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ZAGREB EQUIENERGETIC BIPARTITE GRAPHS

G. H SHIRDEL[1*], S. AHMADI[1], §

ABSTRACT. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and let d_i be the degree of v_i. The Zagreb matrix of the graph G is the square matrix of order n whose (i, j)-entry is equal to $d_i + d_j$ if the vertices v_i and v_j are adjacent, and zero otherwise. The Zagreb energy $ZE(G)$ of G is the sum of the absolute values of the eigenvalues of the Zagreb matrix. Two graphs are said to be Zagreb equienergetic if their Zagreb energies are equal. In this paper, we show how infinitely many pairs of Zagreb equienergetic bipartite graphs can be constructed such that these bipartite graphs are connected, possess an equal number of vertices, an equal number of edges, and are not cospectral.

Keywords: Zagreb energy, Line graph, Complement of graph, Extended double cover of graph, Bipartite graph.

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1. Introduction

In this paper, G is a simple undirected graph with a vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E = E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ represent the graph's order and size, respectively. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u \in V | uv \in E\}$, and the degree of v is $d(v) = |N(v)|$. The graph G is said to be r-regular if the degree of its all vertices is r. A bipartite graph is a graph such that its vertex set can be partitioned into two sets, X and Y, (referred to as the partite sets), such that every edge meets both X and Y. The complement $\overline{G}$ of a graph G is the simple graph whose vertex set is V and whose edges are the pairs of non-adjacent vertices of G. The line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G. The basic properties of line graphs are found in textbooks [9]. The iterated line graphs of G are then defined recursively as $L^2(G) = L(L(G)), L^3(G) = L(L^2(G)), \ldots, L^k(G) = L(L^{k-1}(G)), \ldots$. It is consistent to set $L(G) \equiv L^1(G)$ and $G \equiv L^0(G)$. The basic properties of iterated line graph sequences are summarized in the articles [2, 3]. The extended double cover of G, defined in [1] and denoted by G^* is the bipartite graph with bipartition (X, Y) where

[1] Department of Mathematics, Faculty of Sciences, University of Qom, Qom, Iran.
e-mail: shirdel81math@gmail.com, ORCID: https://orcid.org/0000-0003-2759-4606.
[*] Corresponding author.
e-mail: sara.ahmadi1389@yahoo.com, ORCID: https://orcid.org/0000-0003-0127-409x.
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$X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, in which x_i and y_i are adjacent if and only if $i = j$ or v_i and v_j are adjacent in G . It is easy to see that G^* is connected if and only if G is connected and G^* is regular of degree $r + 1$ if and only if G is regular of degree r.

The adjacency matrix $A(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and zero otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $A(G)$, then the energy of the graph G was first introduced by Gutman [5] in 1978 and is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix,

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The concept of graph energy arose in chemistry (see, e.g., [4, 5]). Two non-isomorphic graphs G_1 and G_2 of the same order are said to be equienergetic if $E(G_1) = E(G_2)$ [11]. The Zagreb indices are widely studied degree-based topological indices and were established by Gutman and Trinajstić [8] in 1972. The Zagreb matrix of a graph G is a square matrix $A_z(G) = (m_{ij})$ of order n, defined in [13], as follows:

$$m_{i,j} = \begin{cases} d_i + d_j & \text{if the vertices } v_i \text{ and } v_j \text{ of } G \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A_z(G)$ labeled as $z_1 \geq z_2 \geq \cdots \geq z_n$ are referred to as the Zagreb eigenvalues of G. The Zagreb energy, represented by $ZE(G)$ and described in [13], is the sum of all absolute Zagreb eigenvalues and is defined as follows:

$$ZE = ZE(G) = \sum_{i=1}^{n} |z_i|.$$

In a manner analogous to equienergetic graphs, two non-isomorphic graphs of the same order are said to be Zagreb equienergetic if they have the same Zagreb energy.

