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THE HOMOGENEOUS q-DIFFERENCE OPERATOR AND THE RELATED POLYNOMIALS

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ABSTRACT. We create the homogeneous q-difference operator $\widetilde{E}(a,b;\theta)$ as an extension of the exponential operator $E(b\theta)$. A new polynomials $h_n(a,b,x|q^{-1})$ are defined as an extension of the q^{-1}-Rogers-Szegö polynomial $h_n(a,b|q^{-1})$. We provide an operator proof of the generating function and its extension, Rogers formula and the invers linearization formula, and Mehler's formula for the polynomials $h_n(a,b|q^{-1})$. The generating function and its extension, Rogers formula and the invers linearization formula, and Mehler's formula for the polynomials $h_n(a,b|q^{-1})$ are deduced by giving special values to parameters of a new polynomial $h_n(a,b,x|q^{-1})$.

Keywords: the homogeneous q-difference operator, the q^{-1}-Rogers-Szegö polynomial, the generating function, the Rogers formula, the invers linearization formula, the Mehler's formula.

AMS Subject Classification: 05A30, 33D45.

1. Introduction

The notations in [8] will be utilized throughout this paper. We assume that $|q| < 1$. The q-shifted factorial is defined as

$$(a;q)_k = \begin{cases} 1, & \text{if } k = 0, \\ (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k = 1,2,3,\cdots. \end{cases}$$

We also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

For multiple q-shifted factorials, we'll use the following notation:

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1,a_2,\ldots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

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The generalized basic hypergeometric series ${}_r\phi_s$ is defined by

$$ {}_r\phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,x) = {}_r\phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,x\right) $$
$$ = \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k\,(b_1;q)_k\cdots(b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} x^k, $$

where $q \neq 0$ when $r > s+1$. Note that

$$ {}_{r+1}\phi_r\left(\begin{array}{c} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{array};q,x\right) = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_{r+1};q)_n}{(q,b_1,\ldots,b_r;q)_n}x^n, \quad |x|<1. $$

The q-binomial coefficients are provided by

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. $$

The Cauchy identity, as well as its special case, will be used frequently [8]

$$ \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x|<1. \tag{1} $$

$$ \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_k} = (x;q)_\infty. \tag{2} $$

Jackson's transformation of ${}_2\phi_1$ series is [8, Appendix III, equation (III.4)]

$$ {}_2\phi_1\left(\begin{array}{c} a,b \\ c \end{array};q,x\right) = \frac{(ax;q)_\infty}{(x;q)_\infty}\,{}_2\phi_2\left(\begin{array}{c} a,c/b \\ c,ax \end{array};q,bx\right). \tag{3} $$

We shall commonly utilize the following identities in this paper [8]:

$$ (a;q)_{n+k} = (a;q)_n(aq^n;q)_k. \tag{4} $$

$$ (q/a;q)_k = a^{-k}(-1)^k q^{\binom{k}{2}+k}\frac{(aq^{-k};q)_\infty}{(a;q)_\infty}. \tag{5} $$

$$ (q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}. \tag{6} $$

$$ \binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + nk. \tag{7} $$

$$ \binom{n-k}{2} = \binom{n}{2} + \binom{k}{2} + k - nk. \tag{8} $$

The Rogers-Szegö polynomials are defined by [6, 7, 13, 11]:

$$ h_n(a,b|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} a^k b^{n-k}. $$

Chen and Liu [5] recalled the operator θ, which appeared in Roman work's [12] as follows:

Definition 1.1. *The operator θ is defined as follows:*

$$ \theta\{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \tag{9} $$

The following identity is the Leibniz rule for the operator θ:

Theorem 1.1. [5, 12]. *For $n \geqslant 0$, we have*

$$\theta^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(a)\}\theta^{n-k}\{g(aq^{-k})\}. \tag{10}$$

The following identities are easy to verify:

Theorem 1.2. [5, 16, 15]. *Let θ be defined as in* (9)*, then*

$$\theta^k\{(at;q)_\infty\} = (-t)^k(at;q)_\infty. \tag{11}$$

$$\theta^k\{a^n\} = \frac{(q;q)_n}{(q;q)_{n-k}}\, a^{n-k}q^{\binom{k}{2}-nk+k}. \tag{12}$$

In 1998, inspired by the Euler identity (2), Chen and Liu [5] defined the q-exponential operator $E(b\theta)$ as follows:

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(b\theta)^n}{(q;q)_n}. \tag{13}$$

In 2010, Liu [11] defined the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$ as follows:

$$h_n(a,b|q^{-1}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-kn}a^kb^{n-k}. \tag{14}$$

Liu [11] proved the following results for $h_n(a,b|q^{-1})$:

