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AN EFFICIENT METHOD FOR SOLVING FRACTIONAL INTEGRAL AND DIFFERENTIAL EQUATIONS OF BRATU TYPE

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ABSTRACT. In this paper, the fractional integral and differential equations of Bratu type, which arise in many important physical phenomena, are investigated by an effective technique, Chebyshev Finite Difference Method with the help of fractional derivative in the concept of Caputo. The effect of the fractional derivative in the outcomes has great agreement with the nonlocality of the problem. The truncation and round off errors and convergence analyzes of the present method are also given. Numerical solutions of illustrative examples of the fractional integral and differential equations of Bratu type are given to highlight the validity and performance of the method. The results of the comparisons are very satisfied and show that the proposed technique is more effective and highly accurate than the other methods.

Keywords: Chebyshev finite difference method, Fractional Bratu type equation, Fractional Integro- Differential Equation, Collocation Method.

AMS Subject Classification: 34A08, 34B15, 65L05, 65L10, 65R20.

1. Introduction

The use of the differential equations for fractional order reveals new non-local model concepts in recent years [1-3]. Fractional analysis reflects physical facts better since it brings some additions due to their definitions as well as resembling the general non-local structure. Fractional analysis can be performed over the time variable in addition to use it on spatial variables. This corresponds to memory, which states that the motion of the material depends not only on the present time but also on the past. Since fractional analysis of both spatial and time variables is much more realistic, studies are getting popular [1-11]. Bratu type of problems occur in study of many physical and chemical phenomena in applied sciences such as the fuel ignition model, chemical reaction theory, Chandrasekhar model of the expansion of the universe, electro spinning process for the manufacturing of the nano fibers, nanotechnology and others [12, 13]. In the last few

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years, studies on the solution of fractional-order Bratu-type equations along with classical Bratu-type equations have become widespread [14-17]. To show the applicability and efficiency of the present method, three fractional integral and differential of Bratu type equations are investigated; the first two problems are given by

$$\left\{ \frac{d^{\alpha}y(x)}{dx^{\alpha}} + \lambda\, e^{y(x)} = 0, \quad 1 < \alpha \leq 2, \;\; 0 \leq x \leq 1 \right. \tag{1}$$

with initial conditions

$$y(0) = a_1, \qquad y'(0) = a_2. \tag{2}$$

or with boundary conditions

$$y(0) = a_3, \qquad y(1) = a_4, \tag{3}$$

where λ is real parameter and $a_i,\ i = 1, 2, 3, 4$ are constants. The fractional form of Bratu-Type of equation (1) reflects the nonlocality of the problem without defining non-local kernels.

The third problem (iii), which is an integral fractional Bratu-type equation is given as

$$\left\{ \frac{d^{\alpha}y(x)}{dx^{\alpha}} + \lambda \int_0^x f(x,t)\; e^{y(x)}dt + g(x) = 0, \quad 0 < \alpha \leq 1, \;\; 0 \leq x \leq 1 \right. \tag{4}$$

with initial condition

$$y(0) = a \tag{5}$$

where λ is a real parameter, a is constant, $f(x,t) \in L^2\left([0,1] \times [0,1]\right)$ and $g(x)$ is an unknown function.

In the solution of initial and boundary value problems, Chebyshev finite difference method (CFDM) has some advantageous. Chebyshev polynomials are orthogonal and the the polynomial approximation is valid throughout the entire interval, not just at certain points of the interval as with many numerical techniques. The use of Chebyshev polynomials [18-24] is advantageous over same-order polynomial approximations as they are the best polynomial approximation that minimize the maximum error in a given interval.

The paper is structured as follows; The definition of Caputo Fractional derivative is given in Section 2 and the Chebyshev Finite Difference Method is provided to deal with fractional order integral and differential equations of Bratu-type in Section 3. In this section, the convergence and the error analyzes of the method are also presented. In Section 4, three fractional Bratu type of equations are solved by using CFDM. The comparison demonstrates that the suggested technique offers approximate global solution with high accuracy and reduces computational costs.

