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SOME HERMITE–HADAMARD INEQUALITIES INVOLVING WEIGHTED INTEGRAL OPERATORS VIA (h, s, m)-CONVEX FUNCTIONS

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Abstract. In this work, we establish new integral inequalities of Hermite–Hadamard type, within the framework of generalized integrals, which contain the Riemann–Liouville fractional integrals as particular case. We use a definition of convexity that includes the classical convex, m-convex, s-convex functions, among others. We show that several known results from the literature are closely related to ours.

Keywords: Hermite–Hadamard integral inequality, generalized integral operators, (h, s, m)-convex functions.

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1. Introduction

A function $f : I \to \mathbb{R}$, $I := [a, b]$ is said to be convex if $f\big(ta+(1-t)b\big) \le tf(a)+(1-t)f(b)$ holds for all $a, b \in I$ and $t \in [0, 1]$. If the above inequality is reversed, then function f is called concave on $[a, b]$. Convex functions have been investigated and generalized widely; these extensions incorporate the m-convex, n-convex, r-convex, h-convex, (h, m)-convex, s-convex functions and numerous others. Readers interested in its multiple extensions and ramifications, can consult e.g. [21], where a fairly complete overview of the development of the convex function concept is presented.

One of the most important inequalities, that has attracted many inequality experts in the last few decades, is the famous Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{a-b}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

holds for any function f convex on the interval $[a, b]$. This inequality was published by Hermite in 1883 and, independently, by Hadamard in 1893. It gives an estimation of

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the mean value of a convex function, and it is important to note that it also provides a refinement to the Jensen inequality.

In the last decades, there has been an intensive development in extending the Hermite–Hadamard inequality. The interested reader is referred to [6, 9, 11, 12, 15, 19, 26, 35] and references therein for more information and extensions. These papers employ various ways of generalization.

An important direction is to utilize the concept of generalized convex functions, see e.g. [16, 24, 36, 37, 38, 42].

The following definition will be the basis of our work.

Definition 1.1. *Let $h : [0,1] \to [0,\infty)$, $h \not\equiv 0$. Function $f : I \subseteq [0,\infty) \to [0,\infty)$ is said to be (h,s,m)-convex on I if inequality*

$$f\left(ta + m(1-t)b\right) \le h^s(t)f(a) + m(1-h(t))^s f(b)$$

is fulfilled for $m \in [0,1]$, $s \in [-1,1]$, for all $a, b \in I$ and $t \in [0,1]$.

Remark 1.1. *Consider special cases of Definition 1.1.*

(1) *If $h(t) = t^\alpha$ with $\alpha \in (0,1]$, then f is an (α,s,m)-convex function on I (see [36]).*
(2) *If $h(t) = t$, then f is an extended (s,m)-convex function on I (see [38]).*
(3) *If $h(t) = t$, $s \in [-1,1]$ and $m = 1$, then f is an extended s-convex function on I (see [37]).*
(4) *If $h(t) = t$, $s \in (0,1]$ and $m = 1$, then f is an s-convex function (in the second sense) on I (see [7, 14]).*
(5) *If $h(t) = t$ and $s = m = 1$, then f is a convex function on I.*

Analyzing the recent extensions of the Hermite–Hadamard inequality, one can see that another important direction of generalization is through the concept of fractional integrals, see e.g. papers [1, 2, 8, 10, 13, 28, 32, 41]. We remark that the concepts of generalized convex functions and fractional integrals are often considered at the same time.

All through this work we utilize the functions $\Gamma(z)$ (see [29, 30, 39, 40]) and $\Gamma_k(z)$ (see [12]):

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \quad \mathrm{Re}(z) > 0,$$
$$\Gamma_k(z) = \int_0^\infty t^{z-1}e^{-t^k/k}\,dt, \quad \mathrm{Re}(z) > 0, k > 0.$$

Obviously, if $k \to 1$, then we have $\Gamma_k(z) \to \Gamma(z)$, furthermore, $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

To encourage comprehension of the subject, we present some specific fractional integral definitions in case of $[a,b] \subseteq [0,\infty)$. The first ones are the classic Riemann–Liouville fractional integrals (see [27]).

