}(fx_{n-1}, fx_n) \\ &= \frac{1+\beta}{2} M_{\sigma}(fx_{n-1}, fx_n), \ \forall n \in \mathbb{N}, \end{aligned} \tag{4}$$

where

$$M_{\sigma}(fx_{n-1}, fx_n) = max \left\{ \begin{array}{l} \sigma(fx_{n-1}, fx_n), \sigma(fx_{n-1}, Tx_{n-1}), \sigma(fx_n, Tx_n), \\ \frac{\sigma(fx_{n-1}, Tx_n) + \sigma(fx_n, Tx_{n-1})}{4} \end{array} \right\}. \tag{5}$$

We now estimate each of the terms on the right hand side of condition (5) separately.

$$\sigma(fx_{n-1}, Tx_{n-1}) \leq \sigma(fx_{n-1}, fx_n), \ as\ fx_n \in Tx_{n-1},$$

$$\sigma(fx_n, Tx_n) \leq \sigma(fx_n, fx_{n+1}), \ as\ fx_{n+1} \in Tx_n$$

and

$$\begin{aligned}\frac{\sigma(fx_{n-1},Tx_n)+\sigma(fx_n,Tx_{n-1})}{4} &\leq \frac{\sigma(fx_{n-1},fx_{n+1})+\sigma(fx_n,fx_n)}{4}\\ &\leq \frac{3\,\sigma(fx_{n-1},fx_n)+\sigma(fx_n,fx_{n+1})}{4}\\ &\leq \max\{\sigma(fx_{n-1},fx_n),\sigma(fx_n,fx_{n+1})\}.\end{aligned}$$

Therefore,

$$\begin{aligned}M_\sigma(fx_{n-1},fx_n) &= \max\left\{\begin{array}{l}\sigma(fx_{n-1},fx_n),\sigma(fx_{n-1},Tx_{n-1}),\sigma(fx_n,Tx_n),\\ \frac{\sigma(fx_{n-1},Tx_n)+\sigma(fx_n,Tx_{n-1})}{4}\end{array}\right\}\\ &\leq \max\left\{\begin{array}{l}\sigma(fx_{n-1},fx_n),\sigma(fx_n,fx_{n+1}),\\ \max\{\sigma(fx_{n-1},fx_n),\sigma(fx_n,fx_{n+1})\}\end{array}\right\}\\ &= \max\{\sigma(fx_{n-1},fx_n),\sigma(fx_n,fx_{n+1})\}.\end{aligned}$$

If $\max\{\sigma(fx_{n-1},fx_n),\sigma(fx_n,fx_{n+1})\}=\sigma(fx_n,fx_{n+1})$, then

$$M_\sigma(fx_{n-1},fx_n)\leq\sigma(fx_n,fx_{n+1}),$$

which contradicts condition (4).
Therefore, $\max\{\sigma(fx_{n-1},fx_n),\sigma(fx_n,fx_{n+1})\}=\sigma(fx_{n-1},fx_n)$ and hence

$$M_\sigma(fx_{n-1},fx_n)\leq\sigma(fx_{n-1},fx_n),\ \text{for all } n\in\mathbb{N}.$$

So, condition (4) implies that

$$\sigma(fx_n,fx_{n+1})\leq\frac{1+\beta}{2}\,\sigma(fx_{n-1},fx_n),\ \text{for all } n\in\mathbb{N}. \tag{6}$$

By repeated use of condition (6), we get

$$\sigma(fx_n,fx_{n+1})\leq\alpha^n\,\sigma(fx_0,fx_1),\ \text{for all } n\in\mathbb{N}, \tag{7}$$

where $\alpha=\frac{1+\beta}{2}\in(0,1)$.

We now prove that the sequence (fx_n) is 0-Cauchy in $f(X)$.

