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THE KANNAN FIXED POINT THEOREM IN INCOMPLETE MODULAR SPACES

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Abstract. In this paper, we prove a fixed point theorem for Kannan contraction mappings definded on incomplete orthogonal modular spaces. Our results improve one of the results given by Abdou and Khamsi [Abdou, A. A. N. and Khamsi, M. A., (2020), Fixed points of Kannan maps in the variable exponent sequence spaces $\ell_{p(.)}$, Mathematics., 8(1), p. 76].

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1. Introduction

In 1950, Nakano [17] initiated the theory of modular spaces in connection with the theory of ordered linear spaces as a generalization of several classes of functions and sequence spaces including $\ell_{p(.)}$, L_p, l_p, and Musielak-Orlicz [16]. Recently, the theory of modular spaces has been extensively investigated [5, 7, 15]. For more details on modular spaces, the reader may consult the book [12]. There was a strong interest to study the fixed point property in modular spaces after the first paper [10] was published in 1990. Many problems in metric fixed point theory can be reformulated in modular spaces although these spaces are not induced by meters. The fixed point property in modular spaces has been defined and investigated by many authors (see for instance, [4, 6, 9, 13, 20]).

The Banach contraction principle [3] is the most widely applied fixed point result in many branches of mathematics and considered as the main source of metric fixed point theory. It is natural to ask if there is an example of a class of mappings with the same fixed point behavior as contractions but that fail to be continuous. To show this, Kannan [11] prepared the following theorem in metric spaces:

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Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying $d(Tx, Ty) \le \lambda(d(x, Tx) + d(y, Ty))$ for all $x, y \in X$, where $0 \le \lambda < \frac{1}{2}$. Then, T has a unique fixed point.*

Eshaghi Gordji et al. [8] introduced the notion of orthogonal set and generalized Banach contarction principle. Fixed point properties in orthogonal spaces have been studied by many authors [2, 10, 18, 19].

This article is motivated by the recent papers [1, 19], where the similar results were obtained for the Meir-Keeler fixed point theorem in incomplete modular spaces and for the Kannan fixed point theorem in the variable exponent sequence spaces $\ell_{p(.)}$. The first aim of this paper is to extend fixed point property of $\ell_{p(.)}$ to the general class of modular spaces. We prove a fixed point theorem for Kannan contraction mappings definded on incomplete orthogonal modular spaces. Our results improve one of the results given by Abdou and Khamsi [1].

2. Preliminaries

We begin by recalling some basic concepts of modular spaces [12].

Definition 2.1. *Let X be a vector space over K ($K = \mathbb{C}$ or $\mathbb{R}$). A function $\rho : X \to [0, \infty]$ is called a modular on X if*

(1) $\rho(x) = 0$ *if and only if* $x = 0$,
(2) $\rho(\alpha x) = \rho(x)$ *for every* $\alpha \in K$ *with* $|\alpha| = 1$,
(3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ *for every* $x, y \in X$ *and* $\alpha, \beta \ge 0$ *with* $\alpha + \beta = 1$.

If we replace (3) *by*

$$\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y),$$

for every $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$, then the modular ρ is called convex. For a convex modular ρ on X one can associate a modular space X_ρ defined as

$$X_\rho = \left\{ x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0 \right\}.$$

Definition 2.2. *Let ρ be a convex modular defined on a vector space X.*

(1) *A sequence (x_n) in X_ρ is said to be ρ-convergent to $x \in X_\rho$ if and only if $\rho(x_n - x) \to 0$ as $n \to \infty$.*
(2) *The sequence (x_n) in X_ρ is said to be ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.*
(3) *X_ρ is said to be ρ-complete if any ρ-Cauchy sequence in X_ρ is a ρ-convergent sequence in X_ρ.*
(4) *A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n) \subset B$ which ρ-convergent to x, then $x \in B$.*
(5) *A subset $B \subset X_\rho$ is said to be ρ-bounded if*
$$\delta_\rho(B) = \sup\{\rho(x - y); x, y \in B\} < \infty.$$
(6) *The ρ-distance between $x \in X_\rho$ and $B \subset X_\rho$ is defined as*
$$d_\rho(x, B) = \inf\{\rho(x - y); y \in B\}.$$
(7) *We say that ρ has the Fatou property if $\rho(x - y) \le \lim_{n\to\infty} \inf \rho(x_n - y_n)$ whenever (x_n) ρ-converges to x and (y_n) ρ-converges to y.*

Note that the ρ-balls $B_\rho(x, r) = \{y \in X_\rho : \rho(x - y) \le r\}$ are ρ-closed if and only if ρ satisfies the Fatou property. A new generalization of Banach's contraction principle is given in [8] by introducing the notion of orthogonal set as follows:

Definition 2.3. *Let X be a non-empty set and $\perp$ be a binary relation defined on $X \times X$. Then $(X, \perp)$ is said to be an orthogonal set (abbreviated as O- set) if there exists $x_0 \in X$ such that*

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

The element x_0 is called an orthogonal element. An orthogonal set may have more than one orthogonal element.