Definition 1.2. *Let $f \in L^1([a,b])$. Then the Riemann–Liouville fractional integrals (right and left, respectively) of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ are defined by*

$${}^{\alpha}I_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt, \quad x > a,$$
$${}^{\alpha}I_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt, \quad x < b.$$

One can also define the k-Riemann–Liouville fractional integrals (see [20]).

Definition 1.3. *If $k > 0$, let $f \in L^1([a, b])$, then the left and right k-Riemann–Liouville fractional integrals of order $\alpha > 0$ are defined by*

$$^{\alpha,k}I_{a^+} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt, \quad x > a,$$
$$^{\alpha,k}I_{b^-} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)\,dt, \quad x < b.$$

Next we present the weighted integral operators that will be the basis of our work.

Definition 1.4. *Let $f \in L^1([a, b])$ and let $w : [0, \infty) \to [0, \infty)$ be a continuous function with first and second order derivatives piecewise continuous on $[0, \infty)$. Then the weighted fractional integrals (right and left, respectively) are defined by*

$$^{n+1}I^w_{a^+} f(x) = \int_a^x w''\left(\frac{x-t}{\frac{b-a}{n+1}}\right) f(t)\,dt, \quad x > a,$$
$$^{n+1}I^w_{b^-} f(x) = \int_x^b w''\left(\frac{t-x}{\frac{b-a}{n+1}}\right) f(t)\,dt, \quad x < b.$$

Remark 1.2. *If we consider $n = 1$ and $w(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$, then the above integral operators $^{n+1}I^w_{a^+}$ and $^{n+1}I^w_{b^-}$ become the classic Riemann–Liouville fractional integral operators (see Definition 1.2). In case of $n = 1$ and $w(t) = \frac{t^{\alpha/k+1}}{k^{\alpha/k}\Gamma(\alpha/k+2)}$, operators $^{n+1}I^w_{a^+}$ and $^{n+1}I^w_{b^-}$ become the k-Riemann–Liouville fractional operators (see Definition 1.3). Other operators can be considered by choosing different weight functions.*

In this paper we derive some Hermite–Hadamard-tpye inequalities via (h, s, m)-convex functions, within the framework of the weighted integral operators of Definition 1.4. Throughout the paper, we assume that weight function $w : [0, \infty) \to [0, \infty)$ is continuous with first and second order derivatives piecewise continuous on $[0, \infty)$.

2. Main results

First, we prove the following preliminary result, which is a basic tool for getting our results.

Lemma 2.1. *Let f be a real function defined on some interval $[a, b] \subset \mathbb{R}$, twice differentiable on (a, b). If $f'' \in L^1([a, b])$, then we have the following equality:*

$$\begin{aligned}
&\left[{}^{n+1}I^w_{a^+} f\left(\frac{na}{n+1} + \frac{b}{n+1}\right) + {}^{n+1}I^w_{b^-} f\left(\frac{a}{n+1} + \frac{nb}{n+1}\right)\right] \\
&- \frac{b-a}{n+1}\left[w'(1)\left(f(a) + f(b)\right) - w'(0)\left(f\left(\frac{na}{n+1} + \frac{b}{n+1}\right) + f\left(\frac{a}{n+1} + \frac{nb}{n+1}\right)\right)\right] \\
&- \frac{(b-a)^2}{(n+1)^2}\left[w(1)\left(f'(a) - f'(b)\right) - w(0)\left(f'\left(\frac{na}{n+1} + \frac{b}{n+1}\right) - f'\left(\frac{a}{n+1} + \frac{nb}{n+1}\right)\right)\right] \\
=&\frac{(b-a)^3}{(n+1)^3}\int_0^1 w(t)\left[f''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) + f''\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right]dt.
\end{aligned}$$

