TWMS J. App. and Eng. Math. V.13, N.4, 2023, pp. 1369-1382

SOLUTION OF THE OPTIMAL PROGRAM TRAJECTORY AND CONTROL OF THE DISCRETIZED EQUATION OF MOTION OF SUCKER-ROD PUMPING UNIT IN A NEWTONIAN FLUID

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Abstract. In this paper, the problem of construction of the optimal trajectory and control for oscillatory systems with liquid dampers in the movement of the sucker-rod pumping unit is considered. Firstly, the equation of motion is reduced to the second order Volterra integral equation, this equation is discretized, odd and even indices are generated for the solution. These indexes is collected in one solution. Then, the quadratic functional is constructed and by means of extended functional the Euler-Lagrange equations are obtained to construct the optimal program trajectory and control. The solution of the problem is found from the obtained system of equations. An algorithm for its calculation process is proposed. The obtained results are realized through an example.

Keywords: sucker-rod pumping unit, optimal program trajectory, control, fractional order derivative, second order Volterra integral equation, Euler-Lagrange equations.

AMS Subject Classification: 49J15, 49J35, 11E04, 15A06.

1. Introduction

The role of fractional derivative is important in solving a number of problems, and they are realized with differential equations that include fractional derivative [6, 7, 15, 25]: sucker-rod pumping unit [23, 24], metal memory determination, oscillatory systems with liquid dampers [1, 19, 22] and others [11, 16, 23], such problems are described by fractional derivative [2, 3]. In the non-interruptible case the equation of the given problem is reduced to the second order Volterra integral equation [18, 29], discretized and the equation of motion is described by difference equations [8, 20]. After that, the boundary conditions

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§ Manuscript received: September 23, 2020; accepted: April 22, 2021.

are given and these boundary conditions perform the construction for the periodic continuation of the problem [21, 24]. Then the process of discretization begins according to the boundary and periodicity conditions [10, 12]. A functional and an extended functional are constructed so that a solution that minimizes the value of this extended functional is sought [23]. In finding this solution, a system of equations consisting of Euler-Lagrange equations is established and the solution of the system is found [4, 14].

2. Problem Statement

Let the motion of oscillatory systems with liquid dampers during oil production by sucker-rod pumping unit is described by the system of ordinary linear differential equations with fractional derivatives [28] and boundary conditions as follows [17]:

$$m_1\ddot{y}(x) + aD^{\alpha}y(x) + by(x) = f(x), 0 < x_0 < x < l + x_0, \tag{1}$$

$$\begin{cases} y(l + x_0 + 0) = y(l + x_0 - 0), \\ \dot{y}(l + x_0 + 0) = -\dot{y}(l + x_0 - 0) + V_1, \end{cases} \tag{2}$$

$$m_2\ddot{y}(x) + aD^{\alpha}y(x) + by(x) = f(x), l + x_0 < x < 2l + x_0, \tag{3}$$

$$\begin{cases} y(2l + x_0) = y(2l + x_0 - 0), \\ \dot{y}(2l + x_0) = -\dot{y}(2l + x_0 - 0) + V_2, \end{cases} \tag{4}$$

where $y(x)$ is a required continuous function, a, b, V_1, V_2, are real numbers, *f(x)* is a continuous scalar function,$\alpha = \frac{p}{q} \in (1, 2)$, p, q are natural numbers.

Let we have the following boundary condition as periodical case

$$\begin{cases} y(2l + x_0) = y(x_0), \\ \dot{y}(2l + x_0) = \dot{y}(x_0). \end{cases} \tag{5}$$