2. Caputo Fractional Derivative

Ross [25] defined the criteria for fractional derivative and then Tarasov [26-28] pointed out that some other properties must be satisfied;

- Fractional derivative operator is linear and fractional derivative of an analytical function is also analytical.
- Fractional order derivative reduces to the integer order derivate while the order, $\alpha \to n$, in consequence, the zero derivate of a function gives itself.
- The fractional derivative cannot be expressed as a finite series of integer derivatives.
- The Leibniz and chain rules do not hold.

Some definitions such as conformable fractional derivative and the alternative fractional derivative do not hold these criteria [28]. Therefore, they cannot be assumed as fractional derivative [26, 27], Caputo definition [29] which satisfies all above is used here, it is

$$ {}_aD_x^{\alpha} f(x) := \begin{cases} \frac{1}{\Gamma(n-a)} \int_a^x \frac{f^{(n)}(s)}{(x-s)^{\alpha-n+1}} ds & n-1 \leq \alpha < n, \\ \frac{d^n}{dt^n} f(x), & \alpha = n, \end{cases} \tag{6} $$

where $\alpha > 0, \quad x > a$ and $a,\ x \in \Re$.
It is clear that when $\alpha \to n$, the Caputo definition reduces to the n^{th} order ordinary derivative

$$ \begin{aligned} \lim_{\alpha \to n} {}_aD_x^{\alpha} f(x) &= \lim_{\alpha \to n} \left(\frac{f^{(n)}(a)(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \int_a^x \frac{(x-s)^{n-\alpha} f^{(n+1)}(x)}{\Gamma(n-\alpha+1)} dx \right) \\ &= f^{(n)}(a) + \int_a^t f^{(n+1)}(x) dx = f^{(n)}(x). \quad (n = 1,\ 2,\ ...) \end{aligned} \tag{7} $$

3. Chebyshev Finite Difference Method

An approximation to a function, $y(x)$, can be given as a sequence of Chebyshev polynomials,

$$ y(x) \cong \sum_{n=0}^{N} b_n\, a_n\, T_n(x) \tag{8} $$

where $b_0 = b_N = \frac{1}{2},\ b_n = 1,\ (n = 1, 2..., N-1)$ and N indicates the approximation polynomial order. The unknown coefficients can be found as

$$ a_n = \frac{2}{N} \sum_{j=0}^{N} b_j\, y(x_j)\, T_n(x_j). \tag{9} $$

$y^{(m)}(x)$ at the point x_k is given as

$$ y^{(m)}(x_k) = \sum_{j=0}^{N} d_{k,j}^{(m)}\, y(x_j) \tag{10} $$

where,

$$ d_{k,j}^{(m)} = \frac{2\,\theta_j}{N} \sum_{n=0}^{N} T_n(x_j)\, T_n^{(m)}(x_k). \tag{11} $$

First two of the coefficients in (11) are given as follows [30]:

$$ d_{k,j}^{(1)} = \frac{4\,\theta_j}{N} \sum_{n=0}^{N} \sum_{\substack{l=0 \\ (n+l)\ odd}}^{n-1} \frac{n\,\theta_n}{c_l} T_n(x_j)\, T_l(x_k), \quad k, j = 0, 1, ..., N, \tag{12} $$

$$ d_{k,j}^{(2)} = \frac{2\,\theta_j}{N} \sum_{n=0}^{N} \sum_{\substack{l=0 \\ (n+l)\ even}}^{n-2} \frac{n\,(n^2-l^2)\,\theta_n}{c_l} T_n(x_j)\, T_l(x_k), \quad k, j = 0, 1, ..., N \tag{13} $$

where

$$ \left. \begin{array}{l} \theta_0 = \theta_n = 1/2,\ \ \theta_j = 1 \\ c_0 = 2,\ \ c_i = 1 \end{array} \right\} \quad j = 1, 2, ..., N-1, \quad i \geq 1. \tag{14} $$

With the help of the polynomial approximation and its derivatives, the given problem is converted into an $(N+1)$ algebraic equations. Any suitable root finding method can be used to solve this system to find $(N+1)$ unknowns of the approximation polynomials. Here, Newton method is used.

Round off and truncation errors and convergence analyzes are given. The proofs of theorems 4.1, 4.2 and 4.3 can be found in [31].