Theorem 1.3. [11]. *Let $h_n(a,b|q^{-1})$ be defined as in* (14)*, then*

- *The generating function for $h_n(a,b|q^{-1})$ is*

$$\sum_{n=0}^{\infty} h_n(a,b|q^{-1})\frac{(-t)^nq^{\binom{n}{2}}}{(q;q)_n} = (at,bt;q)_\infty. \tag{15}$$

- *The Mehler's formula for $h_n(a,b|q^{-1})$ is*

$$\sum_{n=0}^{\infty} h_n(a,b|q^{-1})h_n(c,d|q^{-1})\frac{(-t)^nq^{\binom{n}{2}}}{(q;q)_n} = \frac{(act,adt,bct,bdt;q)_\infty}{(abcdt^2/q;q)_\infty}, \tag{16}$$

provided that $|abcdt^2/q| < 1$.

In 2020, Abdlhusein and Hussein [1] represented the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$ by the operator $E(b\theta)$ as follows:

$$E(b\theta)\{x^n\} = h_n(a,b|q^{-1}). \tag{17}$$

Based on the operator representation (17), Abdlhusein and Hussein [1] retrieved the generating function for $h_n(a,b|q^{-1})$ (15) and the Mehler's formula for $h_n(a,b|q^{-1})$ (16) and found the following identities for $h_n(a,b|q^{-1})$:

Theorem 1.4. [1]. *Let $h_n(a,b|q^{-1})$ be defined as in* (14)*, then*

- *The extended generating function for q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$ is*

$$\sum_{n=0}^{\infty} h_{n+k}(a,b|q^{-1})\frac{(-1)^nq^{\binom{n}{2}}t^n}{(q;q)_n} = b^k(at,bt;q)_\infty\ {}_2\phi_1\left(\begin{matrix} q^{-k}, q/bt \\ 0 \end{matrix}; q, at\right). \tag{18}$$

- *The Rogers formula for the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$ is*

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} & h_{m+n}(a,b|q^{-1})\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} \\ &= \frac{(as,bs,at,bt;q)_\infty}{(abst/q;q)_\infty}. \qquad (19)\\ &= \frac{1}{(abst/q;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_m(a,b|q^{-1})h_n(a,b|q^{-1})\frac{(-1)^m q^{\binom{m}{2}} s^m (-1)^n q^{\binom{n}{2}} t^n}{(q;q)_m(q;q)_n}. \qquad (20)\end{aligned}$$

Also, they derived the invers linearization formula as an applications of the Roger's formula (20) as follows:

$$\begin{aligned} h_{m+n}&(a,b|q^{-1}) \\ &= \sum_{k=0}^{min\{m,n\}} \begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk-mk}(q;q)_k(ab)^k h_{m-k}(a,b|q^{-1})h_{n-k}(a,b|q^{-1}). \qquad (21)\end{aligned}$$

In 2020, Cao and et al. [3] built the new generalized Al-Salam-Carlitz polynomials as follows:

$$\psi_n\begin{pmatrix} a,b,c \\ d,e \end{pmatrix}(x,y|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^k q^{k(k-n)}(a,b,c;q)_k}{(d,e;q)_k}x^{n-k}y^k. \qquad (22)$$

In 2021, Arjika and Mahaman [2] constructed the following generalized trivariate q-Hahn polynomials as follows:

$$\Psi_n^{(a)}(x,y,z|q) = (-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}(a;q)_k P_{n-k}(y,x)z^k. \qquad (23)$$

In 2021, Cao and et al. [4] constructed the following generalized trivariate q-Hahn polynomials as follows:

$$\zeta_n\begin{pmatrix} a,b,c \\ d,e \end{pmatrix}(x,y,z|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{\binom{k}{2}}(a,b,c;q)_k}{(d,e;q)_k}P_{n-k}(y,x)z^k. \qquad (24)$$

In 2021, Srivastava and Arjika [14] a family of generalized q-hypergeometric polynomials is defined by

$$\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q) = (-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} W_k(\mathbf{a},\mathbf{b})P_{n-k}(y,x)z^k, \qquad (25)$$

where $\mathbf{a}=(a_1,a_2,\ldots,a_r)$, $\mathbf{b}=(b_1,b_2,\ldots,b_s)$ and $W_k(\mathbf{a},\mathbf{b}) = \dfrac{(a_1,a_2,\ldots,a_r;q)_k}{(b_1,b_2,\ldots,b_s;q)_k}$

The following is how our paper is structured: In section 2, we construct the homogeneous q-difference operator $\widetilde{E}(a,b;\theta)$ and then establish some of its identities, which will be useful in the next sections. We construct a new polynomials $h_n(a,b,x|q^{-1})$ and derive its generating function and its extension in section 3, then deduce the generating function and its extension for $h_n(a,b|q^{-1})$. In section 4, we obtain the Rogers formula for $h_n(a,b,x|q^{-1})$, and then we deduce the Rogers formula for $h_n(a,b|q^{-1})$. We derive Mehler's formula for $h_n(a,b,x|q^{-1})$ and then infer the Mehler formula for $h_n(a,b|q^{-1})$ in section 5.

