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ON COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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Abstract. We introduce certain subclasses of bi-univalent functions related to the generalized Janowski functions and estimate the general coefficient bound for the newly defined classes. Also, we deduce certain new results and the improvement of known results as special cases of our investigation.

Keywords: Analytic function, bi-univalent functions, bounded variation, Faber polynomial, coefficient estimates.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathcal{E} = \{z : |z| < 1\}$ and having series form as

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k, \quad (n \geq 2) . \tag{1}$$

Further, let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ that are univalent in $\mathcal{E}$ and let $\mathcal{P}$ be the class of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

that are analytic in $\mathcal{E}$ and satisfy the condition $\Re(p(z)) > 0$ in $\mathcal{E}$.
We say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in $\mathcal{E}$ such that $f(z) = g(w(z))$. In addition, if g is univalent in $\mathcal{E}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{E}) \subset g(\mathcal{E})$.

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Analytic functions p in the class $\mathcal{P}[A,B]$ can be defined by using subordination as follows [13].

Let p be analytic in $\mathcal{E}$ with $p(0)=1$. Then $p\in\mathcal{P}[A,B]$, if and only if,

$$p(z)\prec\frac{1+Az}{1+Bz},\qquad -1\le B<A\le 1,\ z\in\mathcal{E}.$$

Noor [16] introduced the class $\mathcal{P}_m[A,B]$ of analytic functions p with $p(0)=1$ such that

$$p(z)=\left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z),$$

where $p_1,p_1\in\mathcal{P}[A,B]$ and $m\ge 2$.

Particularly, for $A=1-2\beta$ and $B=-1$, the class $\mathcal{P}_m[A,B]$ reduces to the class $\mathcal{P}_m(\beta)$ of analytic univalent functions p, normalized with $p(0)=1$ and satisfying

$$\int_0^{2\pi}\left|\frac{\Re(p(z))-\beta}{1-\beta}\right|d\theta\le m\pi,$$

where $m\ge 2$, $\beta\in[0,1)$ and $z\in\mathcal{E}$. Furthermore, for $\beta=0$, we have the class $\mathcal{P}_m(0)=\mathcal{P}_m$, introduced by Pinchuk [17]. Moreover, for $m=2$ we have well known class $\mathcal{P}$ of Caratheodory functions.

It is well known by Koebe one quarter theorem [9] that the image of $\mathcal{E}$ under every function $f\in\mathcal{S}$ contains a disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z))=z,\quad (z\in\mathcal{E})$$

and

$$f\left(f^{-1}(w)\right)=w,\quad (|w|<r_0(f),\ r_0(f)\ge 1/4).$$

The following is the series expansion of the inverse of f, (we say, $g(w)=f^{-1}(w)$),

$$g(w)=f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\dots \tag{2}$$

A function $f\in\mathcal{S}$ is said to be bi-univalent in $\mathcal{E}$ if there exists a function $g\in\mathcal{S}$ such that $g(z)$ is an univalent extension of f^{-1} to $\mathcal{E}$. We denote by Σ the class of bi-univalent in $\mathcal{E}$. The functions $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ are in the class Σ; see [19]. However, the familiar Koebe function is not bi-univalent. Lewin [14] was first who investigated the class Σ and showed that $|a_2|<1.51$. The pioneering work by Srivastava et al. [19] actually revived the study of analytic and bi-univalent functions in recent years. The study of bi-univalent functions gained momentum mainly due to this work. Many researchers [8, 15] recently investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients. Many authors have used the Faber polynomials [10] to determine the general coefficient, $|a_n|$ $(n\ge 3)$, for certain subclasses of bi-univalent functions; see [5, 7, 18]. This problem remained open problem since more than five decades, but recently by employing the Faber polynomials, Al-Refai and Ali [3] estimated $|a_n|$ whenever f is bi-univalent function.

The Faber polynomial expression of analytic function f of the form (1) is used to express the coefficients of its inverse map as,

$$b_n=\frac{1}{n}K_{n-1}^{-n}\left(a_2,a_3,...,a_n\right),$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!\,(n-1)!}a_2^{n-1} + \frac{(-n)!}{[2\,(-n+1)]!\,(n-3)!}a_2^{n-3}a_3 \\ &+\frac{(-n)!}{(-2n+1)!\,(n-4)!}a_2^{n-4}a_4 \\ &+\frac{(-n)!}{[2\,(-n+2)]!\,(n-5)!}a_2^{n-5}\left(a_5+(-n+2)\,a_3^2\right) \\ &+\frac{(-n)!}{(-2n+5)!\,(n-6)!}a_2^{n-6}\left(a_6+(-2n+5)\,a_3a_4\right)+\sum_{j\geq 7}a_2^{n-j}V_j, \end{aligned}$$

