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UPPER BOUND FOR THIRD HANKEL DETERMINANT OF A CLASS OF ANALYTIC FUNCTIONS

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Abstract. We establish upper bounds for second Hankel determinant, the Fekete-Szegö functional and third Hankel determinant for normalized analytic functions $f \in \mathcal{W}_\beta(\alpha, \gamma)$,

$$\mathcal{W}_\beta(\alpha,\gamma) = \left\{ f : \operatorname{Re}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) \right) > \beta \right\},$$

where $\alpha, \gamma \geq 0$ and $\beta < 1$. Also, we show that these bounds reduce to the bounds of some well-known classes for particular choices of parameters α, γ and β.

Keywords: Analytic functions, Coefficient inequalities, Hankel determinant, Fekete-Szegö.

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1. Introduction and Definitions

Let $\mathcal{A}$ be the class of normalized analytic functions f, defined in the unit disc $\mathbb{E} = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{E}$.

Recently, Ali *et al.* [2] defined a class $\mathcal{W}_\beta(\alpha, \gamma)$ of normalized analytic functions defined in $\mathbb{E}$ such that function $f \in \mathcal{W}_\beta(\alpha, \gamma)$ satisfy the condition

$$\operatorname{Re}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) \right) > \beta,$$

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for all $z \in \mathbb{E}$. Here $\alpha, \gamma \geq 0$ and $\beta < 1$. For various choices of α, γ and β, the class $\mathcal{W}_\beta(\alpha, \gamma)$ unify some well-known subclasses of $\mathcal{S}$ as mentioned below:

(1) For $\alpha = 1$, $\gamma = 0$ and $\beta = 0$, the class $\mathcal{W}_\beta(\alpha, \gamma)$ reduces to the well-known class $\mathcal{R}$,
$$\mathcal{R} = \left\{ f \in \mathcal{A} : \mathrm{Re}(f'(z)) > 0 \right\},$$
see [12]. The members of class $\mathcal{R}$ are close-to-convex and hence univalent in $\mathbb{E}$ (see [5, 12]).

(2) For $\alpha = 1 + 2\gamma$ and $\beta = 0$, the class $\mathcal{W}_\beta(\alpha, \gamma)$ reduces to the class $\mathcal{R}_\gamma$, where
$$\mathcal{R}_\gamma = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(f'(z) + \gamma z f''(z) \right) > 0 \right\}.$$
It is well-known that $\mathcal{R}_1$ is a subclass of $\mathcal{S}^*$, the class of univalent starlike functions in $\mathbb{E}$. Also, $\mathcal{R}_1 \not\subset \mathcal{K}$, the class of univalent convex functions in $\mathbb{E}$ (see [17]).

(3) For $\alpha = \gamma = 0$, the class $\mathcal{W}_\beta(\alpha, \gamma)$ reduces to the class $\mathcal{T}_\beta$, where
$$\mathcal{T}_\beta = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{f(z)}{z} \right) > \beta \right\}.$$

(4) For $\gamma = 0$, the class $\mathcal{W}_\beta(\alpha, \gamma)$ reduces to the class
$$\mathrm{P}_\beta(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \right) > \beta \right\}.$$

One can see that $\mathrm{P}_\beta(\alpha_1) \subset \mathrm{P}_\beta(\alpha_2)$ for $\alpha_1 > \alpha_2 \geq 0$. Therefore, for $\alpha \geq 1$, $0 \leq \beta < 1$, $\mathrm{P}_\beta(\alpha) \subset \mathrm{P}_\beta(1) = \{f \in \mathcal{A} : \mathrm{Re} f'(z) > 0\}$ and hence $\mathrm{P}_\beta(\alpha)$ is univalent class (see [5, 12])