2. Some Operator Identities for the Operator $\widetilde{E}(a,b;\theta)$

The homogeneous q-difference operator $\widetilde{E}(a,b;\theta)$ is presented in this section, and some of its operator identities are discovered.

Definition 2.1. *Let θ be defined as in (9), we define the homogeneous q-difference operator $\widetilde{E}(a,b;\theta)$ as follows:*

$$\widetilde{E}(a,b;\theta)=\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(b\theta)^k. \tag{26}$$

Setting $a=0$ in (26), we are led to q-exponential operator $E(b\theta)$ defined by Chen and Liu [5] in (13). This means that the operator $E(b\theta)$ is a special case of the operator $\widetilde{E}(a,b;\theta)$.

Throughout out this paper, we assume that the operator θ acts on the variable b.

Theorem 2.1. *Let the operator $\widetilde{E}(a,x;\theta)$ be defined as in (26), then*

$$\begin{aligned}&\widetilde{E}(a,x;\theta)\left\{(bt,bs;q)_\infty\right\}\\ &=(bt,bs;q)_\infty\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(x;q)_n}{(q;q)_n}(as)^n\ {}_2\phi_1\left(\begin{matrix}xq^n,q/bs\\0\end{matrix};q,atbs/q\right).\end{aligned} \tag{27}$$

Proof.

$$\begin{aligned}&\widetilde{E}(a,x;\theta)\left\{(bt,bs;q)_\infty\right\}\\
&=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(x;q)_n}{(q;q)_n}a^n\theta^n\left\{(bt,bs;q)_\infty\right\}\\
&=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(x;q)_na^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\theta^k\left\{(bt;q)_\infty\right\}\theta^{n-k}\left\{(bsq^{-k};q)_\infty\right\}\quad\text{(by using (10))}\\
&=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(x;q)_na^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-t)^k(bt;q)_\infty\theta^{n-k}\left\{(bsq^{-k};q)_\infty\right\}\quad\text{(by using (11))}\\
&=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\binom{n+k}{2}}(x;q)_{n+k}a^{n+k}}{(q;q)_k(q;q)_n}(-t)^k(bt;q)_\infty\theta^n\left\{(bsq^{-k};q)_\infty\right\}\\
&=(bt;q)_\infty\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}+\binom{k}{2}+nk}(x;q)_n(xq^n;q)_ka^{n+k}}{(q;q)_k(q;q)_n}(-t)^k(-sq^{-k})^n\\
&\quad\times(bsq^{-k};q)_\infty\quad\text{(by using (7) and (11))}\\
&=(bt;q)_\infty\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}+\binom{k}{2}+nk}(x;q)_n(xq^n;q)_ka^{n+k}}{(q;q)_k(q;q)_n}(-t)^k(-sq^{-k})^n(-bs)^k\\
&\quad\times q^{-\binom{k+1}{2}}(q/bs;q)_k(bs;q)_\infty\quad\text{(by using (5))}\\
&=(bt,bs;q)_\infty\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(x;q)_n(xq^n;q)_k}{(q;q)_k(q;q)_n}(-as)^n(atbs)^k(q/bs;q)_kq^{\binom{n}{2}+\binom{k}{2}+nk-kn-\binom{k}{2}-k}\\
&=(bt,bs;q)_\infty\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(x;q)_n(as)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(xq^n;q)_k(q/bs;q)_k}{(q;q)_k}(atbs/q)^k\end{aligned}$$

$$= \quad (bt,bs;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (x;q)_n (as)^n}{(q;q)_n} \ {}_2\phi_1\left(\begin{matrix} xq^n, q/bs \\ 0 \end{matrix} ; q, atbs/q \right).$$

□

Setting $s = 0$ in equation (27), we get the following corollary:

Corollary 2.1. *Let the operator* $\widetilde{E}(a,x;\theta)$ *be defined as in* (26)*, then*

$$\widetilde{E}(a,x;\theta)\left\{(bt;q)_\infty\right\} = (bt;q)_\infty \ {}_1\phi_1\left(\begin{matrix} x \\ 0 \end{matrix} ; q, at \right). \tag{28}$$