such that V_j with $7 \leq j \leq n$ is homogeneous polynomial in the variables $a_2, a_3, ..., a_n$; see [2]. Particularly, the first three terms of K_{n-1}^{-n} are:

$$-\frac{1}{2}K_1^{-2} = a_2,\ \ \frac{1}{3}K_2^{-3} = 2a_2^2 - a_3 \ \text{ and } \ \frac{1}{4}K_3^{-4} = -\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, for any integer p, an expansion of K_{n-1}^{p} is as [1];

$$K_{n-1}^{p} = pa_n + \frac{p\,(p-1)}{2}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + ... + \frac{p!}{(p-n+1)!\,(n-1)!}D_{n-1}^{n-1}, \quad (3)$$

where $D_n^p = D_n^p\,(a_2, a_3, ..., a_n)$, and alternatively; see [20],

$$D_{n-1}^{m}\,(a_2, a_3, ..., a_n) = \sum \frac{m!}{\mu_1!\mu_2!...\mu_{n-1}!}a_2^{\mu_1}a_3^{\mu_2}...a_n^{\mu_{n-1}},$$

where the sum is taken over all nonnegative integers μ_1, μ_1,...,μ_{n-1} satisfying the conditions

$$\begin{cases} \mu_1 + \mu_1 + ... + \mu_{n-1} = m \\ \mu_1 + 2\mu_1 + ... + (n-1)\,\mu_{n-1} = n-1 \end{cases}.$$

Evidently, $D_{n-1}^{n-1}\,(a_2, a_3, ..., a_n) = a_2^{n-1}$.
Motivated by the work on bi-univalent functions as mentioned above, we define a new subclass $\Sigma\mathcal{S}_m^{\gamma,\lambda}[A,B]$ and determine the general coefficient bound $|a_n|$ for $f \in \Sigma\mathcal{S}_m^{\gamma,\lambda}[A,B]$.

Definition 1.1. *For $-1 \leq B < A \leq 1$, $m \geq 2$, $\lambda \in \mathbb{C}$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ of the form* (1) *is said to be in the class $\Sigma\mathcal{S}_m^{\gamma,\lambda}[A,B]$ if the following conditions are satisfied*

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) - 1\right] \in \mathcal{P}_m[A,B], \quad (z \in \mathcal{E})$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) - 1\right] \in \mathcal{P}_m[A,B], \quad (w \in \mathcal{E}),$$

where $g\,(w)$ is given by (2)*.*

Special cases:

(i) For $A = 1-2\beta$ and $B = -1$, we obtain a new class $\Sigma\mathcal{S}_m^{\gamma,\lambda}[1-2\beta,-1] = \Sigma\mathcal{S}_m^{\gamma,\lambda}(\beta)$ of functions $f \in \Sigma$ such that

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) - 1\right] \in \mathcal{P}_m(\beta), \quad (z \in \mathcal{E})$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) - 1\right] \in \mathcal{P}_m(\beta), \quad (w \in \mathcal{E}),$$

where $\beta \in [0,1)$ and g is a function given by (2).

(ii) For $\gamma = 1$, $A = 1-2\beta$ and $B = -1$, we obtain the class $\Sigma\mathcal{S}_m^{1,\lambda}[1-2\beta,-1] = \Sigma\mathcal{S}_m^{\lambda}(\beta)$ introduced in [4].

(iii) For $\gamma = 1$, $m = 2$, $A = 1-2\beta$ and $B = -1$, we obtain the class $\Sigma\mathcal{S}_2^{1,\lambda}[1-2\beta,-1] = \Sigma\mathcal{S}^{\lambda}(\beta)$ introduced in [12].

(iv) For $m = 2$, we obtain the class $\Sigma\mathcal{S}_2^{\gamma,\lambda}[A,B] = \Sigma\mathcal{S}_{\lambda}^{\gamma}[A,B]$ introduced in [6].

(v) For $m = 2$, $\gamma = \lambda = 1$, $A = 1-2\beta$ and $B = -1$, we get the class $\Sigma\mathcal{S}_2^{1,1}[1-2\beta,-1] = \Sigma\mathcal{H}(\beta)$ introduced in [19].

(vi) If we set $m = 2$, $\gamma = 1$ and $\lambda = 0$, we obtain a class introduced in [6].

2. Main Results

In order to derive our main result, we need the following lemmas.