2. Main result

Let G be a $r-$regular graph of order n and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $z_1 \geq z_2 \geq \cdots \geq z_n$ be the eigenvalues and the Zagreb eigenvalues of G respectively. According to the definition of the adjacency matrix $A(G)$ and the Zagreb matrix $A_z(G)$, we have

$$A_z(G) = 2rA(G). \tag{1}$$

Therefore

$$z_i = 2r\lambda_i \quad \text{for } i = 1, 2, \dots, n. \tag{2}$$

If d_{max} is the greatest vertex degree of a graph, then all its eigenvalues belongs to the interval $[-d_{max}, +d_{max}]$ [4]. In particular, the eigenvalues of the $r-$ regular graph G, satisfy the condition $-r \leq \lambda \leq r$ for all $i = 1, 2, \dots, n$. From (2), we have

$$-2r^2 \leq z_i \leq 2r^2 \quad \text{for all} \quad i = 1, 2, \dots, n.$$

On the other hand, $n-$dimension column vector with all one, clearly is an eigenvector of $A(G)$ with corresponding the eigenvalue r. Then the greatest eigenvalue of $A(G)$ is always r, that is $\lambda_1 = r$. Therefore, $z_1 = 2r^2$. By definition, the diagonal elements of the adjacency matrix of graph G are equal to zero. Therefore, the trace of $A(G)$ is zero. From linear algebra, we know that the sum of all the eigenvalues of a square matrix is equal to the trace of the matrix. Therefore, we conclude $\sum_{i=1}^{n} \lambda_i = 0$. Considering (2), we also have $\sum_{i=1}^{n} z_i = 0$.

To prove the main results in this note, we will use the following theorems and their corresponding results.

Theorem 2.1. [2, 3] *The line graph of a regular graph is a regular graph. In particular, the line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = \frac{1}{2}r_0n_0$ and degree $r_1 = 2r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are*

$$n_k = \frac{1}{2}r_{k-1}n_{k-1} \qquad \text{and} \qquad r_k = 2r_{k-1} - 2,$$

where n_{k-1} and r_{k-1} stand for the order and degree of $L^{k-1}(G)$. Therefore

$$r_k = 2^k r_0 - 2^{k+1} + 2$$

and

$$n_k = \frac{n_0}{2^k}\prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k}\prod_{i=0}^{k-1}(2^i r_0 - 2^{i+1} + 2).$$

Corollary 2.1. *If G be a r-regular graph of order n, then the degree and order of $L^2(G)$ are $4r - 6$ and $nr(r-1)/2$, respectively.*

Theorem 2.2. [14] *If $z_1 \geq z_2 \geq \cdots \geq z_n$ be the Zagreb eigenvalues of a regular graph G of order n and degree r, then the Zagreb eigenvalues of $L(G)$ are*

$$\left.\begin{array}{ll} 8 - 8r & n(r-2)/2 \quad \text{times}, \\ 4(r-1)\Big(z_i/2r + r - 2\Big) & i = 1, 2, \ldots, n. \end{array}\right\}$$

Corollary 2.2. *If $z_1 \geq z_2 \geq \cdots \geq z_n$ be the Zagreb eigenvalues of a regular graph G of order n and degree r, then the Zagreb eigenvalues of $L^2(G)$ are*

$$\left.\begin{array}{ll} 24 - 16r & nr(r-2)/2 \quad \text{times}, \\ 8(2r-3)(r-3) & n(r-2)/2 \quad \text{times}, \\ 4(2r-3)(z_i/2r + 3r - 6) & i = 1, 2, \ldots, n. \end{array}\right\} \tag{3}$$

Theorem 2.3. [14] *Let G be a r-regular graph $(r \geq 3)$ of order n with the Zagreb eigenvalues $z_1 \geq z_2 \geq \cdots \geq z_n$. The Zagreb eigenvalues of $A_z(\overline{G})$ are $2(n-r-1)^2$ with multiplicity one and $2(n-r-1)\big(-z_i/2r - 1\big)$, for $i = 2, 3, \ldots, n$.*

Corollary 2.3. *Let G be a r-regular graph $(r \geq 3)$ of order n with Zagreb eigenvalues $z_1 \geq z_2 \geq \cdots \geq z_n$. The Zagreb eigenvalues of $\overline{L^2(G)}$ are*

$$\left.\begin{array}{ll} \big(nr(r-1) - 8r + 10\big) & nr(r-2)/2 \quad \text{times}, \\ \big(nr(r-1) - 8r + 10\big)^2/2 & \text{one time}, \\ \big(nr(r-1) - 8r + 10)(-2r + 5) & n(r-2)/2 \quad \text{times}, \\ \big(nr(r-1) - 8r + 10\big)\big(-z_i/2r - 3r + 5\big) & i = 2, 3, \ldots, n. \end{array}\right\} \tag{4}$$