Theorem 2.2. *Let the operator* $\widetilde{E}(a,x;\theta)$ *be defined as in* (26)*, then*

$$\widetilde{E}(a,x;\theta)\left\{b^k(bt;q)_\infty\right\}$$
$$= b^k(bt;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (x;q)_n (at)^n}{(q;q)_n} \ {}_3\phi_2\left(\begin{matrix} q^{-k}, xq^n, q/bt \\ 0, 0 \end{matrix} ; q, at \right). \tag{29}$$

Proof.

$$\widetilde{E}(a,x;\theta)\left\{b^k(bt;q)_\infty\right\}$$
$$= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(x;q)_n}{(q;q)_n} a^n \theta^n \left\{b^k(bt;q)_\infty\right\}$$
$$= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(x;q)_n a^n}{(q;q)_n} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \theta^j \left\{b^k\right\} \theta^{n-j} \left\{(btq^{-j};q)_\infty\right\} \quad \text{(by using (10))}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{q^{\binom{n}{2}}(x;q)_n a^n}{(q;q)_j (q;q)_{n-j}} \frac{(q;q)_k}{(q;q)_{k-j}} b^{k-j} q^{\binom{j}{2}-kj+j} \theta^{n-j} \left\{(btq^{-j};q)_\infty\right\} \quad \text{(by using (12))}$$
$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{\binom{n+j}{2}}(x;q)_{n+j} a^{n+j}}{(q;q)_j (q;q)_{n}} \frac{(q;q)_k}{(q;q)_{k-j}} b^{k-j} q^{\binom{j}{2}-kj+j} \theta^{n} \left\{(btq^{-j};q)_\infty\right\}$$
$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}+\binom{j}{2}+nj}(x;q)_n (xq^n;q)_j a^{n+j}}{(q;q)_j (q;q)_{n}} \frac{(q;q)_k}{(q;q)_{k-j}} b^{k-j} q^{\binom{j}{2}-kj+j}$$
$$\times(-tq^{-j})^n (btq^{-j};q)_\infty \quad \text{(by using (4), (7) and (11))}$$
$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(x;q)_n (xq^n;q)_j a^{n+j}}{(q;q)_j (q;q)_{n}} \frac{(q;q)_k}{(q;q)_{k-j}} b^{k-j} (-t)^n$$
$$\times q^{\binom{n}{2}+2\binom{j}{2}-kj+j} (bt;q)_\infty (q/bt;q)_j (-bt)^j q^{-\binom{j}{2}-j} \quad \text{(by using (5))}$$
$$= (bt;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(x;q)_n (at)^n}{(q;q)_n}$$
$$\times \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (xq^n, q/bt;q)_j (at)^j}{(q;q)_j} \frac{(q;q)_k q^{-kj}}{(q;q)_{k-j}} b^k$$
$$= b^k (bt;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(x;q)_n (at)^n}{(q;q)_n} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (xq^n, q/bt;q)_j (at)^j}{(q;q)_j}$$

$$\times(-1)^{-j}q^{-\binom{j}{2}}(q^{-k};q)_j \quad \text{(by using (6))}$$
$$= b^k(bt;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(x;q)_n (at)^n}{(q;q)_n} {}_3\phi_2\left(\begin{matrix} q^{-k}, xq^n, q/bt \\ 0,0 \end{matrix} ;q, at\right).$$

□

3. Generating Function for $h_n(a,b,x|q^{-1})$

Polynomials $h_n(a,b,x|q^{-1})$ are defined in this section. The generating function and its extension for the polynomials $h_n(a,b,x|q^{-1})$ generated by employing the operator $\widetilde{E}(a,x;\theta)$. We offer some specific values for the parameters in the generating function as well as its extension for the polynomials $h_n(a,b,x|q^{-1})$ to obtain the generating function and its extension for the polynomials $h_n(a,b|q^{-1})$.

Definition 3.1. *We define the polynomials $h_n(a,b,x|q^{-1})$ as follows:*

$$h_n(a,b,x|q^{-1}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk} a^k b^{n-k} (x,q)_k. \tag{30}$$

- By choosing $x = 0$ in equation (30), we get the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$ (14).
- Upon setting $b = c = d = e = 0$ and $(a,x,y) = (x,b,-a)$, the new generalized Al-Salam-Carlitz polynomials $\psi_n^{\left(\begin{matrix} a,b,c \\ d,e \end{matrix}\right)}(x,y|q)$ defined in (22) reduce to the polynomials $h_n(a,b,x|q^{-1})$.
- By choosing $(a,x,y,z) = (x,b,0,a)$, the generalized trivariate q-Hahn polynomials $\Psi_n^{(a)}(x,y,z|q)$ defined in (23) reduce to the polynomials $h_n(a,b,x|q^{-1})$.
- For $b = c = d = e = 0$ and $(a,x,y,z) = (x,b,0,-a)$ in equation (24), we get

$$\zeta_n^{\left(\begin{matrix} x,0,0 \\ 0,0 \end{matrix}\right)}(b,0,-a|q) = (-1)^n q^{\binom{n}{2}} h_n(a,b,x|q^{-1}).$$

.