Definition 2.4. [8] *Let $(X, \perp)$ be an O- set. A sequence (x_n) in X is an orthogonal sequence (abbreviated as O- sequence) if*

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

Definition 2.5. [19] *Let $(X, \perp)$ be an O- set. A sequence (x_n) in X is called a strongly orthogonal sequence (abbreviated as SO- sequence) if*

$$(\forall n, k \in \mathbb{N}, x_k \perp x_{n+k}) \text{ or } (\forall n, k \in \mathbb{N}, x_{n+k} \perp x_k).$$

From the definition, it is clear that every SO-sequence is an O-sequence, but the converse is not true (see [8]).

Definition 2.6. *Let X_ρ be a modular space and $\perp$ be a binary relation defined on $X \times X$. Then $(X_\rho, \perp)$ is said an orthogonal modular space (abbreviated as O- modular space.*

Definition 2.7. *Let $(X_\rho, \perp)$ be a O- modular space.*

1. *A subset $B \subset X_\rho$ is said to be SO-ρ-closed if for any SO-sequence $(x_n) \subset B$ which ρ-convergent to x, then $x \in B$.*
2. *X_ρ is said to be strongly orthogonal ρ-complete (abbreviated as SO- ρ-complete) if every ρ-Cauchy SO-sequence is ρ-convergent in X_ρ.*

Note that every ρ-complete modular space is SO-ρ-complete modular space, but the converse is not true (see [19]).

Definition 2.8. *Let K be a non-empty subset of O- modular space X_ρ. A mapping $T : K \to K$ is said to be $\perp$-preserving if for each $x, y \in K$ such that $x \perp y$, then $Tx \perp Ty$.*

Definition 2.9. *Let K be a non-empty subset of O- modular space X_ρ. A mapping $T : K \to K$ is said to be kannan ρ-contraction if there exists $0 \leq \lambda < \frac{1}{2}$ such that*

$$\rho(Tx - Ty) \leq \lambda(\rho(x - Tx) + \rho(y - Ty)),$$

for all $x, y \in K$.

3. Kannan ρ-contraction mapping in orthogonal modular spaces

Now we are ready to state our main result.

Theorem 3.1. *Let $(X, \perp, \rho)$ be an SO-ρ-complete orthogonal modular space(not necessarily ρ-complete). Let K be a nonempty, SO-ρ-closed and $T : K \to K$ be a Kannan ρ-contraction, $\perp$-preserving. Let ρ satisfy the Fauto property and x_0 be an orthogonal element such that $\rho(x_0 - Tx_0) < \infty$. Then, there exists $z \in K$ such that the sequence $(T^n(x_0))$ is ρ-convergent to z.*

Proof. Since x_0 is an orthogonal element, we have

$$(\forall y \in X_\rho, x_0 \perp y) \text{ or } (\forall y \in X_\rho, y \perp x_0).$$

This implies that $x_0 \perp T(x_0)$ or $T(x_0) \perp x_0$. Let us consider the sequence (x_n) where $x_n = T^n(x_0)$ for all $n \in \mathbb{N}$. It is clear

$$(\forall n \in \mathbb{N}, x_0 \perp x_n) \text{ or } (\forall n \in \mathbb{N}, x_n \perp x_0).$$

Since T is a $\perp$-preserving map, we have either

$$(\forall n, k \in \mathbb{N}, x_k = T^k(x_0) \perp x_{n+k} = T^k(x_n))$$

or

$$(\forall n, k \in \mathbb{N}, x_{n+k} = T^k(x_n)) \perp x_k = T^k(x_0).$$

This implies that (x_n) is an SO-sequence. Since, T is a Kannan ρ-contraction mapping, we obtain

$$\rho(T^n x_0 - T^{n+1} x_0) \leq \lambda(\rho(T^{n-1} x_0 - T^n x_0) + \rho(T^n x_0 - T^{n+1} x_0)),$$

for all $n \in \mathbb{N}$. Therefor we must have

$$\rho(T^n x_0 - T^{n+1} x_0) \leq \frac{\lambda}{1-\lambda} \rho(T^{n-1} x_0 - T^n x_0).$$