For $m,n\in\mathbb{N}$ with $m>n$, we obtain by using condition (7) that

$$\begin{aligned}\sigma(fx_n,fx_m) &\leq \sigma(fx_n,fx_{n+1})+\sigma(fx_{n+1},fx_{n+2})+\cdots\\ &\quad+\sigma(fx_{m-2},fx_{m-1})+\sigma(fx_{m-1},fx_m)\\ &\leq [\alpha^n+\alpha^{n+1}+\cdots+\alpha^{m-2}+\alpha^{m-1}]\,\sigma(fx_0,fx_1)\\ &< \alpha^n\,[1+\alpha+\alpha^2+\cdots]\,\sigma(fx_0,fx_1)\\ &= \frac{\alpha^n}{1-\alpha}\,\sigma(fx_0,fx_1)\\ &\to 0,\ \text{as } n\to\infty.\end{aligned}$$

This shows that $\lim\limits_{m,n\to\infty}\sigma(fx_n,fx_m)=0$. Therefore, (fx_n) is 0-Cauchy in $f(X)$. As $f(X)$ is 0-complete, there exists $u\in f(X)$ such that $\lim\limits_{n\to\infty}fx_n=u=ft$ for some $t\in X$ with $\sigma(u,u)=0$. That is, $\lim\limits_{n\to\infty}\sigma(fx_n,u)=\sigma(u,u)=0$.

Moreover, since (X,σ,S) is regular, there exists a subsequence (fx_{n_i}) of (fx_n) such that $(fx_{n_i})S(ft)$ for all $i\in\mathbb{N}$. By condition (2), we have

$$H_\sigma(Tx_{n_i},Tt)\leq\beta\,M_\sigma(fx_{n_i},ft),\ \text{for all } i\in\mathbb{N}.$$

This gives that

$$\sigma(fx_{n_i+1}, Tt) \leq H_\sigma(Tx_{n_i}, Tt) \leq \beta\, M_\sigma(fx_{n_i}, ft), \text{ for all } i \in \mathbb{N}. \tag{8}$$

We now prove that there is $k \in \mathbb{N}$ such that for each $i \geq k$,

$$M_\sigma(fx_{n_i}, ft) = \sigma(ft, Tt), \text{ where}$$

$$M_\sigma(fx_{n_i}, ft) = max \left\{ \begin{array}{c} \sigma(fx_{n_i}, ft), \sigma(fx_{n_i}, Tx_{n_i}), \sigma(ft, Tt), \\ \frac{\sigma(fx_{n_i}, Tt)+\sigma(ft, Tx_{n_i})}{4} \end{array} \right\}. \tag{9}$$

We now estimate each of the terms on the right hand side of the above expression.

Suppose that $\sigma(ft, Tt) \neq 0$. Let $\epsilon = \frac{\sigma(ft,Tt)}{4} > 0$. Since $\lim_{i\to\infty} \sigma(fx_{n_i}, ft) = 0$, there exists $k_1 \in \mathbb{N}$ such that

$$\sigma(fx_{n_i}, ft) < \frac{\sigma(ft, Tt)}{4}, \text{ for each } i \geq k_1.$$

As $\sigma(fx_n, ft) \to 0$, there exists $k_2 \in \mathbb{N}$ such that

$$\sigma(fx_{n_i+1}, ft) < \frac{\sigma(ft, Tt)}{4}, \text{ for each } i \geq k_2.$$

So, it must be the case that

$$\sigma(ft, Tx_{n_i}) \leq \sigma(ft, fx_{n_i+1}) < \frac{\sigma(ft, Tt)}{4}, \text{ for each } i \geq k_2.$$

As $\sigma(fx_{n_i}, Tt) \leq \sigma(fx_{n_i}, ft) + \sigma(ft, Tt)$, it follows that

$$\sigma(fx_{n_i}, Tt) < \frac{\sigma(ft, Tt)}{4} + \sigma(ft, Tt) \leq \frac{5}{4}\sigma(ft, Tt), \text{ for each } i \geq k_1.$$