Proof. First note that

$$\int_0^1 w(t)\left[f''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]dt$$
$$=\int_0^1 w(t)f''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt+\int_0^1 w(t)f''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt=I_1+I_2.$$

After integrating by parts twice, we obtain

$$\begin{aligned}I_1=&\frac{n+1}{b-a}\left[-w(1)f'(a)+w(0)f'\left(\frac{na}{n+1}+\frac{b}{n+1}\right)\right]\\&+\frac{(n+1)^2}{(b-a)^2}\left[-w'(1)f(a)+w'(0)f\left(\frac{na}{n+1}+\frac{b}{n+1}\right)\right]\\&+\frac{(n+1)^2}{(b-a)^2}\int_0^1 w''(t)f\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt\\=&\frac{n+1}{b-a}\left[w(0)f'\left(\frac{na}{n+1}+\frac{b}{n+1}\right)-w(1)f'(a)\right]\\&-\frac{(n+1)^2}{(b-a)^2}\left[w'(1)f(a)-w'(0)f\left(\frac{na}{n+1}+\frac{b}{n+1}\right)\right]\\&+\frac{(n+1)^3}{(b-a)^3}\,{}^{n+1}I^w_{a^+}f\left(\frac{na}{n+1}+\frac{b}{n+1}\right),\end{aligned}$$

since substituting $u=\frac{n+t}{n+1}a+\frac{1-t}{n+1}b$ yields

$$\int_0^1 w''(t)f\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt=\frac{n+1}{b-a}\int_a^{\frac{na}{n+1}+\frac{b}{n+1}} w''\left(\frac{\frac{na}{n+1}+\frac{b}{n+1}-u}{\frac{b}{n+1}-\frac{a}{n+1}}\right)f(u)\,du.$$

Analogously,

$$\begin{aligned}I_2=&\frac{n+1}{b-a}\left[w(1)f'(b)-w(0)f'\left(\frac{a}{n+1}+\frac{nb}{n+1}\right)\right]\\&-\frac{(n+1)^2}{(b-a)^2}\left[w'(1)f(b)-w'(0)f\left(\frac{a}{n+1}+\frac{nb}{n+1}\right)\right]\\&+\frac{(n+1)^3}{(b-a)^3}\,{}^{n+1}I^w_{b^-}f\left(\frac{a}{n+1}+\frac{nb}{n+1}\right).\end{aligned}$$

By adding I_1 and I_2, reordering, we obtain the desired result. □

Remark 2.1. *If we put* $w(t)=(1-t)^{\alpha+1}$*, then it is easy to check that from this result,* [23, Lemma 1.3] *is obtained. Lemma 2.1 extends* [18, Lemma 2.1]. *If we consider* $n=1$*,* $w(t)=\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$ *and* $w(t)=\frac{t^{\alpha/k+1}}{k^{\alpha/k}\Gamma(\alpha/k+2)}$*, Lemma 2.1 becomes* [33, Lemma 1] *and* [18, Corollary 2.1]*, respectively.*

Remark 2.2. *With* $w(t)=(1-t)^{\alpha}$ *and* $n=0$*, we can obtain a new result for Riemann–Liouville integrals. By taking* $n=0$ *and* $w(t)=t(1-t)^{\alpha}$*, one can obtain* [5, Lemma 1.5] *(see also* [4]*). Also, if* $n=0$ *and*

$$w(t)=\begin{cases}t^2, & t\in[0,\frac12)\\(1-t)^2, & t\in[\frac12,1]\end{cases},$$

Lemma 2.1 becomes [31, Lemma 2].

Our first main inequality is the following.