If we discretize the problem (1)-(5) and take some notations, then we get [5,27]:

$$W_{i+1} = \psi_i W_i + A_i W_0 + F_i, i = \overline{0, n-1}, \tag{6}$$

$$W_{j+1} = \psi_j W_j + A_j W_0 + F_j,\ j = \overline{n, 2n-1}, \tag{7}$$

$$\begin{array}{l} \begin{pmatrix} -1 & 1 \end{pmatrix} W_n = 0 \\ \begin{pmatrix} 1 & 0 \end{pmatrix} (W_{n+1} - W_n) = hV_1 \\ \begin{pmatrix} -1 & 1 \end{pmatrix} W_{2n} = 0 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} W_0 + \begin{pmatrix} 0 & 1 \end{pmatrix} W_{2n} = 0 \\ \begin{pmatrix} -1 & 1 \end{pmatrix} (W_0 + W_{2n}) = hV_2 \end{array} \tag{8}$$

where $W_n = \begin{pmatrix} y_{2n} \\ y_{2n+1} \end{pmatrix}$

$$A_i = \left(\left(\sum_{k=1}^{m-2} \sum_{1<i_1<i_2<\ldots<i_k \le m-2} \prod_{j=1}^{k} \psi_{i_{k+1-j}} \right) \psi_0 + \psi_0 \right), i = \overline{0, n-1} \tag{9}$$

$$A_j = \left(\left(\sum_{k=1}^{m-2} \sum_{1<j_1<j_2<\ldots<j_k \le m-2} \prod_{g=1}^{k} \psi_{j_{k+1-g}} \right) \psi_0 + \psi_0 \right),\ j = \overline{n, 2n-1} \tag{10}$$

$$\psi_{(n-1)} = \begin{pmatrix} \mathrm{A}_{11}^{(n-1)} & \mathrm{A}_{12}^{(n-1)} \\ \\ \mathrm{A}_{21}^{(n-1)} & \mathrm{A}_{22}^{(n-1)} \end{pmatrix},$$

$$\begin{aligned}
\mathrm{A}_{11}^{(n-1)} &= 1 - h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-4})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-4})\right] + \\
&\quad 2h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-3})\right], \\
\mathrm{A}_{12}^{(n-1)} &= 2 - h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-3})\right] \\
\mathrm{A}_{21}^{(n-1)} &= -2 - 2h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-4})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-4})\right] + \\
&\quad 4h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-3})\right] - \\
&\quad h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-4})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-1}-x_{2n-4})\right] + \\
&\quad 2h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-1}-x_{2n-3})\right] + \\
&\quad h^2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-1}-x_{2n-2})\right] \times \\
&\quad \left\{\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-4})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-4})\right] - \right. \\
&\quad \left. 2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-3})\right]\right\}, \\
\mathrm{A}_{22}^{(n-1)} &= 3 - 2h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-3})\right] - \\
&\quad h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-1}-x_{2n-3})\right] + \\
&\quad h^2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-1}-x_{2n-2})\right] \times \\
&\quad \left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2n-3})\right],
\end{aligned} \tag{11}$$

$$\psi_{(k)} = \begin{pmatrix} \mathrm{A}_{11}^{(k)} & \mathrm{A}_{12}^{(k)} \\ \mathrm{A}_{21}^{(k)} & \mathrm{A}_{22}^{(k)} \end{pmatrix} k = \overline{0, n-2}$$

$$\begin{aligned}
\mathrm{A}_{11}^{(k)} &= -2h\left\{\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2k-2})\right] - \right. \\
&\quad 2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!} + \tfrac{b}{m_1}(x_{2n-2}-x_{2k-1})\right]
\end{aligned}$$

$$+\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k})\right]\Big\},k=\overline{1,n-2},$$

$$\mathrm{A}_{11}^{(0)}=-h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_0)\right],$$

$$\mathrm{A}_{12}^{(k)}=-h\Big\{\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k-1})\right]-$$

$$2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k})\right]+$$

$$\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k+1})\right]\Big\},k=\overline{1,n-2},$$

$$\mathrm{A}_{12}^{(0)}=2h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_0)\right]-$$

$$h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_1)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_1)\right]$$

$$\mathrm{A}_{21}^{(k)}=-h\Big\{2-h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2n-2})\right]\Big\}-$$

$$\Big\{\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k-2})\right]-$$

$$2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k-1})\right]+ \tag{12}$$

$$\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k})\right]\Big\}-$$

$$h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k-2})\right]-$$

$$2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k-1})\right]+$$

$$+\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k})\right]\Big\},k=\overline{1,n-2},$$