Proof. First note that the degree and order of $L^2(G)$ are $4r-6$ and $nr(r-1)/2$, respectively. Then combining Corollary 2.2 and Theorem 2.3, the eigenvalues of $\overline{L^2(G)}$ are obtained immediately. □

Theorem 2.4. *Let G be a r-regular graph of order n and $z_1 \geq z_2 \geq \cdots \geq z_n$ be the Zagreb eigenvalues of G. Then the Zagreb eigenvalues of G^* are $\pm 2(r+1)\big(z_i/2r+1\big)$ for $i = 1, 2, \ldots, n$.*

Proof. Let $A(G)$ and $A_z(G)$ be the adjacency matrix and Zagreb matrix of G, respectively. Then the Zagreb matrix of G^* is as follows:

$$\begin{pmatrix} 0 & 2(r+1)(A(G)+I) \\ 2(r+1)(A(G)+I) & 0 \end{pmatrix},$$

where I is the unit matrix. Since $A(G) = A_z(G)/2r$, the Zagreb matrix of G^* can be written as

$$\begin{pmatrix} 0 & 2(r+1)\big(A_z(G)/2r+I\big) \\ 2(r+1)\big(A_z(G)/2r+I\big) & 0 \end{pmatrix}.$$

Suppose that z is a Zagreb eigenvalue of G and x is an eigenvector corresponding to z that is, $A_z(G)x = zx$. Then we have

$$\begin{pmatrix} 0 & 2(r+1)\big(A_z(G)/2+I\big) \\ 2(r+1)\big(A_z(G)/2+I\big) & 0 \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = 2(r+1)\big(z/2r+1\big) \begin{pmatrix} x \\ x \end{pmatrix}.$$

And

$$\begin{pmatrix} 0 & 2(r+1)\big(A_z(G)/2+I\big) \\ 2(r+1)\big(A_z(G)/2+I\big) & 0 \end{pmatrix} \begin{pmatrix} x \\ -x \end{pmatrix} = -2(r+1)\big(z/2r+1\big) \begin{pmatrix} x \\ -x \end{pmatrix}.$$

As a result, G^* has two Zagreb eigenvalues $\pm 2(r+1)\big(z/2r+1\big)$ corresponding to the Zagreb eigenvalue z of G. This completes the proof. □

Here we offer the method for constructing pairs of Zagreb equienergetic bipartite graphs that is similar to Theorem 2 in [10].
The following is the main result of this note.

Theorem 2.5. *Let G_1 and G_2 be two regular graphs, both on n vertices, both of degree $r \geq 3$. Then*

(1) $(L^2(G_1))^*$ *and* $(L^2(G_1))^*$ *are Zagreb equienergetic bipartite graphs and*

$$ZE((L^2(G_1))^*) = ZE((L^2(G_2))^*) = 2nr(4r-5)(3r-5).$$

(2) $(\overline{L^2(G_1)})^*$ *and* $(\overline{L^2(G_2)})^*$ *are Zagreb equienergetic bipartite graphs and*

$$ZE((\overline{L^2(G_1)})^*) = ZE((\overline{L^2(G_2)})^*) = (nr(r-1)-8r+12)(5nr^2-9nr-16r+24).$$

(3) $\overline{(L^2(G_1)^*)}$ *and* $\overline{(L^2(G_2)^*)}$ *are Zagreb equienergetic graphs and*

$$ZE(\overline{(L^2(G_1)^*)}) = ZE(\overline{(L^2(G_2)^*)}) = 2(nr-4)(r-1)(4nr^2-6nr-8r+8).$$