- When $r = s = 1$, $b_1 = 0$, and $(a_1,x,y,z) = (x,b,0,a)$, the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ defined in (25) reduce to the polynomials $h_n(a,b,x|q^{-1})$.

Proposition 3.1. *Let the operator $\widetilde{E}(a,x;\theta)$ be defined as in* (26)*, then*

$$\widetilde{E}(a,x;\theta)\{b^n\} = h_n(a,b,x|q^{-1}). \tag{31}$$

Proof.

$$\begin{aligned}
\widetilde{E}(a,x;\theta)\{b^n\} &= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(x;q)_k}{(q;q)_k} a^k \theta^k \{b^n\} \quad \text{(by using (26))} \\
&= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(x;q)_k a^k}{(q;q)_k} \frac{(q;q)_n}{(q;q)_{n-k}} b^{n-k} q^{\binom{k}{2}-nk+k} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk}(x;q)_k a^k b^{n-k} \\
&= h_n(a,b,x|q^{-1}).
\end{aligned}$$

□

By substituting $x = 0$ in equation (31), we obtain equation (17), as described in Abdlhusein and Hussein's [1].

Theorem 3.1 (Generating function for $h_n(a, b, x|q^{-1})$)**.** *Let the polynomials $h_n(a, b, x|q^{-1})$ be defined as in* (30)*, then*

$$\sum_{n=0}^{\infty} h_n(a, b, x|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = (bt; q)_\infty \ {}_1\phi_1 \left(\begin{matrix} x \\ 0 \end{matrix} ; q, at \right). \tag{32}$$

Proof.

$$\begin{aligned} &\sum_{n=0}^{\infty} h_n(a, b, x|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} \widetilde{E}(a, x; \theta) \{b^n\} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \qquad \text{(by using (31))} \\ &= \widetilde{E}(a, x; \theta) \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (bt)^n}{(q; q)_n} \right\} \\ &= \widetilde{E}(a, x; \theta) \{(bt; q)_\infty\} \qquad \text{(by using (2))} \\ &= (bt; q)_\infty \ {}_1\phi_1 \left(\begin{matrix} x \\ 0 \end{matrix} ; q, at \right). \qquad \text{(by using (28))} \end{aligned}$$

□

We recover the generating function for polynomials $h_n(a, b|q^{-1})$ found by Liu [11] (equation (15)) by setting $x = 0$ in the generating function for polynomials $h_n(a, b, x|q^{-1})$ (32).

Theorem 3.2 (Extended generating function for $h_n(a, b, x|q^{-1})$)**.** *Let the polynomials $h_n(a, b, x|q^{-1})$ be defined as in* (30)*, then*

$$\begin{aligned} &\sum_{n=0}^{\infty} h_{n+k}(a, b, x|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \\ &= b^k (bt; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (x; q)_n (at)^n}{(q; q)_n} \ {}_3\phi_2 \left(\begin{matrix} q^{-k}, xq^n, q/bt \\ 0, 0 \end{matrix} ; q, at \right). \end{aligned} \tag{33}$$

Proof.

$$\begin{aligned} &\sum_{n=0}^{\infty} h_{n+k}(a, b, x|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} \widetilde{E}(a, x; \theta) \left\{ b^{n+k} \right\} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \qquad \text{(by using (31))} \\ &= \widetilde{E}(a, x; \theta) \left\{ b^k \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (bt)^n}{(q; q)_n} \right\} \\ &= \widetilde{E}(a, x; \theta) \left\{ b^k (bt; q)_\infty \right\} \qquad \text{(by using (2))} \\ &= b^k (bt; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (x; q)_n (at)^n}{(q; q)_n} \ {}_3\phi_2 \left(\begin{matrix} q^{-k}, xq^n, q/bt \\ 0, 0 \end{matrix} ; q, at \right). \qquad \text{(by using (29))} \end{aligned}$$

□

Setting $k = 0$ in (33), we get the generating function (32).

Setting $x = 0$ in the extended generating function for the polynomials $h_n(a, b, x|q^{-1})$ (33), we recover the extended generating function for the polynomials $h_n(a, b|q^{-1})$ obtained by Abdlhusein and Hussein [1] (equation (18)).