Set $k := \frac{\lambda}{1-\lambda}$. It is easily seen that,

$$\rho(T^n x_0 - T^{n+1} x_0) \leq k^n \rho(x_0 - T x_0),$$

for all $n \in \mathbb{N}$.

$$\begin{aligned} \rho(T^n x_0 - T^{n+m} x_0) &\leq \lambda[\rho(T^{n-1} x_0 - T^n x_0) + \rho(T^{n+m-1} x_0 - T^{n+m} x_0)] \\ &\leq \lambda[k^{n-1} \rho(x_0 - T x_0) + k^{n+m-1} \rho(x_0 - T x_0)] \end{aligned}$$

for all $n, m \in \mathbb{N}$. Since $k < 1$ and $\rho(x_0 - T x_0) < \infty$, the sequence (x_n) is a ρ-Cauchy SO-sequence. Since X_ρ is an SO-ρ-complete orthogonal modular space and K is an SO-ρ-closed subset of X_ρ, there exists $z \in K$ such that the sequence $(T^n(x_0))$ is ρ-convergent to z. □

In Theorem 1 of [1], it is shown that if K is a nonempty ρ-closed subset of $\ell_{p(.)}$ and $T : K \to K$ is a Kannan ρ-contraction mapping, then $(T^n(z))$ is ρ-convergent to $w \in K$, where $z \in K$ is such that $\rho(z - Tz) < \infty$. Furthermore it is shown that if $\rho(w - Tw) \neq \infty$, then w is a fixed point of T. Note that the modular ρ defined on $\ell_{p(.)}$ has the Fauto property. As a conseuence of Theorem 3.1 we can obtain the following theorem which shows that Theorem 1 given in [1] for $\ell_{p(.)}$ is still true for the larger class of modular spaces X_ρ.

Theorem 3.2. *Assume that X_ρ is a ρ-complete modular space and ρ satisfies the Fauto property. Let K be a nonempty, ρ-closed and $T : K \to K$ be a Kannan ρ-contraction. If there exist $x_0 \in K$ such that $\rho(x_0 - T x_0) < \infty$. Then, there exists $z \in K$ such that the sequence $(T^n(x_0))$ is ρ-convergent to z. Moreover, if $\rho(z - Tz) \neq \infty$, then z is a fixed point of T.*

Proof. For all $x, y \in K$, we define a binary relation $\perp$ on K by

$$x \perp y \text{ iff } \rho(Tx - Ty) \leq \lambda(\rho(x - Tx) + \rho(y - Ty)).$$

Since T is a Kannan ρ-contraction, it is clear that for all $x, y \in K$, $x \perp y$. So $\perp$ is an orthogonal relation on K. Since X_ρ is ρ-complete and K is ρ-closed, hence K is an SO-ρ-closed subset of X_ρ. Let $x, y \in K$ and $x \perp y$. Using the binary relation $\perp$ defined on K and the definition of Kannan ρ-contraction, we have

$$\rho(T(Tx) - T(Ty)) \leq \lambda(\rho(Tx - T(Tx) + \rho(Ty - T(Ty)),$$

hence $Tx\perp Ty$. This implies that T is $\perp$-preserving. The hypothesises of Theorem 3.1 are satisfied. Therefore there exists $z \in K$ such that the sequence $(T^n(x_0))$ is ρ-convergent to z. On the other hand, $\rho(z-Tz) \neq \infty$, we will prove that $\rho(z-Tz)=0$. Since

$$\begin{aligned}\rho(T^n x_0 - Tz) &\leq \lambda(\rho(T^{n-1}x_0 - T^n x_0) + \rho(z-Tz))\\ &\leq \lambda(k^{n-1}\rho(x_0 - Tx_0) + \rho(z-Tz))\end{aligned}$$

for any $n \geq 1$, using the Fatou's property, we get

$$\begin{aligned}\rho(z-Tz) &\leq \lim_{n\to\infty} \inf \rho(T^n x_0 - Tz)\\ &\leq \lambda\rho(z-Tz).\end{aligned}$$

Since $\lambda < \frac{1}{2}$, we conclude that $\rho(z-Tz)=0$, i.e., z is the fixed point of T. □

The following example shows that Theorem 3.2 is a particular case of our main theorem.