$$\mathrm{A}_{21}^{(0)}=-h\Big\{2-h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2n-2})\right]\Big\}\times$$

$$\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_0)\right]-$$

$$h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_0)\right],$$

$$\mathrm{A}_{22}^{(k)}(a,n,h)=-h\Big\{2-h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2n-2})\right]\Big\}\times$$

$$\Big\{\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k-1})\right]-$$

$$-2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k})\right]+$$

$$\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_{2k+1})\right]-$$

$$h\left\{\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k-1})\right]-\right.$$

$$2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k})\right]+$$

$$\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k-1})\right]-$$

$$2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k})\right]+$$

$$\left.\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2k+1})\right]\right\}, k=\overline{1,n-2},$$

$$\mathrm{A}_{22}^{(0)}=h\left\{2-h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_{2n-2})\right]\right\}\times$$

$$\left\{2\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_0)\right]\right\}-$$

$$\left[\tfrac{a}{m_1}\tfrac{(x_{2n-2}-x_1)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-2}-x_1)\right]+2h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_0)\right]-$$

$$h\left[\tfrac{a}{m_1}\tfrac{(x_{2n-1}-x_1)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_1}(x_{2n-1}-x_1)\right],$$

$$\psi_{(n-1)}=\begin{pmatrix}\mathrm{A}_{11}^{(n-1)} & \mathrm{A}_{12}^{(n-1)}\\ \mathrm{A}_{21}^{(n-1)} & \mathrm{A}_{22}^{(n-1)}\end{pmatrix},$$

$$\mathrm{A}_{11}^{(n-1)}=1-h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_{2n-4})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_{2n-4})\right]+$$

$$2h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_{2n-3})\right],$$

$$\mathrm{A}_{12}^{(n-1)}=2-h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_{2n-3})\right] \tag{13}$$

$$\mathrm{A}_{21}^{(n-1)}=-2-2h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_{2n-4})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_{2n-4})\right]+$$

$$h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_{2n-3})\right]-$$

$$h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2n-4})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2n-4})\right]+$$

$$42h\left[\frac{a}{m_2}\frac{(x_{2n-1}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-1}-x_{2n-3})\right]+$$

$$h^2\left[\frac{a}{m_2}\frac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-1}-x_{2n-2})\right]\times$$

$$\left\{\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2n-4})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2n-4})\right]-\right.$$

$$\left.2\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2n-3})\right]\right\},$$

$$\mathrm{A}_{22}^{(n-1)}=3-2h\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2n-3})\right]-$$

$$h\left[\frac{a}{m_2}\frac{(x_{2n-1}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-1}-x_{2n-3})\right]+$$

$$h^2\left[\frac{a}{m_2}\frac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-1}-x_{2n-2})\right]\times$$

$$\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2n-3})\right],$$

$$\psi_{(p)}=\begin{pmatrix}\mathrm{A}_{11}^{(p)} & \mathrm{A}_{12}^{(p)}\\ \mathrm{A}_{21}^{(p)} & \mathrm{A}_{22}^{(p)}\end{pmatrix}p=\overline{n-1,2n-2},$$

$$\mathrm{A}_{11}^{(0)}=-h\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_0)\right],$$

$$\mathrm{A}_{12}^{(p)}=-h\left\{\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2p-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2p-1})\right]-\right.$$

$$2\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2p})\right]+$$

$$\left.\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2p+1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2p+1})\right]\right\},p=\overline{n-1,2n-2},$$

$$\mathrm{A}_{12}^{(0)}=2h\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_0)\right]- \tag{14}$$

$$h\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_1)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_1)\right],$$

$$\mathrm{A}_{21}^{(p)}=-h\left\{2-h\left[\frac{a}{m_2}\frac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2n-2})\right]\right\}-$$

$$\left\{\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2p-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2p-2})\right]-\right.$$