Proof. First, we need to remind that the degree and order of $L^2(G)$, $(L^2(G))^*$ and $\overline{L^2(G)}$ are as follows:

$$L^2(G) : 4r-6, \quad \frac{nr(r-1)}{2};$$

$$L^2(G)^* : 4r-5, \quad nr(r-1);$$

$$\overline{L^2(G)} : \frac{nr(r-1)-8r+10}{2}, \quad \frac{nr(r-1)}{2}.$$

Let $z_1 \geq z_2 \geq \cdots \geq z_n$ be the Zagreb eigenvalues of G. Combining Corollary 2.2 and Theorem 2.4, the eigenvalues of $(L^2(G))^*$ are obtained as

$$\left.\begin{array}{cc} \pm 2(4r-5) & nr(r-2)/2 \text{ times}, \\ \pm 2(4r-5)(2r-5) & n(r-2)/2 \text{ times}, \\ \pm 2(4r-5)\big(z_i/2r+3r-5\big) & i=1,2,\ldots,n. \end{array}\right\} \tag{5}$$

Evidently, $(4r-5)$ and $(2r-5)$ are positive-valued for $r \geq 3$. In order to determine the sign of $z_i/2r+3r-5$, recall that all Zagreb eigenvalues of a regular graph of degree r lie in the interval $[-2r^2, 2r^2]$. Therefore, $z_i/2r \geq -r$, i.e.,

$$z_i/2r + r \geq 0. \tag{6}$$

Because $r \geq 3$, we have

$$2r-5 > 0. \tag{7}$$

Summing (6) and (7), we obtain $z_i/2r+3r-5 > 0$ for $i=1,2,\ldots,n$.
Knowing the signs of all eigenvalues of $(L^2(G))^*$, from (5) we can obtain the Zagreb energy of $(L^2(G))^*$ as

$$ZE((L^2(G))^*) = 2(4r-5)\left[2\frac{nr(r-2)}{2} + 2(2r-5)\frac{n(r-2)}{2} + 2\sum_{i=1}^{n}(\frac{z_i}{2r}+3r-5)\right]$$

$$= 2nr(4r-5)(3r-5)$$

(recall that $\sum_{i=1}^{n} z_i = 0$).

Combining Corollary 2.3 and Theorem 2.4, the Zagreb eigenvalues of $(\overline{L^2(G)})^*$ are obtained as

$$\left.\begin{array}{ll} \pm 2(nr(r-1)-8r+12) & nr(r-2)/2 \text{ times}, \\ \pm (nr(r-1)-8r+10)^2/2 & \text{one time}, \\ \pm (nr(r-1)-8r+12)(-2r+6) & n(r-2)/2 \text{ times}, \\ \pm (nr(r-1)-8r+12)\big(-z_i/2r-3r+6\big) & i=2,3,\ldots,n. \end{array}\right\} \quad (8)$$

The quantity $\big(nr(r-1)-8r+10\big)/2$ is necessarily positive-valued, because it is equal to the degree of $\overline{L^2(G)}$. Therefore, $nr(r-1)-8r+12$ is always positive too. In the same way as in the proof of previous part, we conclude $-2r+6 \leq 0$ and $-z_i/2r-3r+6 \leq 0$ for $i=1,2,\ldots,n$.
Knowing the signs of all eigenvalues of $\overline{L^2(G)}^*$, from (8) we can obtain the Zagreb energy of $\overline{L^2(G)}^*$ as

$$\begin{aligned} ZE(\overline{L^2(G)}^*) = {} & (nr(r-1)-8r+12)^2 \\ & + 2(nr(r-1)-8r+12)\Bigg[(2r-6)(\frac{n(r-2)}{2}) \\ & + 2\frac{nr(r-2)}{2} + \sum_{i=2}^{n}(\frac{z_i}{2r}+3r-6)\Bigg]. \end{aligned}$$

Considering

$$\sum_{i=2}^{n}(\frac{z_i}{2r}) = \sum_{i=1}^{n}(\frac{z_i}{2r}) - \frac{z_1}{2r}, \quad (9)$$

we have

$$\begin{aligned} ZE(\overline{L^2(G)}^*) = {} & (nr(r-1)-8r+12)^2 + 2(nr(r-1)-8r+12)\Bigg[n(r-2)(r-3) \\ & + nr(r-2) + \sum_{i=1}^{n}\frac{z_i}{2r} - \frac{z_1}{2r} + 3(r-2)(n-1)\Bigg] \end{aligned}$$