Put $k = max\{k_1,\, k_2\}$. Then, for $i \geq k$, we have

$$\sigma(fx_{n_i}, Tx_{n_i}) \leq \sigma(fx_{n_i}, fx_{n_i+1}) \leq \sigma(fx_{n_i}, ft) + \sigma(ft, fx_{n_i+1}) < \frac{\sigma(ft, Tt)}{2}$$

and

$$\frac{\sigma(fx_{n_i}, Tt) + \sigma(ft, Tx_{n_i})}{4} < \frac{1}{4}(\frac{5}{4} + \frac{1}{4})\,\sigma(ft, Tt) = \frac{3}{8}\sigma(ft, Tt) < \sigma(ft, Tt).$$

Then, for $i \geq k$, it follows from condition (9) that $M_\sigma(fx_{n_i}, ft) = \sigma(ft, Tt)$. Therefore, for $i \geq k$, we obtain from condition (8) that

$$\sigma(fx_{n_i+1}, Tt) \leq \beta\, \sigma(ft, Tt). \tag{10}$$

By condition (10), for $i \geq k$, we get

$$\sigma(ft, Tt) \leq \sigma(ft, fx_{n_i+1}) + \sigma(fx_{n_i+1}, Tt) \leq \sigma(ft, fx_{n_i+1}) + \beta\, \sigma(ft, Tt).$$

Taking limit as $i \to \infty$, we have $\sigma(ft, Tt) \leq \beta\, \sigma(ft, Tt) < \sigma(ft, Tt)$. Thus, we arrive at a contradiction. Therefore, $\sigma(ft, Tt) = 0$. Since Tt is σ-closed, it follows that $u = ft \in Tt$. That is, u is a point of coincidence of f and T with $\sigma(u, u) = 0$. □

Corollary 3.1. *Let (X, σ) be a dislocated metric space. Let $T : X \to CB^\sigma(X)$ and $f : X \to X$ be such that $T(X) \subseteq f(X)$ and $f(X)$ a 0-complete subspace of X. Assume that there exists $\beta \in [0, 1)$ such that*

$$H_\sigma(Tx, Ty) \leq \beta\, M_\sigma(fx, fy),$$

for all $x,\, y \in X$. Then f and T have a point of coincidence u(say) in $f(X)$ with $\sigma(u, u) = 0$.

Proof. The proof follows from Theorem 3.1 by taking $\rho = X \times X$. $\square$

The following result is a generalization of Theorem 2 [8].

Corollary 3.2. *Let (X, σ) be a 0-complete dislocated metric space and let $T : X \to CB^{\sigma}(X)$ be a multi-valued mapping. Assume that there exists $\beta \in [0, 1)$ such that*

$$H_{\sigma}(Tx, Ty) \leq \beta\, M_{\sigma}(x, y),$$

for all x, $y \in X$, where $M_{\sigma}(x, y) = max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}$. Then T has a fixed point u(say) in X with $\sigma(u, u) = 0$.

Proof. The proof follows from Theorem 3.1 by taking $f = I$, the identity map on X and $\rho = X \times X$. $\square$

Corollary 3.3. *Let $(X, \sigma, \preceq)$ be a partially ordered 0-complete dislocated metric space. Let $T : X \to CB^{\sigma}(X)$ be a multi-valued mapping with the property that if x, $y \in X$ and x, y are comparable, then z_1, z_2 are comparable for all $z_1 \in Tx$, $z_2 \in Ty$. Assume that there exists $\beta \in (0, 1)$ such that*

$$H_{\sigma}(Tx, Ty) \leq \beta\, M_{\sigma}(x, y),$$

for all x, $y \in X$ with $x \preceq y$ or $y \preceq x$. Suppose also that the triple $(X, \sigma, \preceq)$ has the following property:

If (x_n) is a sequence in X such that $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = 0$ and x_n, x_{n+1} are comparable for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i}, x are comparable for all $i \geq 1$.

If there exists $x_0 \in X$ such that x_0, z are comparable for some $z \in Tx_0$, then T has a fixed point u(say) in X with $\sigma(u, u) = 0$.