$$2\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2p-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-2}-x_{2p-1})\right]+$$

$$\left.\left[\frac{a}{m_2}\frac{(x_{2n-2}-x_{2p})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_2}(x_{2n-1}-x_{2p})\right]\right\}-$$

$$h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2p-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2p-2})\right]-$$

$$2\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2p-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2p-1})\right]+$$

$$+\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2p})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2p})\right]\Bigg\}, p=\overline{n-1,2n-2},$$

$$\mathrm{A}_{21}^{(0)}=-h\left\{2-h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2n-2})\right]\right\}\times$$

$$\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_0)\right]-$$

$$h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_0)\right],$$

$$\mathrm{A}_{22}^{(p)}=-h\left\{2-h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2n-2})\right]\right\}\times$$

$$\Bigg\{\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_{2p-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_{2p-1})\right]-$$

$$2\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_{2p})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_{2p})\right]+$$

$$\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_{2p+1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_{2p+1})\right]-$$

$$h\Bigg\{\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2p-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2p-1})\right]-$$

$$2\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2p})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2p})\right]+$$

$$\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2p-1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2p-1})\right]-$$

$$2\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2p})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2p})\right]+$$

$$\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2p+1})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2p+1})\right]\Bigg\}, p=\overline{n-1,2n-2},$$

$$\mathrm{A}_{22}^{(0)}=h\left\{2-h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_{2n-2})^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_{2n-2})\right]\right\}\times$$

$$\left\{2\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_0)\right]\right\}-$$

$$\left[\tfrac{a}{m_2}\tfrac{(x_{2n-2}-x_1)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-2}-x_1)\right]+$$

$$2h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_0)\right]-$$

$$h\left[\tfrac{a}{m_2}\tfrac{(x_{2n-1}-x_1)^{1-\alpha}}{(1-\alpha)!}+\tfrac{b}{m_2}(x_{2n-1}-x_1)\right],$$

It is required to find the minimum of the following quadratic functional [13]

$$J=\sum_{i=0}^{n-1} W_i' Q W_i+\sum_{j=n}^{2n-1} W_j' Q W_j+\sum_{i=0}^{2n-1} F_i' C F \rightarrow \min \tag{15}$$

where $Q=Q' \geq 0$ is a symmetric matrix of dimension $2n \times 2n$, $C \geq 0$ is a matrix of dimension $2n \times 2n$, the prime means the transposition operation.

3. Method of solving the problem

For solving the problem (6)-(8), (15) we construct the extended functional as follows [2, 9]:

$$\overline{J}=\alpha((-1\,1)W_n)+\beta((1\,0)(W_{n+1}-W_n)-V_1)+\gamma((-1\,1)W_{2n})+\xi((-1\,0)W_0+(0\,1)W_{2n})+$$

$$\eta((-1\,1)\left(W_0+W_{2n}-V_2\right)+\frac{1}{2}\left(\sum_{i=0}^{n-1} W_i' Q W_i+\lambda_{i+1}^T\left(\psi_i W_i+A_i W_0+F_i-W_{i+1}\right)\right)+$$

$$\sum_{j=n}^{2n-1}\left(W_j' Q W_j+\lambda_{j+1}^T(\psi_j W_j+A_j W_0+F_j-W_{j+1}))+\sum_{k=0}^{2n-1} F_k' C F_k\right) \tag{16}$$

where $\alpha, \beta, \gamma, \eta, \xi$ are scalars, λ_i- is 1×2 dimensional column vector.

$$\frac{\partial \overline{J}}{\partial W_0}=0, \frac{\partial \overline{J}}{\partial W_n}=0, \frac{\partial \overline{J}}{\partial W_{n+1}}=0, \frac{\partial \overline{J}}{\partial W_{2n}}=0, \frac{\partial \overline{J}}{\partial W_i}=0, \frac{\partial \overline{J}}{\partial W_j}=0, \frac{\partial \overline{J}}{\partial F_k}=0$$