In 1976, Noonan and Thomas [15] defined the qth Hankel determinant $H_q(n)$ of f for $q \geq 1$ and $n \geq 1$ by
$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \vdots \\ \vdots & & \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In literature, much attention has been given to find upper bounds for the Hankel determinant whose elements are the coefficients of univalent functions, see e.g. [6, 8, 9, 16, 18]. The correct order of growth for $H_q(n)$ when $f \in \mathcal{S}$ is as yet unknown [16], whereas exact bounds have been obtained in the case $q = 2$ and $n = 2$ for a variety of subclasses of $\mathcal{S}$, most of these stemming from the method used in [11]. In 2007, Babalola [3] studied the third Hankel determinant $H_3(1)$ for some subclasses of analytic functions. By definition, $H_3(1)$ is given by
$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in \mathcal{A}$,
$$H_3(1) = a_3\left(a_2 a_4 - a_3^2\right) + a_4\left(a_2 a_3 - a_4\right) + a_5\left(a_3 - a_2^2\right), \quad a_1 = 1.$$
By triangle inequality,
$$|H_3(1)| \leq |a_3|\left|a_2 a_4 - a_3^2\right| + |a_4|\left|a_2 a_3 - a_4\right| + |a_5|\left|a_3 - a_2^2\right|. \tag{2}$$
Here, $\left|a_3 - a_2^2\right|$ is the well-known Fekete-Szegö functional and $|a_2 a_4 - a_3^2|$ is the second hankel determinant $H_2(2)$. In this paper, we will establish upper bounds for $H_2(2)$, Fekete-Szegö functional and $H_3(1)$ for $f \in \mathcal{W}_\beta(\alpha, \gamma)$. We will also show that these bounds reduce

to the bounds of some well-known classes for particular choices of parameters.

Let $\mathcal{P}$ be the family of all functions $p(z)$ given by

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

analytic in $\mathbb{E}$ for which $\mathrm{Re}\,(p(z)) > 0$ for $z \in \mathbb{E}$. It is well-known that for $p \in \mathcal{P}$, $|p_k| \leq 2$ for each $k \geq 1$.
Following lemma due to Libera and Zlotkiewicz [10, 11] is instrumental in proving our main result.

Lemma 1.1. *Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ belongs to $\mathcal{P}$. Then*

$$\begin{aligned} 2p_2 &= {p_1}^2 + x(4 - {p_1}^2) \quad \text{and} \\ 4p_3 &= {p_1}^3 + 2xp_1(4 - {p_1}^2) - x^2 p_1(4 - {p_1}^2) + 2\zeta(1 - |x|^2)(4 - {p_1}^2) \end{aligned}$$

for some x, ζ such that $|x| \leq 1$ and $|\zeta| \leq 1$.

The following two lemmas due to Ali [1] are also required to prove our results.

Lemma 1.2. *If $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ belongs to $\mathcal{P}$, then*

$$\left|p_2 - v p_1^2\right| \leq 2 \max\{1, |2v - 1|\}.$$

Lemma 1.3. *Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ belongs to $\mathcal{P}$. If $0 \leq B \leq 1$ and $B(2B - 1) \leq D \leq B$, then*

$$\left|p_3 - 2Bp_1p_2 + Dp_1^3\right| \leq 2.$$

We use the notations introduced in [2]. Let $\mu \geq 0$ and $\nu \geq 0$ satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \tag{3}$$

- When $\gamma = 0$, then μ is chosen to be 0, in this case, $\nu = \alpha \geq 0$.
- When $\alpha = 1 + 2\gamma$, then $\mu + \nu = 1 + \gamma = 1 + \mu\nu$ or $(\mu - 1)(1 - \nu) = 0$.
  - i. For $\gamma > 0$, then choosing $\mu = 1$ gives $\nu = \gamma$.
  - ii. For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

Theorem 1.1. *Let $0 \leq \mu \leq 1$, $0 \leq \nu \leq 1$ satisfy (3). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$ with $0 \leq \beta < 1$, then*

$$H_2(2) \leq \frac{4\,(1-\beta)^2}{(1+2\mu)^2\,(1+2\nu)^2}. \tag{4}$$

Proof. Since $f \in \mathcal{W}_\beta(\alpha, \gamma)$, therefore

$$\frac{\left((1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z) - \beta\right)}{1 - \beta} \in \mathcal{P}.$$

There exist $p(z) \in \mathcal{P}$, where $p(z) = 1 + p_1 z + p_2 z^2 + \dots$, such that

$$(1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z) - \beta = (1 - \beta)\,p(z).$$

In view of (3) the above equation becomes

$$(1 + \mu\nu - \mu - \nu)\frac{f(z)}{z} + (\mu + \nu - \mu\nu)f'(z) + \mu\nu z f''(z) - \beta = (1 - \beta)\,p(z). \tag{5}$$

Substituting $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in (5), we get

$$(1-\beta)+\sum_{n=2}^{\infty}\left[\mu\nu n^2+(\mu+\nu-2\mu\nu)n+(1+\mu\nu-\mu-\nu)\right]a_n z^{n-1}=(1-\beta)\left(1+\sum_{n=1}^{\infty}p_n z^n\right).$$