Proof. The proof can be obtained from Theorem 3.1 by taking $f = I$ and $\rho = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. $\square$

Corollary 3.4. *Let (X, σ) be a 0-complete dislocated metric space. Let ρ be a binary relation over X and let $S = \rho \cup \rho^{-1}$. Suppose $T : X \to CB^{\sigma}(X)$ is S-preserving and there exists $\beta \in (0, 1)$ such that*

$$H_{\sigma}(Tx, Ty) \leq \beta\, M_{\sigma}(x, y),$$

for all x, $y \in X$ with xSy. Suppose also that the following conditions hold:

(i) *(X, σ, S) is regular;*
(ii) *there exists $x_0 \in X$ such that $x_0 S z$ for some $z \in Tx_0$.*

Then T has a fixed point u(say) in X with $\sigma(u, u) = 0$.

Proof. The proof follows from Theorem 3.1 by taking $f = I$. $\square$

We obtain some results as an application of Theorem 3.1.

The result stated below is a generalization of Nadler's fixed point theorem [28].

Theorem 3.2. *Let (X, σ) be a dislocated metric space and let $T : X \to CB^{\sigma}(X)$ and $f : X \to X$ be a hybrid pair of mappings such that $T(X) \subseteq f(X)$ and $f(X)$ a 0-complete subspace of X. Assume that there exists $\beta \in [0, 1)$ such that*

$$H_{\sigma}(Tx, Ty) \leq \beta\, \sigma(fx, fy)$$

for all x, $y \in X$. Then f and T have a point of coincidence u(say) in $f(X)$ with $\sigma(u, u) = 0$.

Proof. As $\sigma(fx, fy) \le M_\sigma(fx, fy)$ for all x, $y \in X$, the result follows from Theorem 3.1 by considering $\rho = X \times X$. $\square$

Theorem 3.3. *Let (X, σ) be a 0-complete dislocated metric space and let $T : X \to CB^\sigma(X)$ be a multi-valued mapping such that*

$$H_\sigma(Tx, Ty) \le a_1\, \sigma(x, y) + a_2\, \sigma(x, Tx) + a_3\, \sigma(y, Ty) + a_4\, [\sigma(x, Ty) + \sigma(y, Tx)] \tag{11}$$

for all x, $y \in X$, where a_1, a_2, a_3, $a_4 \ge 0$ and $a_1 + a_2 + a_3 + 4a_4 < 1$. Then T has a fixed point u(say) in X with $\sigma(u, u) = 0$.

Proof. Condition (11) gives that

$$H_\sigma(Tx, Ty) \le (a_1 + a_2 + a_3 + 4a_4) M_\sigma(x, y) \tag{12}$$

for all x, $y \in X$, where $0 \le a_1 + a_2 + a_3 + 4a_4 < 1$. Taking $\beta = a_1 + a_2 + a_3 + 4a_4$, it follows from condition (12) that

$$H_\sigma(Tx, Ty) \le \beta\, M_\sigma(x, y)$$

for all x, $y \in X$, where $\beta \in [0, 1)$ is a constant. Now applying Corollary 3.2, we can obtain the desired result. $\square$

We now present Nadler's fixed point theorem in dislocated metric spaces.

Theorem 3.4. *Let (X, σ) be a 0-complete dislocated metric space and let $T : X \to CB^\sigma(X)$ be a multi-valued mapping. Assume that there exists $\beta \in [0, 1)$ such that*

$$H_\sigma(Tx, Ty) \le \beta\, \sigma(x, y) \tag{13}$$

for all x, $y \in X$. Then T has a fixed point u(say) in X with $\sigma(u, u) = 0$.

Proof. Condition (13) implies that $H_\sigma(Tx, Ty) \le \beta\, \sigma(x, y) \le \beta\, M_\sigma(x, y)$ for all x, $y \in X$ where $\beta \in [0, 1)$ is a constant. The result now follows from Corollary 3.2. $\square$

Remark 3.1. *We note that Theorem 3.1 is a proper generalization(see Example 3.1) of some multi-valued fixed point theorems including Nadler's fixed point theorem* [28].