Then similarly [9] we get the Euler-Lagrange equation for the problem (6)-(8), (15) will have the form [12]:

$$\begin{pmatrix} W_{i+1} \\ \lambda_i \end{pmatrix}=\begin{pmatrix} \psi_i & -C^{-1} \\ Q & \psi_i{}^T \end{pmatrix}\begin{pmatrix} W_i \\ \lambda_{i+1} \end{pmatrix}+\begin{pmatrix} A_i W_0 \\ \alpha(-1\ 1)^T-\beta(1\ 0)^T \end{pmatrix},\ i=\overline{0, n-1}$$

$$\begin{pmatrix} W_{j+1} \\ \lambda_j \end{pmatrix}=\begin{pmatrix} \psi_j & -C^{-1} \\ Q & \psi_j{}^T \end{pmatrix}\begin{pmatrix} W_j \\ \lambda_{j+1} \end{pmatrix}+\left(A_j W_0 \beta(1\ 0)^T+(\gamma+\eta)(-1\ 1)^T+\xi(0\ 1)^T\right),\ j=\overline{n, 2n-1} \tag{17}$$

We can solve the system of equations [19, 21] (17) by means of the MATLAB software package [26, 28].

4. Examples

1. Let consider the following example [17] applying the above method. Suppose the problem is given in the following way:

$$y''(x)+a D^\alpha y(x)=f(x),\ \ 0<x_0<x<l+x_0 \tag{18}$$

$$\left\{\begin{array}{l} y(l+x_0)=y(l+x_0-0), \\ \dot{y}(l+x_0)=-\dot{y}(l+x_0-0)+V, \end{array}\right. \tag{19}$$

$$\left\{\begin{array}{l} y(l+x_0)=y(x_0), \\ \dot{y}(l+x_0)=\dot{y}(x_0). \end{array}\right. \tag{20}$$

After discretizing the problem (18)-(20) analogously (6)-(8) we get

$$W_n = \phi_{n-1} W_{n-1} + P_n W_0 + R_n, n \geq 2 \tag{21}$$

$$\begin{cases} \begin{pmatrix} -1 & 1 \end{pmatrix} W_n = 0, \\ \begin{pmatrix} 1 & 0 \end{pmatrix} W_0 + \begin{pmatrix} 0 & 1 \end{pmatrix} W_n = 0, \\ \begin{pmatrix} 1 & -1 \end{pmatrix} (W_0 + W_n) = 1, \end{cases} \tag{22}$$

It is required to find the minimum of the following quadratic functional

$$J = \sum_{i=0}^{n-1} W_i' W_i + R_i' R_i \to \min \tag{23}$$

Now let solve this problem:

$$1. n = 5, m_1 = m_2 = 1, a = 3, b = 1, \alpha = \frac{5}{3}, Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V = 1.$$

2. $A_k, k = \overline{0, n-1}$ let's calculate the values of the expressions.

$$3. W_{i+1} = \psi_i W_i + P_i W_0 + R_i, i = \overline{0, n-1}$$

$$. \begin{cases} \begin{pmatrix} -1 & 1 \end{pmatrix} W_n = 0, \\ \begin{pmatrix} 1 & 0 \end{pmatrix} W_0 + \begin{pmatrix} 0 & 1 \end{pmatrix} W_n = 0, \\ \begin{pmatrix} 1 & -1 \end{pmatrix} (W_0 + W_n) = 1, \end{cases}$$

4. Extended functional:

$$\overline{J} = \alpha \left(\begin{pmatrix} -1 & 1 \end{pmatrix} W_n \right) + \beta \left(\begin{pmatrix} 1 & 0 \end{pmatrix} W_0 + \begin{pmatrix} 0 & 1 \end{pmatrix} W_n \right) + \gamma \left(\begin{pmatrix} 1 & -1 \end{pmatrix} (W_0 + W_n) - V \right) +$$
$$+ \frac{1}{2} \sum_{i=0}^{n-1} \left(W_i' W_i + R_i' R_i + \lambda_{i+1}^T \left(\psi_i W_i + P_i W_0 + R_i - W_{i+1} \right) \right)$$