4. Rogers Formula for $h_n(a, b, x|q^{-1})$

We will provide an operator approach to Rogers formula for the polynomials $h_n(a, b, x|q^{-1})$ in this section. By incorporating special values for variables in the Rogers formula for $h_n(a, b, x|q^{-1})$, the Rogers formula for the q^{-1}-Rogers-Szegö polynomials $h_n(a, b|q^{-1})$ is obtained.

Theorem 4.1 (Rogers formula for $h_n(a, b, x|q^{-1})$)**.** *Let the polynomials $h_n(a, b, x|q^{-1})$ be defined as in* (30)*, then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(a, b, x|q^{-1})\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}$$
$$= (bt, bs; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (x;q)_n (as)^n}{(q;q)_n} {}_2\phi_1\left(\begin{matrix} xq^n, q/bs \\ 0 \end{matrix}; q, atbs/q\right) \tag{34}$$
$$= \sum_{k=0}^{\infty} \frac{(x;q)_k (atbs/q)^k}{(q;q)_k}$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(a, b, xq^k|q^{-1}) h_m(a, b, xq^{k+n}|q^{-1})\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}. \tag{35}$$

Proof. By using (31), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(a, b, x|q^{-1})\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \widetilde{E}(a, x; \theta)\left\{b^{n+m}\right\} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}$$
$$= \widetilde{E}(a, x; \theta)\left\{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (bt)^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (bs)^m}{(q;q)_m}\right\}$$
$$= \widetilde{E}(a, x; \theta)\left\{(bt, bs; q)_\infty\right\} \qquad \text{(by using (2))}$$
$$= (bt, bs; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (x;q)_n (as)^n}{(q;q)_n} {}_2\phi_1\left(\begin{matrix} xq^n, q/bs \\ 0 \end{matrix}; q, atbs/q\right).$$
$$\text{(by using (27))}$$

Hence the proof of (34) is completed.

To prove (35), replace $a = xq^n$, $b = q/bs$, $c = 0$ and $x = atbs/q$, respectively, in Jackson's transformations [8, Appendix III, equation (III.4)] (equation (3)), we get

$${}_2\phi_1\left(\begin{matrix} xq^n, q/bs \\ 0 \end{matrix}; q, atbs/q\right)$$

$$= \frac{(atbsxq^n/q;q)_\infty}{(atbs/q;q)_\infty} {}_2\phi_2\left(\begin{matrix} xq^n, 0 \\ 0, atbsxq^n/q \end{matrix}; q, at\right)$$
$$= \frac{(atbsxq^n/q;q)_\infty}{(atbs/q;q)_\infty} {}_1\phi_1\left(\begin{matrix} xq^n \\ atbsxq^n/q \end{matrix}; q, at\right)$$
$$= \frac{(atbsxq^n/q;q)_\infty}{(atbs/q;q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}}(xq^n;q)_m (at)^m}{(q;q)_m (atbsxq^n/q;q)_m}. \tag{36}$$

Substitute (36) into (34), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(a,b,x|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} \frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}$$
$$= (bt,bs;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(x;q)_n (as)^n}{(q;q)_n} \frac{(atbsxq^n/q;q)_\infty}{(atbs/q;q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}}(xq^n;q)_m (at)^m}{(q;q)_m (atbsxq^n/q;q)_m}$$
$$= (bt,bs;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(x;q)_n (as)^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}}(xq^n;q)_m (at)^m}{(q;q)_m}$$
$$\times \sum_{k=0}^{\infty} \frac{(xq^{n+m};q)_k (atbs/q)^k}{(q;q)_k} \quad \text{(by using (1))}$$
$$= \sum_{k=0}^{\infty} \frac{(x;q)_k (atbs/q)^k}{(q;q)_k} (bt;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(xq^k;q)_n (at)^n}{(q;q)_n}$$
$$\times (bs;q)_\infty \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}}(xq^{k+n};q)_m (as)^m}{(q;q)_m}$$
$$= \sum_{k=0}^{\infty} \frac{(x;q)_k (atbs/q)^k}{(q;q)_k}$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(a,b,xq^k|q^{-1}) h_m(a,b,xq^{k+n}|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} \frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}.$$

□

Setting $x = 0$ in the Rogers formulas for the polynomials $h_n(a,b,x|q^{-1})$ (34) and (35), we recover Rogers formulas for the polynomials $h_n(a,b|q^{-1})$ (19) and (20) derived by Abdlhusein and Hussein [1].