Remark 3.2. *The results of this study are obtained under the weaker assumption that the underlying dislocated metric space is 0-complete. In view of Lemma 2.1(b), these are also valid if the space is complete.*

We now present an example to justify the validity of our main result. It should be noticed that a generalized version of Nadler's Theorem can not explain the existence of points of coincidence in the following example.

Example 3.1. *Let $X = [0, \infty)$ and $\sigma : X \times X \to [0, \infty)$ be defined by*

$$\sigma(x, y) = \begin{cases} 0, & if\ \ x = y \in (3, 9], \\ \\ x + y, & otherwise. \end{cases}$$

Then (X, σ) is a 0-complete dislocated metric space. Let $T : X \to CB^\sigma(X)$ be defined by

$$Tx = \begin{cases} \{0, \frac{x}{3}\}, & if\ \ 0 \le x < 1, \\ \\ \{0\}, & if\ \ x = 1, \\ \\ [x^2, x^2 + 6], & if\ \ x > 1 \end{cases}$$

and $fx = 3x$ for all $x \in X$. Obviously, $T(X) \subseteq f(X) = X$. It is easy to check that each Tx is σ-closed and bounded in (X, σ).

If we consider the usual metric $d(x, y) =| x - y |$ for all $x, y \in X$, then (X, d) is a complete metric space. For $x = 0,\ y = 2$, we have $fx = 0,\ fy = 6,\ Tx = \{0\},\ Ty = [4, 10]$. Therefore, $H(Tx, Ty) = \max\{4, 10\} = 10 > k\, d(fx, fy)$ for any $k \in (0, 1)$ and hence condition (1) of Theorem 2.4 does not hold true. So, Theorem 2.4 can not explain the existence of points of coincidence of f and T. However, our main result can explain it.

Let $\rho = \{(0, \frac{1}{3^n}) : n = 1, 2, 3, \cdots\} \cup \{(0, 0)\}$. For $x = 0,\ y = \frac{1}{3^{n+1}},\ n \in \mathbb{N}$ and $S = \rho \cup \rho^{-1}$, we have $fx = 0,\ fy = \frac{1}{3^n},\ Tx = \{0\},\ Ty = \{0, \frac{1}{3^{n+2}}\}$ and so $(fx)S(fy)$ which implies that $(Tx)S(Ty)$. Therefore, T is S-preserving w.r.t. f and $x_0 = 0 \in X$ such that $(fx_0)Sz$ for $z = 0 \in Tx_0$. Moreover, for $x = 0,\ y = \frac{1}{3^{n+1}},\ n \in \mathbb{N}$, we have $H_\sigma(Tx, Ty) = \max\{0, \frac{1}{3^{n+2}}\} = \frac{1}{3^{n+2}}$ and

$$M_\sigma(fx, fy) = \max\left\{\begin{array}{l} \sigma(fx, fy), \sigma(fx, Tx), \frac{\sigma(fy,Ty)}{2}, \\ \\ \frac{\sigma(fx,Ty)+\sigma(fy,Tx)}{2} \end{array}\right\} = \max\left\{\frac{1}{3^n},\ 0,\ \frac{1}{2}\cdot\frac{1}{3^n}\right\} = \frac{1}{3^n}.$$

Therefore, $H_\sigma(Tx, Ty) = \frac{1}{3^{n+2}} = \frac{1}{9}\, M_\sigma(fx, fy)$. In case $x = y = 0$, we have $H_\sigma(Tx, Ty) = 0 = M_\sigma(fx, fy)$. Thus, $H_\sigma(Tx, Ty) = \frac{1}{9}\, M_\sigma(fx, fy)$ for all $x,\ y \in X$ with $(fx)S(fy)$. Also, any sequence (x_n) in X with the property $x_n S x_{n+1}$ must be either $x_n = 0, \forall n \in \mathbb{N}$ or a sequence of the following form

$$x_n = \begin{cases} 0, & if\ \ n \text{ is odd}, \\ \\ \frac{1}{3^n}, & if\ \ n \text{ is even}, \end{cases}$$

where the words 'odd' and 'even' are interchangeable. Therefore, it follows immediately that (X, σ, S) is regular. Thus, all the hypotheses of Theorem 3.1 hold true. Then the existence of points of coincidence of f and T follows from Theorem 3.1.