Theorem 3.1 (Convergence). *If $y(x) \in L_w^2(-1,1)$ and $|y''(x)| \leq C$, then its Chebyshev expansion uniformly converges to $y(x)$.*

Theorem 3.2 (Round-off error). *The rounding error effect on $d_{k,j}^{(1)}$ and $d_{k,j}^{(2)}$ is limited to the following formulas [32],*

$$\begin{cases} d_{k,j}^{(1)*} - d_{k,j}^{(1)} \leq 4\theta_j\,(\delta - \mathrm{O}(\frac{1}{N^2}\delta))\,(\frac{N^2}{3} + \frac{1}{6}) \\ d_{k,j}^{(2)*} - d_{k,j}^{(2)} \leq \frac{4\theta_j}{3}\,(\delta - \mathrm{O}(\frac{1}{N^2}\delta))\,(\frac{N^4}{5} - \frac{1}{5}) \end{cases}. \tag{15}$$

Theorem 3.3 (Truncation error). *Suppose the function $y(x)$ is approached by $P_N\, y = \sum\limits_{m=0}^{N} a_m T_m$. For $\forall\, y(x) \in H_w^m(-1,1)$, $m \geq 0$, the truncation error,*

$$\|y - P_N\, y\|_{L_w^2(-1,1)} \leq C\, N^{-m}\, \|y\|_{H_w^{m,N}(-1,1)} \tag{16}$$

is satisfied.

4. Numerical Examples

In this section, three integral and differential fractional nonlinear problems are investigated by using CFDM. For each problem, the given interval ($x \in [0,1]$) is transformed to the interval $t \in [-1,1]$ by

$$x = \frac{(b-a)}{2}t + \frac{(b+a)}{2}. \tag{17}$$

Example 1:

$$\begin{cases} \frac{d^\alpha y(x)}{dx^\alpha} - 2\,e^{y(x)} = 0, \quad 1 < \alpha \leq 2, \;\; 0 \leq x \leq 1 \\ y(0) = y'(0) = 0. \end{cases} \tag{18}$$

The exact solution for $\alpha = 2$ is $y(x) = -2\ln(\cos x)$.

By taking $\alpha = 2$ and $N = 9$, $y(x)$ is obtained by using CFDM as

$$\begin{aligned} y(x) &= 4.6629 \times 10^{-15}x + 0.9999x^2 + 0.00008x^3 + 0.1645x^4 + 0.0177x^5 \\ &-0.0256x^6 + 0.1430x^7 - 0.1357x^8 + 0.0671x^9. \end{aligned} \tag{19}$$

For $N = 9$, the numerical solutions obtained by CFDM for different values of α and the exact solution ($\alpha = 2$) are given in Fig. 1.

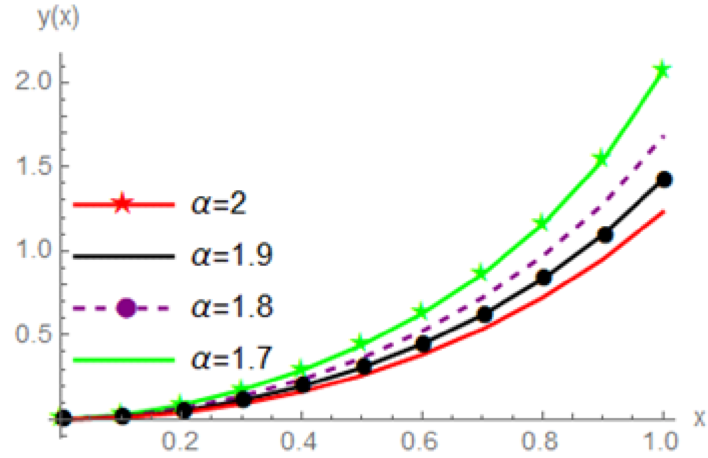


Fig. 1 The numerical solution by CFDM for different values of α and $N = 9$ for Example 1.

In Table 1, the comparison of the absolute errors of presented method ($N = 9$) with the other methods is provided.