Example 3.1. *For given function $p:\mathbb{N}\to[1,\infty)$, we define the following vector space*

$$\ell_{p(.)} := \{(x_n)\in\mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty}\tfrac{1}{p(n)}|\alpha x_n|^{p(n)} < \infty \text{ for some } \alpha > 0\ \}.$$

Let the function $\rho:\ell_{p(.)}\to[0,\infty]$ be defined by

$$\rho(x) = \rho((x_n)) = \sum_{n=0}^{\infty}\frac{1}{p(n)}|x_n|^{p(n)}.$$

Then ρ defines a convex modular on $\ell_{p(.)}$ For more details the reader may consult [14, 17, 21]. *By setting $p(n)=1$ for all $n\in\mathbb{N}$ and $\alpha=1$,it is obvious that the following set*

$$K = \{(x_n)\in\mathbb{R}^{\mathbb{N}};\exists n_1,n_2,...,n_j;\forall n\neq n_1,n_2,...,n_j, x_n=0\}$$

is a subspace of $\ell_{p(.)}$. For all $a,b\in K$, we define $a\perp b$ if and only if there exists $\gamma\in\{0,1\}$ such that $a=\gamma b$ or $b=\gamma a$. Then, for all $a\in K$ we have, $0\perp a$. Hence, $(K,\perp)$ is an O-modular space. Note that K is not ρ-complete, for instance, sequence $U_n=(1,\frac{1}{2},\frac{1}{3},...,\frac{1}{n},0,0,0,...)$, $n\in\mathbb{N}$ is a ρ-Cauchy sequence not converging to any element in K. We assert that K is SO-ρ-complete space. Let (U_n) be ρ-Cauchy SO-sequence in K and let there exists $n_0\in\mathbb{N}$ such that $U_{n_0}\neq 0$. From the definition of $\perp$ it follows for every $n\in\mathbb{N}$, $U_n=\gamma_n U_{n_0}$ where $\gamma_n = 0$ or 1. For every $i,j\in\mathbb{N}$ we have

$$|\gamma_i-\gamma_j|\,\rho(U_{n_0}) = \rho(\gamma_i U_{n_0} - \gamma_j U_{n_0}) = \rho(U_i - U_j).$$

Since (U_n) is ρ-Cauchy, seuence (γ_n) is Cauchy in $\mathbb{R}$. Let $\lim_{n\to\infty}\gamma_n=\gamma$. We get $\gamma_n = 0$ or 1. Put $U=\gamma U_{n_0}$. On the one hand,

$$\rho(U_n - U) = \rho(\gamma_n U_{n_0} - \gamma U_{n_0}) = (\gamma_n-\gamma)\rho(U_{n_0})\rho(U_{n_0}).$$

Thus (U_n) is ρ-convergent to $U\in X$. Hence K is SO-ρ-complete. Let mapping $T:K\to K$ be defined as follows:

$$T(x) = \begin{cases}\{\frac{x_n}{4}\} & \text{if } 0\leq\rho(x)\leq\frac{1}{4},\\ 0 & \text{if } \rho(x)>\frac{1}{4}.\end{cases}$$

It is obvious that T is $\perp$-preserving, because if for each $x,y\in K$ such that $x\perp y$, then $x=0$ or $y=0$ or $x=y$, hence $Tx=0$or $Ty=0$ or $Tx=Ty$. In each case $Tx\perp Ty$. Now, we claim that T is Kannan ρ-contraction. Let $x,y\in K$ such that $x\perp y$. We have *the following cases:*

- *Case 1:* $x = 0$ *and* $\rho(y) \le \frac{1}{4}$ *(or* $y = 0$ *and* $\rho(x) \le \frac{1}{4}$*). Then for* $\lambda = \frac{5}{12}$ *we obtain*

$$\rho(Tx - Ty) = \rho(Ty) = \frac{1}{4}\rho(y) \le \frac{5}{16}\rho(y) = \frac{5}{12} \times \frac{3}{4}\rho(y) = \frac{5}{12}(\rho(x - Tx) + \rho(y - Ty)).$$

- *Case 2:* $x = 0$ *and* $\rho(y) > \frac{1}{4}$ *(or* $y = 0$ *and* $\rho(x) > \frac{1}{4}$*). Then for* $\lambda = \frac{5}{12}$ *we obtain*

$$\rho(Tx - Ty) = 0 \le \frac{5}{12}(\rho(x - Tx) + \rho(y - Ty)).$$

- *Case 3:* $x = y$ *and* $\rho(x) = \rho(y) \le \frac{1}{4}$ *(or* $x = y$ *and* $\rho(x) = \rho(y) > \frac{1}{4}$*). Then for* $\lambda = \frac{5}{12}$ *we obtain*

$$\rho(Tx - Ty) = \rho(Ty) = \frac{1}{4}\rho(y) \le \frac{5}{16}\rho(y) = \frac{5}{12} \times \frac{3}{4}\rho(y) = \frac{5}{12}(\rho(x - Tx) + \rho(y - Ty)).$$

Hence, for $\lambda = \frac{5}{12}$*, all conditions of Theorem 3.1 are satisfied.*

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