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° (the interior of I) such that $f'' \in L^1\left(\left[a, \frac{b}{m}\right]\right)$. Under the assumptions of Lemma 2.1, if $|f''|$ is (h, s, m)-convex on $\left[a, \frac{b}{m}\right]$, we have the following inequality:*

$$\begin{aligned}&\left|{}^{n+1}I^w_{a^+}f\left(\frac{na}{n+1}+\frac{b}{n+1}\right)+{}^{n+1}I^w_{b^-}f\left(\frac{a}{n+1}+\frac{nb}{n+1}\right)-A\right|\\ \le&\frac{(b-a)^2}{(n+1)^2}\left(B_1\left|f''(a)\right|+mB_2\left|f''\left(\frac{b}{m}\right)\right|\right)\end{aligned}\tag{1}$$

with

$$\begin{aligned}A=&\frac{b-a}{n+1}\left[w'(1)\left(f(a)+f(b)\right)-w'(0)\left(f\left(\frac{na}{n+1}+\frac{b}{n+1}\right)+f\left(\frac{a}{n+1}+\frac{nb}{n+1}\right)\right)\right]\\ &\frac{(b-a)^2}{(n+1)^2}\left[w(1)\left(f'(a)-f'(b)\right)-w(0)\left(f'\left(\frac{na}{n+1}+\frac{b}{n+1}\right)-f'\left(\frac{a}{n+1}+\frac{nb}{n+1}\right)\right)\right],\end{aligned}$$

and

$$\begin{aligned}B_1&=\int_0^1 w(t)\left[h^s\left(\frac{n+t}{n+1}\right)+h^s\left(\frac{1-t}{n+1}\right)\right]dt,\\ B_2&=\int_0^1 w(t)\left[\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s+\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s\right]dt.\end{aligned}$$

Proof. From Lemma 2.1, we obtain

$$\begin{aligned}&\left|\int_0^1 w(t)\left[f''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]dt\right|\\ \le&\int_0^1 w(t)\left|f''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt+\int_0^1 w(t)\left|f''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt.\end{aligned}\tag{2}$$

Using the (h, s, m)-convexity of $|f''|$, we get

$$\begin{aligned}&\int_0^1 w(t)\left|f''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt\\ \le&\int_0^1 w(t)\left[h^s\left(\frac{n+t}{n+1}\right)\left|f''(a)\right|+m\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s\left|f''\left(\frac{b}{m}\right)\right|\right]dt\\ =&\left|f''(a)\right|\int_0^1 w(t)h^s\left(\frac{n+t}{n+1}\right)dt+m\left|f''\left(\frac{b}{m}\right)\right|\int_0^1 w(t)\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s dt.\end{aligned}\tag{3}$$

In the same way,

$$\begin{aligned}&\int_0^1 w(t)\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt\\ \le&\left|f''(a)\right|\int_0^1 w(t)h^s\left(\frac{1-t}{n+1}\right)dt+m\left|f''\left(\frac{b}{m}\right)\right|\int_0^1 w(t)\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s dt.\end{aligned}\tag{4}$$

From (3) and (4), we easily obtain (1). The theorem is proved. ☐

Remark 2.3. *Theorem 2.1 extends* [18, Theorem 2.1]. *If in Theorem 2.1 we set $n = 1$, in case of f'' is convex and put $w(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$, we get* [33, Theorem 3]. *In case of f'' is s-convex (in the second sense) and $w(t) = (1-t)^{\alpha+1}$, we obtain* [23, Theorem 2.1].

Refinements of the previous results, can be obtained by imposing new additional conditions on $|f''|^q$.