Table 1: Comparison of CFDM ($N = 9$) with the other methods in terms of maximum absolute errors for $\alpha = 2$ for the Example 1.

x	[33]	[34]	[17] (a)	[17] (b)	Present Method
0.1	$2.98E-4$	$1.41E-5$	$1.41E-5$	$1.78E-7$	$7.53E-9$
0.2	0	$2.93E-5$	$3.22E-5$	$4.50E-7$	$1.77E-8$
0.3	$1.69E-4$	$1.79E-5$	$5.12E-5$	$7.19E-7$	$7.21E-8$
0.4	$1.10E-4$	$1.19E-4$	$7.14E-5$	$1.00E-6$	$5.35E-8$
0.5	0	$6.59E-4$	$9.28E-5$	$1.31E-6$	$2.16E-7$
0.6	0	$2.21E-3$	$1.17E-4$	$1.66E-6$	$2.66E-7$
0.7	$7.75E-5$	$6.00E-3$	$1.44E-4$	$2.06E-6$	$2.70E-7$
0.8	0	$1.43E-2$	$1.77E-4$	$2.25E-6$	$1.15E-6$
0.9	$3.47E-3$	$3.13E-2$	$2.17E-4$	$3.12E-6$	$2.21E-6$
1	0	$6.43E-2$	$2.69E-4$	$3.63E-6$	$3.9E-5$

Example 2:

$$\begin{cases} \frac{d^\alpha y(x)}{dx^\alpha} + 2\, e^{y(x)} = 0, \quad 1 < \alpha \leq 2, \;\; 0 \leq x \leq 1 \\ y(0) = y(1) = 0 \end{cases} \tag{20}$$

The analytical solution of the problem for $\alpha = 2$ is $y(x) = -2\ln\left[\frac{\cosh(0.5(x-0.5)\theta)}{\cosh(\frac{\theta}{4})}\right]$, where θ satisfies the equation $\theta = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$.

For $\alpha = 2$ and $N = 16$, the solution $y(x)$ is obtained by using CFDM as

$$\begin{aligned} y(x) &= -1.9872 \times 10^{-14} + 1.2481x - 0.9999x^2 - 0.4160x^3 + 0.0368x^4 \\ &+0.1340x^5 + 0.0441x^6 - 0.0293x^7 - 0.0244x^8 - 0.0151x^9 + 0.0429x^{10} \\ &-0.0521x^{11} + 0.0726x^{12} - 0.0711x^{13} + 0.0398x^{14} - 0.0118x^{15} + 0.0014x^{16}. \end{aligned} \tag{21}$$

The numerical solutions of Example 2 by CFDM for different values of α are given in Fig. 2.

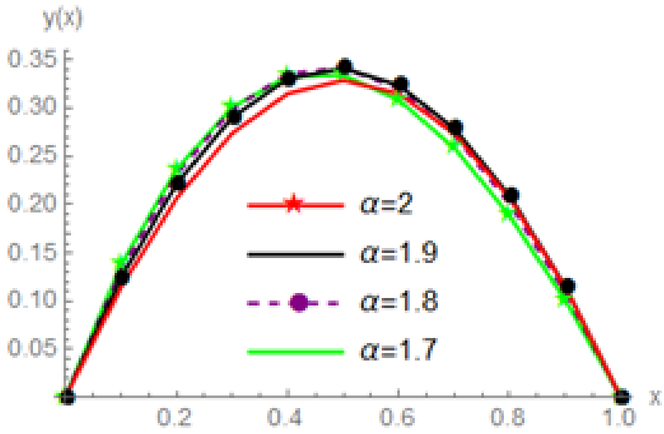


Fig. 2 The numerical solution by CFDM for $N = 9$ and different values of α for Example 2.

The comparison of the absolute errors of present with the other methods is provided in Table 2.

Table 2: Comparison of CFDM ($N = 16$ and $\alpha = 2$) with the other methods in terms of maximum absolute errors for Example 2.

x	[35]	[36]	[37]	[15]	Present Method
0.1	$2.12E-3$	$1.71E-5$	$4.03E-6$	$4.08E-13$	$1.21E-14$
0.2	$4.20E-3$	$3.25E-5$	$5.70E-6$	$8.04E-13$	$7.20E-14$
0.3	$6.18E-3$	$4.48E-5$	$5.22E-6$	$1.18E-12$	$1.62E-14$
0.4	$8.00E-3$	$5.28E-5$	$3.07E-6$	$1.52E-12$	$5.44E-14$
0.5	$9.59E-3$	$5.26E-5$	$1.45E-6$	$1.83E-12$	$2.78E-14$
0.6	$1.09E-2$	$5.28E-5$	$3.04E-6$	$2.08E-12$	$5.41E-14$
0.7	$1.19E-2$	$4.48E-5$	$5.19E-6$	$2.28E-12$	$1.60E-14$
0.8	$1.23E-2$	$3.25E-5$	$5.67E-6$	$2.42E-12$	$7.14E-14$
0.9	$1.08E-2$	$1.71E-5$	$4.01E-6$	$2.50E-12$	$1.25E-14$