Equivalently,

$$(1-\beta)+\sum_{n=2}^{\infty}(1+(n-1)\mu)(1+(n-1)\nu)\,a_n z^{n-1}=(1-\beta)\left(1+\sum_{n=1}^{\infty}p_n z^n\right).$$

On equating the corresponding coefficients, we have

$$\begin{aligned}
a_2 &= \frac{(1-\beta)}{(1+\mu)(1+\nu)}p_1\\
a_3 &= \frac{(1-\beta)}{(1+2\mu)(1+2\nu)}p_2\\
a_4 &= \frac{(1-\beta)}{(1+3\mu)(1+3\nu)}p_3\\
a_5 &= \frac{(1-\beta)}{(1+4\mu)(1+4\nu)}p_4.
\end{aligned} \tag{6}$$

Let $L = (1+\mu)(1+\nu)(1+3\mu)(1+3\nu)$ and $M = (1+2\mu)^2(1+2\nu)^2$. Note that for $0 \le \mu \le 1$ and $0 \le \nu \le 1$,

$$M > 0, \quad L > 0, \quad M - L \ge 0 \quad \text{and} \quad M - 2L < 0.$$

Using (6) together with the values of L and M, the second hankel determinant becomes

$$H_2(2) = |a_2a_4 - a_3^2| = (1-\beta)^2\left|\frac{p_1p_3}{L} - \frac{p_2^2}{M}\right| = \frac{(1-\beta)^2}{LM}\left|Mp_1p_3 - Lp_2^2\right|.$$

Making use of Lemma 1.1 the above equation becomes

$$\begin{aligned}
H_2(2) = \frac{(1-\beta)^2}{4LM}\big|&(M-L)p_1^4 + 2(M-L)p_1^2x\left(4-p_1^2\right) - Mp_1^2\left(4-p_1^2\right)x^2\\
&-Lx^2\left(4-p_1^2\right)^2 + 2Mp_1\left(4-p_1^2\right)\left(1-|x|^2\right)\zeta\big|.
\end{aligned}$$

Now, without loss of generality, normalise p_1 so that $p_1 = p$, for $0 \le p \le 2$. Using the triangle inequality, we get

$$\begin{aligned}
H_2(2) \le \frac{(1-\beta)^2}{4LM}\Big\{&(M-L)p^4 + 2(M-L)p^2|x|\left(4-p^2\right) + Mp^2\left(4-p^2\right)|x|^2\\
&+L|x|^2\left(4-p^2\right)^2 + 2Mp\left(4-p^2\right)\left(1-|x|^2\right)\Big\}\\
:= \frac{(1-\beta)^2}{4LM}&\phi(|x|).
\end{aligned}$$

Differentiating $\phi(|x|)$ with respect to $|x|$, we have

$$\phi'(|x|) = 2(M-L)p^2\left(4-p^2\right) + 2|x|\left(4-p^2\right)(2-p)(2L-p(M-L)).$$

One can see that for $0 \leq \mu \leq 1$, $0 \leq \nu \leq 1$ and $0 \leq p \leq 2$, $\phi'(|x|) \geq 0$. Thus $\phi(|x|) \leq \phi(1)$ and hence

$$H_2(2) \leq \frac{(1-\beta)^2}{4LM}\left\{(M-L)\,p^4 + 2\,(M-L)\,p^2\left(4-p^2\right) + Mp^2\left(4-p^2\right) + L\left(4-p^2\right)^2\right\}$$
$$:= \frac{(1-\beta)^2}{4LM}g(p).$$

Solving $g'(p) = 0$ we have

$$p = 0, \quad p = \sqrt{\frac{3M-4L}{M-L}} \quad \text{and} \quad p = -\sqrt{\frac{3M-4L}{M-L}}.$$

Since $3M - 4L < 0$ for $0 \leq \mu, \nu \leq 1$, therefore $g(p)$ has only one critical point at $p = 0$. Further

$$g''(p)\big|_{p=0} = 8\,(3M-4L) < 0.$$

Thus $g(p)$ attains its maximum value at $p = 0$, i.e. $g(p) \leq g(0)\ \forall\ p \in [0, 2]$. Hence

$$H_2(2) \leq \frac{(1-\beta)^2}{4LM}16L = \frac{4\,(1-\beta)^2}{M} = \frac{4\,(1-\beta)^2}{(1+2\mu)^2\,(1+2\nu)^2}.$$

This completes the proof of Theorem 1.1. □

For particular values of α and γ, we will get various known results from Theorem 1.1. Letting $\alpha = \gamma = 0$ (which means $\mu = \nu = 0$) in Theorem 1.1, we obtain the following result of Hayami and Owa [7].