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L^1\left(\left[a, \frac{b}{m}\right]\right)$. Under the assumptions of Lemma 2.1 if $|f''|^q$, $q > 1$, is (h, s, m)-convex on $\left[a, \frac{b}{m}\right]$, we have*

$$\left| {}^{n+1}I^w_{a^+} f\left(\frac{na}{n+1} + \frac{b}{n+1}\right) + {}^{n+1}I^w_{b^-} f\left(\frac{a}{n+1} + \frac{nb}{n+1}\right) - A\right|$$
$$\leq \frac{(b-a)^2}{(n+1)^2} C_q \left[\left(C_{11}|f''(a)|^q + mC_{12}\left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}} + \left(C_{21}|f''(a)|^q + mC_{22}\left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

with A as before, $C_q = \left(\int_0^1 w^p(t)\,dt\right)^{\frac{1}{p}}$, $C_{11} = \int_0^1 h^s\left(\frac{n+t}{n+1}\right) dt$, $C_{12} = \int_0^1 \left(1 - h\left(\frac{n+t}{n+1}\right)\right)^s dt$, $C_{21} = \int_0^1 h^s\left(\frac{1-t}{n+1}\right) dt$, $C_{22} = \int_0^1 \left(1 - h\left(\frac{1-t}{n+1}\right)\right)^s dt$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. As in the previous result, Lemma 2.1 yields (2). From Hölder's inequality, we get

$$\begin{aligned} &\int_0^1 w(t)\left|f''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right| dt \\ &\leq \left(\int_0^1 w^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\int_0^1 w(t)\left|f''\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right| dt \\ &\leq \left(\int_0^1 w^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f''\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{6}$$

Using the (h, s, m)-convexity of $|f''|^q$, we obtain from (5) and (6):

$$\begin{aligned} &\int_0^1 \left|f''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right|^q dt \\ &\leq |f''(a)|^q \int_0^1 h^s\left(\frac{n+t}{n+1}\right) dt + m\left|f''\left(\frac{b}{m}\right)\right|^q \int_0^1 \left(1 - h\left(\frac{n+t}{n+1}\right)\right)^s dt, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\int_0^1 \left|f''\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right|^q dt \\ &\leq |f''(a)|^q \int_0^1 h^s\left(\frac{1-t}{n+1}\right) dt + m\left|f''\left(\frac{b}{m}\right)\right|^q \int_0^1 \left(1 - h\left(\frac{1-t}{n+1}\right)\right)^s dt. \end{aligned} \tag{8}$$

Denoting $C_q = \left(\int_0^1 w^p(t)\,dt\right)^{\frac{1}{p}}$, substituting (7) and (8) in (5) and (6) results in the required inequality. □

Remark 2.4. *If we consider $|f''|^q$ as an s-convex function (in the second sense) and $w(t) = (1-t)^{\alpha+1}$, then this result becomes* [23, Theorem 2.2].

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L^1\left(\left[a, \frac{b}{m}\right]\right)$. Under the assumptions of Lemma 2.1, if $|f''|^q$, $q > 1$, is (h,s,m)-convex on $\left[a, \frac{b}{m}\right]$, we have the following inequality:*

$$\left| {}^{n+1}I^w_{a^+} f\left(\frac{na}{n+1} + \frac{b}{n+1}\right) + {}^{n+1}I^w_{b^-} f\left(\frac{a}{n+1} + \frac{nb}{n+1}\right) - A \right|$$
$$\le \frac{(b-a)^2}{(n+1)^2} D_q \left[\left(D_{11}|f''(a)|^q + mD_{12}\left|f''\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} + \left(D_{21}|f''(a)|^q + mD_{22}\left|f''\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \right]$$

with A as before, $D_q = \left(\int_0^1 w(t)\,dt\right)^{\frac{1}{p}}$, $D_{11} = \int_0^1 w(t)h^s\left(\frac{n+t}{n+1}\right)dt$, $D_{12} = \int_0^1 w(t) \cdot \left(1 - h\left(\frac{n+t}{n+1}\right)\right)^s dt$, $D_{21} = \int_0^1 w(t)h^s\left(\frac{1-t}{n+1}\right)dt$, $D_{22} = \int_0^1 w(t)\left(1 - h\left(\frac{1-t}{n+1}\right)\right)^s dt$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking into account Lemma 2.1, we have (2). By using well-known power mean inequality, we obtain

$$\begin{aligned} &\int_0^1 w(t)\left|f''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right|dt \\ &\le \left(\int_0^1 w(t)\,dt\right)^{\frac{1}{p}} \left(\int_0^1 w(t)\left|f''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\int_0^1 w(t)\left|f''\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right|dt \\ &\le \left(\int_0^1 w(t)\,dt\right)^{\frac{1}{p}} \left(\int_0^1 w(t)\left|f''\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{10}$$