Example 3:

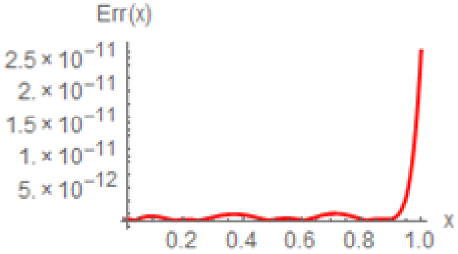
$$\begin{cases} \frac{d^{0.5}y(x)}{dx^{0.5}} + \int_0^x x\left(t-\frac{1}{2}\right)e^{y(t)}dt + g(x) = 0, \quad 0 \le x \le 1 \\ y(0) = 0 \end{cases} \tag{22}$$

where $g(x) = -\frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5} + \frac{\Gamma(2)}{\Gamma(1.5)}x^{0.5} - \frac{x}{2}\left(e^{x^2-x}-1\right)$. The exact solution is $y(x) = x^2 - x$.

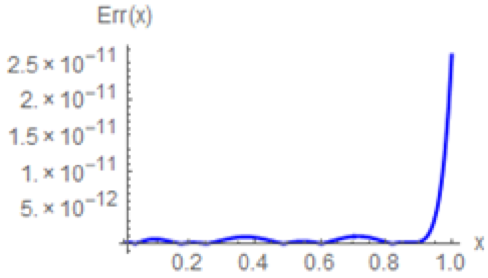
The numerical solution for $N = 9$, $y(x)$ is obtained by using CFDM as

$$\begin{aligned} y(x) = & -4.9960 \times 10^{-16} - 0.9999x + 0.9999x^2 + 1.7240 \times 10^{-8}x^3 \\ & -1.0453 \times 10^{-7}x^4 + 3.4324 \times 10^{-7}x^5 - 6.4836 \times 10^{-7}x^6 \\ & +7.0450 \times 10^{-7}x^7 - 4.0927 \times 10^{-7}x^8 + 9.8505 \times 10^{-8}x^9. \end{aligned} \tag{23}$$

The absolute error of the present method is given in Fig. 3.



(a)



(b)

Fig. 3 The absolute error $(Err(x))$ of the CFDM $(N = 9)$ in (a) and its logarithmic value on base $10(\log_{10}(Err(x)))$ in (b) for Example 3.

The comparison of absolute errors of presented method with the other methods is given in Table 3.

Table 3: The comparison of CFDM $(N = 9)$ with the other methods in terms of maximum absolute errors for Example 3.

x	[38]	[17] (a)	[17] (b)	Present Method
0	$5.91E-3$	$6.30E-3$	0	0
$\frac{1}{6}$	$8.61E-3$	$1.98E-3$	$1.56E-3$	$7.59E-14$
$\frac{2}{6}$	$9.25E-3$	$1.57E-3$	$4.03E-4$	$8.09E-13$
$\frac{3}{6}$	$1.74E-3$	$1.22E-3$	$4.97E-4$	$1.24E-13$
$\frac{4}{6}$	$3.14E-3$	$1.18E-3$	$2.46E-3$	$7.73E-13$
$\frac{5}{6}$	$3.71E-2$	$1.10E-3$	$5.73E-3$	$1.30E-13$

5. Conclusions

CFDM is used to obtain the numerical solutions of the integral and differential fractional order Bratu-Type of equations. Some theorems are given for the convergence and the error analyzes of the presented method. The effect of the fractional derivative in the outcomes has great agreement with the nonlocality of the problem. Moreover, It can be easily seen from the Fig 1 and Fig 2, the solutions of fractional order Bratu type of equations get close to the classical result when $(\alpha \to 2)$. The numerical outcomes prove that the CFDM is more accurate, less computational and highly efficient technique for solving this type of equations.

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