Corollary 4.1 (The inverse linearization formula for $h_n(a,b,x|q^{-1})$)**.** *We have*

$$h_{m+n}(a,b,x|q^{-1}) = \sum_{k=0}^{min\{m,n\}} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk-mk} (q;q)_k (ab)^k (x;q)_k$$
$$\times h_{m-k}(a,b,xq^k|q^{-1}) h_{n-k}(a,b,xq^m|q^{-1}). \tag{37}$$

Proof. From equation (35),we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{m+n}(a,b,x|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} \frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(x;q)_k (ab/q)^k}{(q;q)_k}$$
$$\times h_m(a,b,xq^k|q^{-1}) h_n(a,b,xq^{k+m}|q) \frac{(-1)^n q^{\binom{n}{2}} t^{n+k}}{(q;q)_n} \frac{(-1)^m q^{\binom{m}{2}} s^{m+k}}{(q;q)_m}. \tag{38}$$

Comparing the coefficients of $t^n s^m$ in equation (38), we get

$$h_{m+n}(a,b,x|q^{-1})\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}}}{(q;q)_m}$$
$$=\sum_{k=0}^{\infty}\frac{(x;q)_k(ab/q)^k}{(q;q)_k}h_{m-k}(a,b,xq^k|q^{-1})h_{n-k}(a,b,xq^m|q)\frac{(-1)^{n-k}q^{\binom{n-k}{2}}}{(q;q)_{n-k}}\frac{(-1)^{m-k}q^{\binom{m-k}{2}}}{(q;q)_{m-k}}$$
$$=\sum_{k=0}^{\infty}\frac{(x;q)_k(ab)^k q^{-k}}{(q;q)_k}h_{m-k}(a,b,xq^k|q^{-1})h_{n-k}(a,b,xq^m|q)$$
$$\times\frac{(-1)^n q^{\binom{n}{2}+\binom{k}{2}+k-nk}}{(q;q)_{n-k}}\frac{(-1)^m q^{\binom{m}{2}+\binom{k}{2}+k-mk}}{(q;q)_{m-k}}. \qquad \text{(by using (8))}$$

Hence

$$h_{m+n}(a,b,x|q^{-1})$$
$$=\sum_{k=0}^{min\{m,n\}}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}q^{k^2-nk-mk}(q;q)_k(ab)^k(x;q)_k h_{m-k}(a,b,xq^k|q^{-1})h_{n-k}(a,b,xq^m|q^{-1}).$$

□

Setting $x=0$ in the inverse linearization formula for the polynomials $h_n(a,b,x|q^{-1})$ (37), we recover the inverse linearization formula for the polynomials $h_n(a,b|q^{-1})$ obtained by Abdlhusein and Hussein [1] (equation (21)).

5. Mehler's Formula for $h_n(a,b,x|q^{-1})$

We will show an operator approach to Mehler's formula for the polynomials $h_n(a,b,x|q^{-1})$ in this section. The Miller's formula for the q^{-1}-Rogers-Szeg"o polynomials $h_n(a,b|q^{-1})$ is obtained by supplying special values for variables in the Mehler's formula for $h_n(a,b,x|q^{-1})$.

Theorem 5.1 (The Mehler's formula for $h_n(a,b,x|q^{-1})$)**.** *Let the polynomials $h_n(a,b,x|q^{-1})$ be defined as in* (30)*, then*

$$\sum_{n=0}^{\infty}h_n(a,b,x|q^{-1})h_n(c,d,y|q^{-1})\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}$$
$$=(bdt;q)_\infty\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+n}q^{\binom{k}{2}+\binom{n}{2}}(y;q)_k(x;q)_n}{(q;q)_k(q;q)_n}(bct)^k(adt)^n$$
$$\times\ {}_3\phi_2\left(\begin{matrix}q^{-k},xq^n,q/bdt\\0,0\end{matrix};q,adt\right). \tag{39}$$

Proof. By using (31), we get

$$\sum_{n=0}^{\infty}h_n(a,b,x|q^{-1})h_n(c,d,y|q^{-1})\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty}\widetilde{E}(a,x;\theta)\{b^n\}h_n(c,d,y|q^{-1})\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}$$
$$=\widetilde{E}(a,x;\theta)\left\{\sum_{n=0}^{\infty}h_n(c,d,y|q^{-1})\frac{(-1)^n q^{\binom{n}{2}}(bt)^n}{(q;q)_n}\right\}$$

$$= \widetilde{E}(a,x;\theta)\left\{(dbt;q)_\infty \ {}_1\phi_1\left(\begin{matrix} y \\ 0 \end{matrix} ;q,cbt\right)\right\} \quad \text{(by using (32))}$$