By the (h,s,m)-convexity of $|f''|^q$, we get

$$\begin{aligned} &\int_0^1 w(t)\left|f''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right|^q dt \\ &\le \int_0^1 w(t)\left[h^s\left(\frac{n+t}{n+1}\right)|f''(a)|^q + m\left(1 - h\left(\frac{n+t}{n+1}\right)\right)^s \left|f''\left(\frac{b}{m}\right)\right|^q\right]dt \\ &= |f''(a)|^q \int_0^1 w(t)h^s\left(\frac{n+t}{n+1}\right)dt + m\left|f''\left(\frac{b}{m}\right)\right|^q \int_0^1 w(t)\left(1 - h\left(\frac{n+t}{n+1}\right)\right)^s dt \end{aligned} \tag{11}$$

and similarly

$$\begin{aligned} &\int_0^1 w(t)\left|f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right|^q dt \\ &\le |f''(a)|^q \int_0^1 w(t)h^s\left(\frac{1-t}{n+1}\right)dt + m\left|f''\left(\frac{b}{m}\right)\right|^q \int_0^1 w(t)\left(1 - h\left(\frac{1-t}{n+1}\right)\right)^s dt. \end{aligned} \tag{12}$$

If we put (11) and (12), in (9) and in (10), it allows us to obtain the required inequality. The proof is complete. □

Remark 2.5. *The result* [23, Theorem 2.4] *can be obtained from Theorem 2.3 by considering $|f''|^q$ being an s-convex function (in the second sense). If additionally, $n = 1$, the last two results yield Theorems 3 and 4 of* [22].

3. Conclusions

In this work we have obtained some inequalities, using a certain weighted integral, which contain several already published results. Apart from the remarks made, we can point out the strength of our approach, due to the fact that we considered general convex functions such as s-convex or h-convex functions.

Moreover, we can cover some known results other than the above remarks. The following example is such. Consider the continuous function $w : [0,1] \to [0,\infty)$ with first and second order derivatives piecewise continuous on $[0,1]$ so that $w(0) = w(1) = 0$. Then we can formulate the following result that can be proved very similarly to Lemma 2.1, putting $n = 0$.

Proposition 3.1. *Let function w be as above, $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on (a,b). If $f'' \in L^1([a,b])$, then we have the following equalities:*

$$w'(0)f(b) - w'(1)f(a) + \frac{1}{b-a} I^w_{a^+} f(b) = (b-a)^2 \int_0^1 w(t) f''(ta + (1-t)b)\, dt,$$
$$w'(0)f(a) - w'(1)f(b) + \frac{1}{b-a} I^w_{b^-} f(a) = (b-a)^2 \int_0^1 w(t) f''((1-t)a + tb)\, dt.$$

This result contains as a particular case [3, Lemma 1] (see also [12]) by setting $w(t) = t(1-t)$.

We remark that we can reformulate our main results by considering the following concept of generalized convex functions (see [17, 25]).

Definition 3.1. *Let $h : [0,1] \to [0,\infty)$, $h \not\equiv 0$. Function $f : I \subseteq [0,\infty) \to [0,\infty)$ is said to be (h,m)-convex on I if inequality*

$$f(ta + m(1-t)b) \le h(t)f(a) + mh(1-t)f(b)$$

is fulfilled for $m \in [0,1]$, for all $a, b \in I$ and $t \in [0,1]$.