5. Then we get the Euler-Lagrange equation for the problem (21)-(23) will have the form

$$\begin{pmatrix} W_{i+1} \\ \lambda_i \end{pmatrix} = \begin{pmatrix} \psi_i & -C^{-1} \\ Q & \psi_i^T \end{pmatrix} \begin{pmatrix} W_i \\ \lambda_{i+1} \end{pmatrix} + \begin{pmatrix} A_i W_0 \\ \beta \begin{pmatrix} 1 & 0 \end{pmatrix}^T + \gamma \begin{pmatrix} 1 & -1 \end{pmatrix}^T \end{pmatrix}, i = \overline{0, n-1}$$

6. We can use the MATLAB program package to build the solution matrix of the problem and find the unknowns. Then we calculate $J_{\min}$.

Now let plot the graph of dependence of the controller [22], general solution [26] for each n in MATLAB:

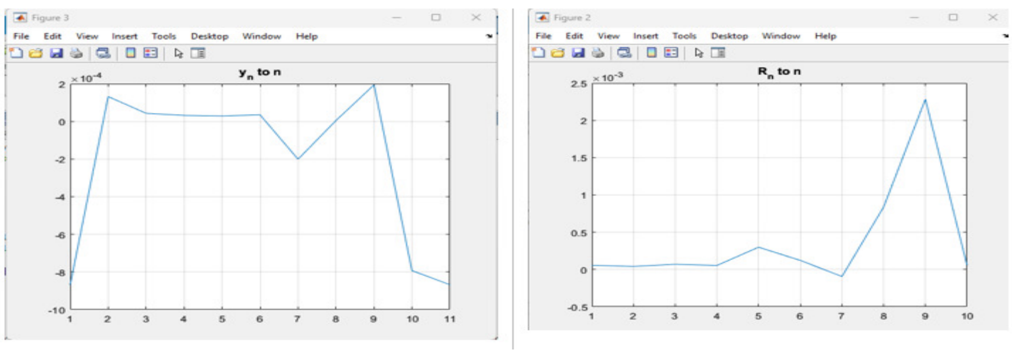

Figure 1. Graph of dependence of y_n on n and Graph of dependence of R_n on n.

Steps	l
10^{-1}	2.636643731144403e-01
10^{-2}	4.625551306168188e-02
10^{-3}	1.073514619372248e-02
10^{-4}	1.035232859585342e-02
10^{-5}	1.033074019816137e-02
10^{-6}	1.032905426276624e-02
10^{-7}	1.032860519396068e-02

Lets the problem is given in the following way:

$$m_1\ddot{y}(x) + aD^{\alpha}y(x) + by(x) = f(x), \tag{24}$$

$$0 < x_0 < x < l + x_0 \begin{cases} y(l+x_0) = y(l+x_0-0), \\ \dot{y}(l+x_0) = -\dot{y}(l+x_0-0) + V, \end{cases} \tag{25}$$

$$\begin{cases} y(l+x_0) = y(x_0), \\ \dot{y}(l+x_0) = \dot{y}(x_0). \end{cases} \tag{26}$$

After discretizing the problem (24)-(26) analogously (6)-(8) we get

$$W_{i+1} = \psi_i W_i + P_i W_0 + R_i, i = \overline{0, n-1}, \tag{27}$$

$$\begin{cases} \begin{pmatrix} -1 & 1 \end{pmatrix} W_n = 0, \\ \begin{pmatrix} 1 & 0 \end{pmatrix} W_0 + \begin{pmatrix} 0 & 1 \end{pmatrix} W_n = 0, \\ \begin{pmatrix} 1 & -1 \end{pmatrix}(W_0 + W_n) = 1, \end{cases} \tag{28}$$