Corollary 1.1. *If $f \in \mathcal{A}$ satisfies*

$$Re\frac{f(z)}{z} > \beta$$

with $0 \leq \beta < 1$, then

$$H_2(2) \leq 4\,(1-\beta)^2.$$

If $\gamma = \beta = 0$, then $\mu = 0$ and $\nu = \alpha = 1$, we get the following result of Janteng et. al. [8].

Corollary 1.2. *If $f \in \mathcal{A}$ satisfies $Ref'(z) > 0$ then $H_2(2) \leq \frac{4}{9}$.*

If $\alpha = 1 + 2\gamma$ with $\gamma > 0$ and $\mu = 1$ then $\nu = \gamma > 0$. In this case, we get the following result obtained by Mohamed et. al. [13].

Corollary 1.3. *If $f \in \mathcal{A}$ satisfies $Re\,(f'(z) + \gamma z f''(z)) > 0$ for $\gamma \geq 0$ then*

$$H_2(2) \leq \frac{4}{9(1+2\gamma)^2}.$$

If $\gamma = \beta = 0$, then $\mu = 0$ and $\nu = \alpha > 0$, we get the result due to Murugusundaramoorthy and Magesh [14].

Corollary 1.4. *If $f \in \mathcal{A}$ satisfies $Re\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right) > 0$ then*

$$H_2(2) \leq \frac{4}{(1+2\alpha)^2}.$$

Theorem 1.2. *Let $0 \leq \mu \leq 1$, $0 \leq \nu \leq 1$ satisfy (3). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$ with $0 \leq \beta < 1$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{2\,(1-\beta)}{(1+2\mu)\,(1+2\nu)}. \tag{7}$$

Proof. In the view of (6), one can see that

$$\left|a_3 - a_2^2\right| = \left|(1-\beta)\frac{p_2}{(1+2\mu)(1+2\nu)} - (1-\beta)^2\frac{p_1^2}{(1+\mu)^2(1+\nu)^2}\right|. \tag{8}$$

Let $Q = (1+2\mu)(1+2\nu)$ and $R = (1+\mu)^2(1+\nu)^2$. Note that for $0 \le \mu \le 1$ and $0 \le \nu \le 1$,

$$Q > 0, \quad R > 0, \quad R - Q \ge 0 \quad \text{and} \quad R - 2Q > 0.$$

Using Q and R, the equation (8) becomes

$$\left|a_3 - a_2^2\right| = \frac{(1-\beta)}{Q}\left|p_2 - \frac{(1-\beta)Q}{R}p_1^2\right|.$$

Letting $v = \frac{(1-\beta)Q}{R}$ in Lemma 1.2, we get

$$\begin{aligned}\left|p_2 - \frac{(1-\beta)Q}{R}p_1^2\right| &\le 2\max\left\{1, \left|\frac{2(1-\beta)Q}{R} - 1\right|\right\} \\ &= 2\max\left\{1, \left|\frac{2(1-\beta)Q - R}{R}\right|\right\}.\end{aligned}$$

Since $R - Q \ge 0$ and $0 \le \beta < 1$, therefore $-R < 2(1-\beta)Q - R \le 2Q - R \le 0$, and so

$$\left|\frac{2(1-\beta)Q - R}{R}\right| \le 1.$$

Thus

$$\left|p_2 - \frac{(1-\beta)Q}{R}p_1^2\right| \le 2\max\left\{1, \left|\frac{2(1-\beta)Q - R}{R}\right|\right\} = 2.$$

Hence

$$\left|a_3 - a_2^2\right| = \frac{(1-\beta)}{Q}\left|p_2 - \frac{(1-\beta)Q}{R}p_1^2\right| \le \frac{2(1-\beta)}{Q} = \frac{2(1-\beta)}{(1+2\mu)(1+2\nu)}.$$

□

Taking various permissible values of α, γ and β, we obtain several special cases of above result.

Remark 1.1.

i. *For $\alpha = \gamma = 0$, that is $\mu = \nu = 0$, Theorem 1.2 yields a particular case of Theorem 3.1 in* [7].

ii. *For $\gamma = \beta = 0$ with $\mu = 0$ and $\nu = \alpha = 1$, Theorem 1.2 gives a result of Babalola and Opoola* [4].