Lemma 2.1. [11] *Let $p \in \mathcal{P}[A,B]$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then, for $-1 \le A < B \le 1$ and $n \ge 1$,*

$$|p_n| \le A - B.$$

Lemma 2.2. *Let $m \ge 2$, $-1 \le A < B \le 1$ and let $p \in \mathcal{P}_m[A,B]$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then*

$$|p_n| \le \frac{m}{2}(A - B).$$

Proof. We can easily prove by using the definition of $\mathcal{P}_m[A,B]$ together with Lemma 2.1. □

Lemma 2.3. [3] *Let $f \in \Sigma$ with $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$; $(n \ge 2)$, and $f^{-1}(w) = w + \sum_{k=n}^{\infty} b_k w^k$ $(|w| < r_0(f),\ r_0(f) \ge 1/4)$. Then*

$$b_{2n-1} = na_n^2 - a_{2n-1} \quad \text{and} \quad b_k = -a_k \quad \text{for} \quad (n \le k \le 2n-2).$$

Theorem 2.1. *Let $f \in \Sigma\mathcal{S}_m^{\gamma,\lambda}[A,B]$ be given by (1). Then, for $n \ge 2$,*

$$|a_n| \le \min\left\{\sqrt{\frac{m(A-B)|\gamma|}{2|1+2(n-1)\lambda|n}}; \frac{m(A-B)|\gamma|}{2|1+(n-1)\lambda|}\right\}$$

and

$$\left|na_n^2 - a_{2n-1}\right| \le \frac{m(A-B)|\gamma|}{2|1+(n-1)\lambda|},$$

with $\lambda \in \mathbb{C}\backslash\left\{-\frac{1}{n-1}, -\frac{1}{2(n-1)}\right\}$, $\gamma \in \mathbb{C}\backslash\{0\}$ and $-1 \le B < A \le 1$.

Proof. Let $f \in \Sigma\mathcal{S}_m^{\gamma,\lambda}[A,B]$ be given by (1). Then there exists two analytic functions $p, q \in \mathcal{P}_m[A,B]$ with

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \tag{4}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \dots \tag{5}$$

such that

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) - 1\right] = p(z) \tag{6}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) - 1\right] = q(w), \tag{7}$$

where $g(w)$ is given by (2).

On the other hand, for $(k \geq n \geq 2)$

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right]=1+\sum_{k=n}^{\infty}\frac{[1+(k-1)\lambda]}{\gamma}a_k z^k, \tag{8}$$

and

$$\begin{aligned}1+\frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)-1\right] &= 1+\sum_{k=n}^{\infty}\frac{[1+(k-1)\lambda]}{\gamma}b_k w^k \\ &= 1+\sum_{k=n}^{\infty}\frac{[1+(k-1)\lambda]}{\gamma} \\ &\quad\times\left[\frac{1}{k}K_{k-1}^{-k}(a_2,a_3,...,a_k)\right]w^k,\end{aligned} \tag{9}$$

Comparing the corresponding coefficients of (4) and (8) gives

$$\frac{[1+(k-1)\lambda]}{\gamma}a_k=p_{k-1},\ (k\geq n\geq 2). \tag{10}$$

Similarly, from (5) and (9), we get

$$\frac{[1+(k-1)\lambda]}{\gamma}K_{k-1}^{-k}(a_2,a_3,...,a_k)=q_{k-1},\ (k\geq n\geq 2). \tag{11}$$

From (11) and (3), we can write

$$\frac{[1+(k-1)\lambda]}{\gamma}b_k=q_{k-1},\ (k\geq n\geq 2). \tag{12}$$

By using Lemma 2.3, (10) and (12) implies

$$|a_k|\leq\frac{m(A-B)|\gamma|}{2|1+(k-1)\lambda|};\quad \text{for } (k\geq n\geq 2) \text{ and } \lambda\in\mathbb{C}\backslash\left\{\frac{-1}{k-1}\right\} \tag{13}$$

and

$$|b_k|\leq\frac{m(A-B)|\gamma|}{2|1+(k-1)\lambda|};\quad \text{for } (k\geq n\geq 2) \text{ and } \lambda\in\mathbb{C}\backslash\left\{\frac{-1}{k-1}\right\}. \tag{14}$$

Particularly, we have

$$|a_n|\leq\frac{m(A-B)|\gamma|}{2|1+(n-1)\lambda|};\ \text{for } \lambda\in\mathbb{C}\backslash\left\{\frac{-1}{n-1}\right\} \tag{15}$$

and

$$|a_{2n-1}|\leq\frac{m(A-B)|\gamma|}{2|1+2(n-1)\lambda|} \text{ and } |b_{2n-1}|\leq\frac{m(A-B)|\gamma|}{2|1+2(n-1)\lambda|}. \tag{16}$$

Therefore, on making use of Lemma 2.3 along with (15), we find

$$|a_n|\leq\sqrt{\frac{|a_{2n-1}|+\ |b_{2n-1}|}{n}}\leq\sqrt{\frac{m(A-B)|\gamma|}{|1+2(n-1)\lambda|n}};\quad \text{for } \lambda\in\mathbb{C}\backslash\left\{-\frac{1}{2(n-1)}\right\}. \tag{17}$$

and

$$\left|na_n^2-a_{2n-1}\right|\leq|b_{2n-1}|\leq\frac{m(A-B)|\gamma|}{2|1+2(n-1)\lambda|}.$$

This proves our result. □

Taking $A=1-2\beta$ and $B=-1$ in the above theorem, we get the following new result.