$$= \widetilde{E}(a,x;\theta)\left\{(dbt;q)_\infty \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(y;q)_k}{(q;q)_k}(cbt)^k\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(y;q)_k}{(q;q)_k}(ct)^k \widetilde{E}(a,x;\theta)\left\{b^k(dbt;q)_\infty\right\} \qquad \text{(by using (2))}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(y;q)_k}{(q;q)_k}(bct)^k (dbt;q)_\infty \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(x;q)_n}{(q;q)_n}(adt)^n$$

$$\times \ {}_3\phi_2\left(\begin{matrix} q^{-k}, xq^n, q/bdt \\ 0,0 \end{matrix} ;q,adt\right) \quad \text{(by using (29))}$$

$$= (dbt;q)_\infty \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+n} q^{\binom{k}{2}+\binom{n}{2}}(y;q)_k(x;q)_n}{(q;q)_k(q;q)_n}(bct)^k(adt)^n$$

$$\times \ {}_3\phi_2\left(\begin{matrix} q^{-k}, xq^n, q/bdt \\ 0,0 \end{matrix} ;q,adt\right)$$

□

Setting $x = 0$ and $y = 0$ in Mehler's formula for the polynomials $h_n(a,b,x|q^{-1})$ (39), we recover Mehler's formula for the polynomials $h_n(a,b|q^{-1})$ obtained by Liu [11] (equation (16)) as follows:

Corollary 5.1. *Let $h_n(a,b|q^{-1})$ be defined as in* (14), *then*

$$\sum_{n=0}^{\infty} h_n(a,b|q^{-1})h_n(c,d|q^{-1})\frac{(-1)^{-1}q^{\binom{n}{2}}t^n}{(q;q)_n} = \frac{(act,adt,bct,bdt;q)_\infty}{(abcdt^2/q;q)_\infty},$$

provided that $|abcdt^2/q| < 1$.

Proof. Setting $x = 0$ and $y = 0$ in (39), we get

$$\sum_{n=0}^{\infty} h_n(a,b|q^{-1})h_n(c,d|q^{-1})\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}$$

$$= (bdt;q)_\infty \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+n} q^{\binom{k}{2}+\binom{n}{2}}}{(q;q)_k(q;q)_n}(bct)^k(adt)^n \ {}_3\phi_2\left(\begin{matrix} q^{-k}, 0, q/bdt \\ 0,0 \end{matrix} ;q,adt\right)$$

$$= (bdt;q)_\infty \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+n} q^{\binom{k}{2}+\binom{n}{2}}}{(q;q)_k(q;q)_n}(bct)^k(adt)^n \ \sum_{m=0}^{k}\frac{(q^{-k},q/bdt;q)_m}{(q;q)_m}(adt)^m$$

$$= (bdt;q)_\infty \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}(adt)^n \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}(bct)^k \ \sum_{m=0}^{k}\frac{(q^{-k},q/bdt;q)_m}{(q;q)_m}(adt)^m$$

$$= (bdt,adt;q)_\infty \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}(bct)^k \ \sum_{m=0}^{k}\frac{(q;q)_k}{(q;q)_{k-m}}(-1)^m q^{\binom{m}{2}-km}\frac{(q/bdt;q)_m}{(q;q)_m}(adt)^m$$

$$= (bdt,adt;q)_\infty \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+m} q^{\binom{k+m}{2}}}{(q;q)_k}(bct)^{k+m} \ (-1)^m q^{\binom{m}{2}-(k+m)m}\frac{(q/bdt;q)_m}{(q;q)_m}(adt)^m$$

$$= (bdt, adt; q)_\infty \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+m} q^{\binom{k}{2}+\binom{m}{2}+km}}{(q;q)_k} (bct)^k \ (-1)^m q^{\binom{m}{2}-(k+m)m}$$

$$\times \frac{(q/bdt;q)_m}{(q;q)_m} (abcdt^2)^m \quad \text{(by using (7))}$$

$$= (bdt, adt; q)_\infty \sum_{m=0}^{\infty} \frac{(q/bdt;q)_m}{(q;q)_m} (abcdt^2/q)^m \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (bct)^k$$

$$= \frac{(bdt, adt, bct, act; q)_\infty}{(abcdt^2/q; q)_\infty}. \quad \text{(by using (1) and (2))}$$

□

6. Conclusions

(1) The operator $\widetilde{E}(a, b; \theta)$ is an extension of the operator $E(b\theta)$.
(2) The polynomials $h_n(a, b, x|q^{-1})$ is an extension of the q^{-1}-Rogers-Szegö polynomials $h_n(a, b|q^{-1})$.
(3) The identities of the polynomials $h_n(a, b, x|q^{-1})$ are an extension of the identities of the polynomials q^{-1}-Rogers-Szegö polynomials $h_n(a, b|q^{-1})$.

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Husam Luti Saad for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.12, N.2.

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