which, bearing in mind

$$\sum_{i=1}^{n} z_i = 0 \quad and \quad z_1 = 2r^2 \quad (10)$$

yields the formula

$$ZE(\overline{L^2(G)}^*) = (nr(r-1)-8r+12)(5nr^2-9nr-16r+24).$$

Combining (5) and Theorem 2.3, the Zagreb eigenvalues of $\overline{L^2(G)^*}$ are obtained as

$$\left.\begin{array}{ll} 2((nr-4)(r-1))^2 & \text{one time,} \\ 2(nr-4)(r-1)(4r-6) & \text{one time,} \\ 2(nr-4)(r-1)(\pm 1-1) & nr(r-2)/2 \text{ times,} \\ 2(nr-4)(r-1)\Big(\pm(2r-5)-1\Big) & n(r-2)/2 \text{ times,} \\ 2(nr-4)(r-1)\Big(\pm(z_i/2r+3r-5)-1\Big) & i=2,3,\ldots,n. \end{array}\right\} \tag{11}$$

In the same way as in the proof of part 1, we know the signs of all quantities in (11). Therefore we can obtain the Zagreb energy of $\overline{L^2(G_1)^*}$ as

$$\begin{aligned} ZE(\overline{L^2(G_1)^*}) &= 2\big((nr-4)(r-1)\big)\Big[(nr-4)(r-1)+4r-6+nr(r-2) \\ &\quad + n(2r-5)(r-2)+(6r-10)(n-1)+\sum_{i=2}^{n} z_i/r\Big] \\ &= 2(nr-4)(r-1)\Big[4nr^2-6nr-6r+8-z_1/r\Big] \\ &= 2(nr-4)(r-1)\Big[4nr^2-6nr-8r+8\Big]. \end{aligned}$$

This completes the proof. □

Remark 2.1. *If G be a regular graph of degree $r = 1$, then $L(G)$ consists of isolated vertices, and $L^2(G)$ is the graph without vertices. If G be a regular graph of degree $r = 2$, then G and $L(G)$ are isomorphic. Consequently, if $r = 2$, then G and $L^k(G)$ are isomorphic for all $k \geq 1$.*

3. Discussion

Corollary 3.1. *Let G_1 and G_2 be two regular graphs, both on n vertices, both of degree $r \geq 3$. Then for any $k \geq 2$, the following pairs of graphs are Zagreb equienergetic*

(1) $(L^k(G_1))^*$ *and* $(L^k(G_2))^*$;

(2) $\overline{(L^k(G_1))}^*$ *and* $\overline{(L^k(G_2))}^*$;

(3) $\overline{(L^k(G_1))^*}$ *and* $\overline{(L^k(G_2))^*}$.

Corollary 3.2. *Let G_1 and G_2 be two connected and non-cospectral regular graphs, both on n vertices, both of degree $r \geq 3$. Then for any $k \geq 2$, both $(L^k(G_1))^*$ and $(L^k(G_2))^*$ are regular, bipartite, connected, non-cospectral, and Zagreb equienergetic. Furthermore, $(L^k(G_1))^*$ and $(L^k(G_2))^*$ possess the same number of vertices, and the same number of edges.*

Within Theorem 2.5, we obtained the expression (in terms of n and r) for the Zagreb energy of the extended double cover of the second iterated line graph of a regular graph. Analogous (yet much less simple) expressions could be calculated also for $ZE((L^k(G))^*)$; $k \geq 2$, i.e., the energy of the extended double cover of the k-th iterated line graph, $k \geq 2$, of a regular graph on n vertices and of degree $r \geq 3$ is also fully determined by the parameters n and r.

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Gholam Hassan Shirdel received his B.Sc. Degree from the Ferdowsi University of Mashhad in 1993, M.Sc.(in 1996) and Ph.D. (in 2003) both from University of Tehran. He is currently an associate professor in the department of mathematics and computer sciences at University of Qom (Qom, Iran). His main research interests include network flows, optimization, graph theory, operations research, fuzzy set theory, fuzzy optimization, simulation, combinatorics and combinatorial optimization.

Sara Ahmadi received her B.Sc. from the Razi University of Kermanshah in 2009 and the M.Sc. Degree from Azerbaijan Shahid Madani University of Tabriz in 2012. Now she is the forth-year applied mathematics Ph.D. student at the University of Qom, Iran. Her field of work is graph theory and topology indices. She is currently working as a high school math teacher. She is interested in graph theory and algorithm design.