It is required to find the minimum of the following quadratic functional

$$J = \sum_{i=0}^{n-1} W_i' W_i + R_i' R_i \to \min, \tag{29}$$

Now let's solve this problem:

$$1. n = 5, m_1 = m_2 = 3, a = 7, b = 5, \alpha = \frac{7}{5}, Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V = 3.$$

$$2. W_{i+1} = \psi_i W_i + P_i W_0 + R_i, i = \overline{0, n-1} n \geq 2$$

$$\begin{cases} \begin{pmatrix} -1 & 1 \end{pmatrix} W_n = 0, \\ \begin{pmatrix} 1 & 0 \end{pmatrix} W_0 + \begin{pmatrix} 0 & 1 \end{pmatrix} W_n = 0, \\ \begin{pmatrix} 1 & -1 \end{pmatrix}(W_0 + W_n) = 1, \end{cases}$$

3. After constructing extended functional, we get the Euler-Lagrange equation for the problem (27)-(29) and we can use the MATLAB program package to build the solution matrix of the problem.

Now let's plot the graph of dependence of the controller, general solution for each n in MATLAB:

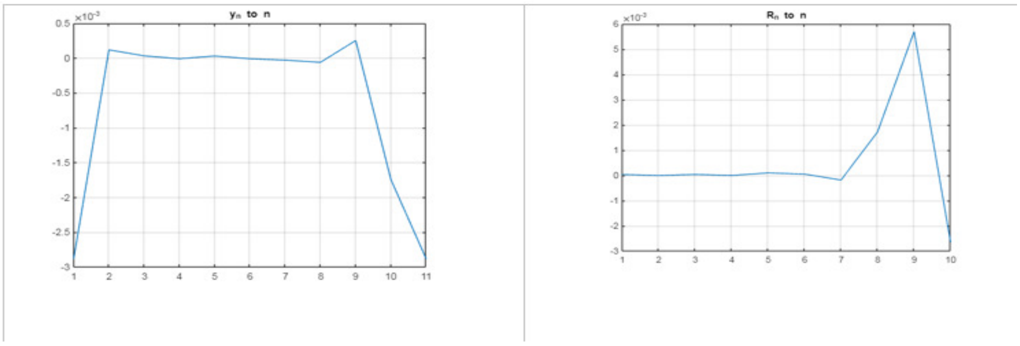

Figure 2. Graph of dependence of y_n on n and Graph of dependence of R_n on n.

Let's consider the following example applying the above method. Suppose the problem is given in the following way:

$$m_1\ddot{y}(x) + aD^{\alpha}y(x) = f(x), 0 < x_0 < x < l + x_0, \tag{30}$$

$$\begin{cases} y(l + x_0 + 0) = y(l + x_0 - 0), \\ \dot{y}(l + x_0 + 0) = -\dot{y}(l + x_0 - 0) + V_1, \end{cases} \tag{31}$$

$$m_2\ddot{y}(x) + aD^{\alpha}y(x) = f(x), l + x_0 < x < 2l + x_0, \tag{32}$$

$$\begin{cases} y(2l + x_0) = y(2l + x_0 - 0), \\ \dot{y}(2l + x_0) = -\dot{y}(2l + x_0 - 0) + V_2, \end{cases} \tag{33}$$

And boundary condition as periodical case

$$\begin{cases} y(2l + x_0) = y(x_0), \\ \dot{y}(2l + x_0) = \dot{y}(x_0). \end{cases} \tag{34}$$

After discretizing the problem (30)-(34) analogously (6)-(8) [23,26] we get

$$W_{i+1} = \psi_i W_i + P_i W_0 + R_i, i = \overline{0, n-1}, \tag{35}$$

$$W_{j+1} = \psi_j W_j + P_j W_0 + R_j, j = \overline{n, 2n-1} \tag{36}$$

$$\begin{array}{l} \begin{pmatrix} -1 & 1 \end{pmatrix} W_n = 0 \\ \begin{pmatrix} 1 & 0 \end{pmatrix} (W_{n+1} - W_n) = hV_1 \\ \begin{pmatrix} -1 & 1 \end{pmatrix} W_{2n} = 0 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} W_0 + \begin{pmatrix} 0 & 1 \end{pmatrix} W_{2n} = 0 \\ \begin{pmatrix} -1 & 1 \end{pmatrix} (W_0 + W_{2n}) = hV_2 \end{array} \tag{37}$$