It is worthy to note that Theorem 3.1 can not assure the uniqueness of a point of coincidence. It is obvious that f and T have infinitely many points of coincidence in $f(X)$. In fact, for every $x \in (1, 3]$, fx is a point of coincidence of f and T with $\sigma(fx, fx) = 0$.

Now we show that Theorem 3.1 remains invalid without regularity property of (X, σ, S).

Example 3.2. *Let $X = [0, \infty)$ with $\sigma(x, y) = x + y$ for all $x,\ y \in X$. Then (X, σ) is a 0-complete dislocated metric space. Let $T : X \to CB^\sigma(X)$ be defined by*

$$Tx = \begin{cases} \{1\}, & if\ \ x = 0, \\ \\ \{\frac{x}{9}\}, & if\ \ x \neq 0 \end{cases}$$

and $fx = \frac{x}{3}$ for all $x \in X$. Then, $T(X) \subseteq f(X) = X$. Let $\rho = \{(x, y) : (x, y) \in (0, 1] \times (0, 1],\ x \leq y\}$ and $S = \rho \cup \rho^{-1}$. For $x,\ y \in X$ and

$$\begin{aligned} (fx)S(fy) \ &\Rightarrow\ 0 < fx,\ fy \leq 1 \Rightarrow 0 < x,\ y \leq 3 \\ &\Rightarrow\ Tx = \{\frac{x}{9}\},\ Ty = \{\frac{y}{9}\},\ 0 < \frac{x}{9},\ \frac{y}{9} \leq \frac{1}{3} \Rightarrow (Tx)S(Ty). \end{aligned}$$

This proves that T is S-preserving w.r.t. f. Obviously, for $x_0 = 3 \in X$, we have $(fx_0)Sz$ for $z = \frac{1}{3} \in Tx_0$. Further, we note that for $x,\ y \in X$ with $(fx)S(fy)$, we have

$0 < x,\, y \leq 3,\; Tx = \{\frac{x}{9}\}, Ty = \{\frac{y}{9}\}$. *Therefore,*

$$H_\sigma(Tx, Ty) = \sigma(\frac{x}{9}, \frac{y}{9}) = \frac{1}{9}\,(x+y) = \frac{1}{3}\,\sigma(fx, fy) \leq \frac{1}{3}\,M_\sigma(fx, fy),$$

for all $x,\, y \in X$ *with* $(fx)S(fy)$. *Thus, condition (2) of Theorem 3.1 holds true. But* (X, σ, S) *is not regular. In fact, the sequence* (x_n) *where* $x_n = \frac{1}{n}$ *is such that* $\lim_{n\to\infty} \sigma(x_n, 0) = \sigma(0, 0) = 0$ *and* $x_n S x_{n+1}$ *for all* $n \in \mathbb{N}$. *But there exists no subsequence* (x_{n_i}) *of* (x_n) *such that* $(x_{n_i}, 0) \in S$. *Thus, all the hypotheses of Theorem 3.1 are fulfilled except regularity property. As a result, we observe that there exists no point of coincidence of* f *and* T *in* $f(X)$.

4. Conclusions

In this work, we obtained some coincidence point and fixed point results by using an arbitrary binary relation. Our main result generalized the well known Nadler's fixed point theorem in the setting of dislocated metric spaces. Finally, some examples are provided to justify that our main result is a proper generalization of some well known comparable results in the existing literature.

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