Corollary 2.1. *Let* $f \in \Sigma\mathcal{S}_m^{\gamma;\lambda}(\beta)$ *be given by* (1). *Then, for* $n \geq 2$,

$$|a_n| \leq \min\left\{\sqrt{\frac{m(1-\beta)|\gamma|}{|1+2(n-1)\lambda|\,n}};\frac{m(1-\beta)|\gamma|}{|1+(n-1)\lambda|}\right\}$$

and

$$\left|na_n^2 - a_{2n-1}\right| \leq \frac{m(1-\beta)|\gamma|}{|1+(n-1)\lambda|},$$

with $\lambda \in \mathbb{C}\backslash\left\{-\frac{1}{n-1},-\frac{1}{2(n-1)}\right\}$, $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\beta \in [0,1)$.

If we choose $\gamma = 1$ in the above corollary, we get

Corollary 2.2. *Let* $f \in \Sigma\mathbf{S}_m^{\lambda}(\beta)$ *be given by* (1). *Then, for* $n \geq 2$,

$$|a_n| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{|1+2(n-1)\lambda|\,n}};\frac{m(1-\beta)}{|1+(n-1)\lambda|}\right\}$$

and

$$\left|na_n^2 - a_{2n-1}\right| \leq \frac{m(1-\beta)}{|1+(n-1)\lambda|},$$

with $\lambda \in \mathbb{C}\backslash\left\{-\frac{1}{n-1},-\frac{1}{2(n-1)}\right\}$ *and* $\beta \in [0,1)$.

Furthermore, for $m = 2$, the above corollary gives the following result, which is the improvement of the Theorem 1 proved by Jahangiri et al. [12].

Corollary 2.3. *Let* $f \in \Sigma\mathcal{S}^{\lambda}(\beta)$ *be given by* (1). *Then, for* $n \geq 2$,

$$|a_n| \leq \min\left\{\sqrt{\frac{2(1-\beta)}{|1+2(n-1)\lambda|\,n}};\frac{2(1-\beta)}{|1+(n-1)\lambda|}\right\}$$

and

$$\left|na_n^2 - a_{2n-1}\right| \leq \frac{2(1-\beta)}{|1+(n-1)\lambda|},$$

with $\lambda \in \mathbb{C}\backslash\left\{-\frac{1}{n-1},-\frac{1}{2(n-1)}\right\}$ *and* $\beta \in [0,1)$.

If we set $m = 2$ in Theorem 2.1, we get the following corollary, which gives the bound for general coefficient of the functions in class introduced in [6].

Corollary 2.4. *Let* $f \in \Sigma\mathcal{S}_{\lambda}^{\gamma}[A,B]$ *be given by* (1). *Then, for* $\gamma \in \mathbb{C}\backslash\{0\}$

$$|a_n| \leq \min\left\{\sqrt{\frac{(A-B)|\gamma|}{|1+2(n-1)\lambda|\,n}};\frac{(A-B)|\gamma|}{|1+(n-1)\lambda|}\right\}$$

and

$$\left|na_n^2 - a_{2n-1}\right| \leq \frac{(A-B)|\gamma|}{|1+(n-1)\lambda|},$$

with $\lambda \in \mathbb{C}\backslash\left\{-\frac{1}{n-1},-\frac{1}{2(n-1)}\right\}$, $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $-1 \leq B < A \leq 1$.

If we take $\gamma = \lambda = 1$ and $m = 2$ in the Corollary 2.1, we deduce general coefficient bound for the functions in class introduced by Srivastava et al. [19].

Corollary 2.5. [19] *Let* $f \in \sum \mathcal{H}(\beta)$ *be given by* (1). *Then, for* $n \geq 2$,

$$|a_n| \leq \min\left\{\sqrt{\frac{2(1-\beta)}{(2n-1)n}}; \frac{2(1-\beta)}{n}\right\}$$

and

$$\left|na_n^2 - a_{2n-1}\right| \leq \frac{2(1-\beta)}{n},$$

with $\beta \in [0,1)$.

Remark 2.1. *In particular, the coefficient estimates* $|a_2|$ *and* $|a_3|$ *obtained by Theorem* 2.1 *improves the estimates proved by the authors in* [6, 12, 19].

3. Conclusion

We have introduced certain subclasses of bi-univalent functions by using the notion of generalized Janowski functions. The general coefficient bounds for the functions in these classes are investigated. It is shown that the bound estimates in the main result are the improvements of the bound estimates already proved in the literature.

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