It is required to find the minimum of the following quadratic functional

$$J = \sum_{i=0}^{n-1} W_i' W_i + R_i' R_i + \sum_{j=n}^{2n-1} W_j' W_j + R_j' R_j \to \min, \tag{38}$$

Now let's solve this problem:

$$1. n = 4, m_1 = 2, m_2 = 3, a = 2, \alpha = \frac{5}{4}, Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V_1 = 1, V_2 = 2.$$

$$2.W_{i+1} = \psi_i W_i + P_i W_0 + R_i, i = \overline{0, n-1},$$

$$W_{j+1} = \psi_j W_j + P_j W_0 + R_j, j = \overline{n, 2n-1}$$

$$\begin{aligned}
&\begin{pmatrix} -1 & 1 \end{pmatrix} W_n = 0 \\
&\begin{pmatrix} 1 & 0 \end{pmatrix} (W_{n+1} - W_n) = hV_1 \\
&\begin{pmatrix} -1 & 1 \end{pmatrix} W_{2n} = 0 \\
&\begin{pmatrix} -1 & 0 \end{pmatrix} W_0 + \begin{pmatrix} 0 & 1 \end{pmatrix} W_{2n} = 0 \\
&\begin{pmatrix} -1 & 1 \end{pmatrix} (W_0 + W_{2n}) = hV_2
\end{aligned}$$

3. After constructing extended functional, we get the Euler-Lagrange equation for the problem (35)-(38) and we can use the MATLAB program package to build the solution matrix of the problem.
Now let's plot the graph of dependence of the controller, general solution for each n in MATLAB:

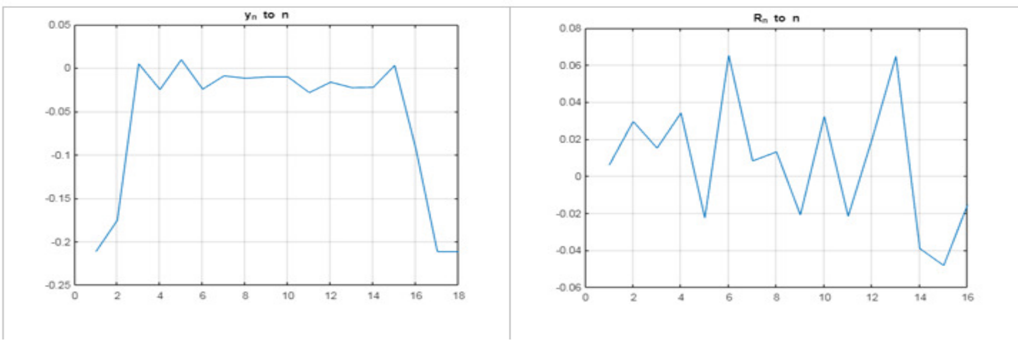

Figure 3. Graph of dependence of y_n on n and Graph of dependence of R_n on n.

Compare. Let compare the result of the given discrete case with the continuous case [17] using the following table. In the table$l = \|x_i - y(l_i)\|$, where x_i is the phase coordinate of the object in continuous case and $y(l_i)$ is the phase coordinate of the object in discrete case. The table shows that the obtained results differ from each others 10^{-2} order.

5. Conclusions

In this paper the optimal program trajectory and control for oscillatory systems with liquid dampers by periodic boundary condition in discrete case, where the second term contains fractional order derivative have been constructed. After discretizing the equation and boundary conditions the extended functional has been constructed and the Euler-Lagrange equations have been found. Then the obtained Euler-Lagrange equations have been solved by means of the MATLAB software package. At the end the numerical examples are proposed, the graphs of optimal program trajectory and control have been shown.

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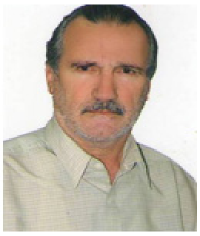
Aliev Fikret for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.7, N.1.



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