Remark 3.1. *If in the previous definition, $m = 1$, then f is an h-convex function on I (see [34]).*

In a similar way as the proofs of Theorems 2.1, 2.2 and 2.3, we get analogous results.

Proposition 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L^1\left(\left[a, \frac{b}{m}\right]\right)$. Under the assumptions of Lemma 2.1, if $|f''|$ is (h,m)-convex on $\left[a, \frac{b}{m}\right]$, we have the following inequality:*

$$\begin{aligned}&\left| {}^{n+1}I^w_{a^+} f\left(\frac{na}{n+1} + \frac{b}{n+1}\right) + {}^{n+1}I^w_{b^-} f\left(\frac{a}{n+1} + \frac{nb}{n+1}\right) - A \right| \\ \le& \frac{(b-a)^2}{(n+1)^2} \mathcal{B} \left(|f''(a)| + m \left| f''\left(\frac{b}{m}\right) \right| \right)\end{aligned}$$

with A as before, $\mathcal{B} = \int_0^1 w(t) \left[h\left(\frac{n+t}{n+1}\right) + h\left(\frac{1-t}{n+1}\right) \right] dt$.

Proposition 3.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L^1\left(\left[a, \frac{b}{m}\right]\right)$. Under the assumptions of Lemma 2.1 if $|f''|^q$, $q > 1$, is (h,m)-convex* *on* $\left[a, \frac{b}{m}\right]$*, we have*

$$\left| {}^{n+1}I_{a^+}^{w} f\left(\frac{na}{n+1}+\frac{b}{n+1}\right) + {}^{n+1}I_{b^-}^{w} f\left(\frac{a}{n+1}+\frac{nb}{n+1}\right) - A \right|$$
$$\leq \frac{(b-a)^2}{(n+1)^2} C_q \mathcal{C} \left[\left(\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} + \left(\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \right]$$

with A, C_q *as before, and* $\mathcal{C} = \left(\int_0^1 h\left(\frac{n+t}{n+1}\right) dt\right)^{\frac{1}{q}}$.

Proposition 3.4. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be twice differentiable function on* I° *such that* $f'' \in L^1\left(\left[a, \frac{b}{m}\right]\right)$*. Under the assumptions of Lemma 2.1, if* $|f''|^q$*,* $q > 1$*, is* (h, m)*-convex on* $\left[a, \frac{b}{m}\right]$*, we have the following inequality:*

$$\left| {}^{n+1}I_{a^+}^{w} f\left(\frac{na}{n+1}+\frac{b}{n+1}\right) + {}^{n+1}I_{b^-}^{w} f\left(\frac{a}{n+1}+\frac{nb}{n+1}\right) - A \right|$$
$$\leq \frac{(b-a)^2}{(n+1)^2} D_q \left[\left(\mathcal{D}_1 \left|f''(a)\right|^q + m\mathcal{D}_2\left|f''\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} + \left(\mathcal{D}_2 \left|f''(a)\right|^q + m\mathcal{D}_1\left|f''\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \right]$$

with A, D_q *as before,* $\mathcal{D}_1 = \int_0^1 w(t)h\left(\frac{n+t}{n+1}\right) dt$ *and* $\mathcal{D}_2 = \int_0^1 w(t)h\left(\frac{1-t}{n+1}\right) dt$.

Other formulations of our results can be obtained in two directions. First, by imposing additional conditions on the weight function w. Second, by using other notions of convexity. These directions may lead to future research.

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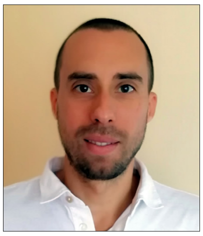

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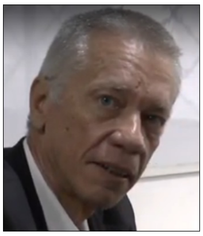

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