Theorem 1.3. *Let $0 \le \mu \le 1$, $0 \le \nu \le 1$ satisfy* (3). *If $f \in \mathcal{W}_\beta(\alpha, \gamma)$ with $0 \le \beta < 1$, then*

$$|a_4 - a_2a_3| \le \frac{2(1-\beta)}{(1+3\mu)(1+3\nu)}. \tag{9}$$

Proof. In the view of (6), one can see that

$$|a_4 - a_2a_3| = \left|\frac{(1-\beta)p_3}{(1+3\mu)(1+3\nu)} - \frac{(1-\beta)^2p_1p_2}{(1+\mu)(1+\nu)(1+2\mu)(1+2\nu)}\right|.$$

Let $S = (1+3\mu)(1+3\nu)$ and $T = (1+\mu)(1+\nu)(1+2\mu)(1+2\nu)$. Note that for $0 \le \mu, \nu \le 1$,

$$S > 0, \quad T > 0 \quad \text{and} \quad T - S \ge 0. \tag{10}$$

Thus

$$|a_4 - a_2a_3| = \frac{(1-\beta)}{S}\left|p_3 - \frac{(1-\beta)S}{T}p_1p_2\right|.$$

Applying Lemma 1.3 with $2B = \frac{(1-\beta)S}{T}$ and $D = 0$, we have

$$\left|p_3 - \frac{(1-\beta)S}{T}p_1p_2\right| \leq 2,$$

provided

$$0 \leq B \leq 1 \quad \text{and} \quad B(2B-1) \leq D \leq B.$$

Using (10) and the fact that $0 \leq \beta < 1$, we have

$$0 < B = \frac{(1-\beta)S}{2T} \leq \frac{1}{2} < 1.$$

Consequently, for $D = 0$ we have

$$B(2B-1) \leq D \leq B.$$

Finally, in the view of Lemma 1.3, we have

$$|a_4 - a_2a_3| = \frac{(1-\beta)}{S}\left|p_3 - \frac{(1-\beta)S}{T}p_1p_2\right| \leq \frac{2(1-\beta)}{S} = \frac{2(1-\beta)}{(1+3\mu)(1+3\nu)}.$$

This completes the proof of Theorem 1.3. □

Setting $\gamma = \beta = 0$ with $\mu = 0$ and $\nu = \alpha = 1$ in Theorem 1.3, we get Theorem 3.1 of [3].

In the view of (4), (6), (7) and (9), one can see that for $f \in \mathcal{W}_\beta(\alpha,\gamma)$ with $0 \leq \beta < 1$, we have the following information:

$$|a_3| \leq \frac{2(1-\beta)}{(1+2\mu)(1+2\nu)},$$

$$|a_4| \leq \frac{2(1-\beta)}{(1+3\mu)(1+3\nu)},$$

$$|a_5| \leq \frac{2(1-\beta)}{(1+4\mu)(1+4\nu)}.$$

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4(1-\beta)^2}{(1+2\mu)^2(1+2\nu)^2},$$

$$\left|a_3 - a_2^2\right| \leq \frac{2(1-\beta)}{(1+2\mu)(1+2\nu)},$$

$$|a_4 - a_2a_3| \leq \frac{2(1-\beta)}{(1+3\mu)(1+3\nu)}.$$

Substituting all these values in (2), we have

$$|H_3(1)| \leq \frac{8(1-\beta)^3}{(1+2\mu)^3(1+2\nu)^3} + \frac{4(1-\beta)^2}{(1+3\mu)^2(1+3\nu)^2} + \frac{4(1-\beta)^2}{(1+2\mu)(1+2\nu)(1+4\mu)(1+4\nu)}.$$

Theorem 1.4. *Let $0 \leq \mu \leq 1$, $0 \leq \nu \leq 1$ satisfy (3). If $f \in \mathcal{W}_\beta(\alpha,\gamma)$ with $0 \leq \beta < 1$, then*

$$|H_3(1)| \leq \frac{8(1-\beta)^3}{(1+2\mu)^3(1+2\nu)^3} + \frac{4(1-\beta)^2}{(1+3\mu)^2(1+3\nu)^2} + \frac{4(1-\beta)^2}{(1+2\mu)(1+2\nu)(1+4\mu)(1+4\nu)}.$$

Remark 1.2. *Setting* $\gamma = \beta = 0$ *with* $\mu = 0$ *and* $\nu = \alpha = 1$ *in Theorem 1.4, we get Corollary 3.2 of* [3].

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