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SOFT BORNOLOGY

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Abstract. In this paper we present the parameterized extension of the concept of bornology. In doing so, we introduce the notions of soft bornology and an M-valued soft bornology where the parametrization plays the key role. Furthermore, we introduce boundedness and parameterized degree of boundedness for soft sets in such spaces, respectively. We observe some of the elementary properties of the proposed spaces. Finally, we looked bounded soft mappings in the corresponding spaces.

Keywords: Bornology, boundedness, soft set, soft mapping.

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1. Introduction

In 1949, S.T. Hu [7] defined the concept of bornology (or so called abstract boundedness) which is a collection of classical sets satisfying some certain axioms. The elements of a bornology are called bounded sets of the space. The collection of totally bounded sets in a uniform space and the collection of relatively compact sets in a topological space are the examples of a bornology. This invention helped researchers to define boundedness for sets in any topological spaces even if their topology are not metrizable. After Hogbe-Nlend [6] published his work, the popularity of this new kind of structure has taken attention and several authors applied this structure to their own areas such as hyper spaces, selecion principles and fuzzy set theory [1, 4, 15, 16].

In 1999, Molodtsov [8] introduced the theory of soft sets in order to emphasizes the key role of the parametrization tool in the modeling of the real life problems. Since these kinds of modelings are based on uncertain data. This point of view impressed many researchers working in diverse fields of mathematics and they applied this set theory to their own branches [2, 3, 13, 14, 17], effectively.

In any soft metric space, one can easily describe boundedness for soft sets, but the term "boundedness" makes no sense for soft sets in a soft topological space, since there is no distance between points of soft topological spaces. Since the bounded soft sets take important place for some applications as in the classical cases, we aimed the term "boundedness" is meaningful for soft sets by defining soft bornology. For this aim, we consider Hu's bornology [7] and Sostak and Uljane's M-valued bornology [15] in the soft

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setting, respectively. First, we introduce bounded soft sets by extending the Hu's definition to the soft setting and later, we introduce M-valued (graded) boundedness for soft sets by extending Sostak and Uljane's definition to the soft setting. Our study is arranged in the following manner: In Section 2, we summarize the basic notions and notations of soft sets and bornological structures which are used in the next section. In Section 3, we define soft bornology by giving several examples and proposed basic structural properties. Then we define M-valued soft bornolgy and examine some of the structural characteristics.

In what follows, X refers to a nonempty initial universe and E denotes an arbitrary nonempty set viewed on the sets of parameters. By M we denote the complete completely distributive lattice $M = (M, \leq, \wedge, \vee)$ whose the bottom and the top elements are denoted by 0_M and 1_M, respectively. For a complete lattice M and $\alpha, \beta \in M$, the wedge-below relation $\lhd$ is defined on M as follows: $\beta \lhd \alpha \Leftrightarrow$ if $K \subseteq M$ and $\alpha \leq \bigvee K$ then $\exists \gamma \in K, \beta \leq \gamma$. For more details about the lattices, see [5, 12].

An element α in M is said to be coprime if $\alpha \leq \beta \vee \gamma$ implies that $\alpha \leq \beta$ or $\alpha \leq \gamma$. The set of all nonzero coprime elements of M is denoted by $c(M)$.

2. Preliminaries

In this section, we recall the main definitions and results to make the paper clear to understand for the readers. For this reason, we divide this section into two subsections; soft sets and bornological structures.

2.1. Soft sets.

In this subsection, we recall the notion of a soft set which is a parameterized collection of crisp sets and we remind operations on the soft sets.

Definition 2.1. [8] *A pair (F, E) is called a soft set over X if F is a mapping from E into the set of all subsets of X; i.e., $F : E \to 2^X$. In what follows a soft set (F, E) over X is denoted by a triple (F, E, X). Sometimes the mapping $F : E \to 2^X$ is referred to a soft structure over the pair (E, X).*

Definition 2.2. [11] *Let (F_1, E, X) and (F_2, E, X) be two soft structures such that for any $e \in E, F_1(e) \subseteq F_2(e)$. Then F_1 is said to be coarser than F_2 and denoted by $F_1 \preceq F_2$.*

Definition 2.3. [2]Let (F, E, X) and (G, E, X) be two soft sets, then

1. (F, E, X) is a subset of (G, E, X) denoted by $(F, E, X) \sqsubseteq (G, E, X)$ if $F(e) \subseteq G(e)$, for each $e \in E$. In this case $(F, E, X) = (G, E, X)$ iff (F, E, X) is subset of (G, E, X) and vice-versa.
2. the union $(K, E, X) = (F, E, X) \sqcup (G, E, X)$ is defined by $K(e) = F(e) \cup G(e)$ for all $e \in E$.
3. the intersection $(H, E, X) = (F, E, X) \sqcap (G, E, X)$ is defined by $H(e) = F(e) \cap G(e)$ for all $e \in E$.
4. the complement of the soft set (F, E, X) is denoted by $(F, E, X)^c = (F^c, E, X)$, where $F^c : E \to 2^X$ is a mapping given by $F^c(e) = X \setminus F(e)$ for all $e \in E$.
5. (F, E, X) is said to be the null soft set and denoted by Φ iff $F(e) = \emptyset$ for each $e \in E$.
6. (F, E, X) is said to be the universal soft set and denoted by $\widetilde{X}$ iff $F(e) = X$ for each $e \in E$.

Theorem 2.1. [10] Let $S(X, E)$ denotes the family of all soft structures over (E, X) which equipped with the partial order $\preceq$. Then $(S(X, E), \preceq)$ is a complete lattice.

Definition 2.4. [10]Let $\varphi : X \to Y$ and $\psi : E_1 \to E_2$ be two crisp functions. Then the pair (φ, ψ) is called a soft mapping from (F, E_1, X) to (G, E_2, Y) and denoted by

$(\varphi, \psi) : (F, E_1, X) \to (G, E_2, Y)$, whenever $\varphi^{\to} \circ F \geq G \circ \psi$. Here, $\varphi^{\to}$ denotes the Zadeh image operator from 2^X to 2^Y.

(1) The image of (F, E_1, X) under the soft mapping (φ, ψ) is denoted by
$(\varphi, \psi)(F, E_1, X) = (\varphi(F), \psi(E_1), Y)$,
where $\psi(E_1)$ is the image of E_1 and $\varphi(F)$ is defined by follows: for each $e_2 \in E_2$,

$$\varphi(F)(e_2) = \bigcup_{e_2 = \psi(e_1)} \varphi(F(e_1)).$$

(2) The pre-image of (G, E_2, Y) under the soft mapping (φ, ψ) is denoted by
$(\varphi, \psi)^{-1}(G, E_2, Y) = (\varphi^{-1}(G), \psi^{-1}(E_2), X)$,
where $\psi^{-1}(E_2)$ is the inverse image of E_2 and $\varphi^{-1}(G)$ is defined by follows: for each $e_1 \in E_1$.

$$\varphi^{-1}(G)(e_1) = (\varphi^{\leftarrow} \circ G \circ \psi)(e_1) = \varphi^{-1}(G(\psi(e_1))).$$

If the crisp functions φ, ψ are both injective (resp., surjective), then the soft mapping (φ, ψ) is said to be injective (resp., surjective).

Let (φ_1, ψ_1) and (φ_2, ψ_2) be two soft mappings where, $\varphi_1 : X \to Y, \varphi_2 : Y \to Z$ and $\psi_1 : E_1 \to E_2, \psi_2 : E_2 \to E_3$ are crisp functions. Then their composition is defined as follows, $(\varphi_2, \psi_2) \circ (\varphi_1, \psi_1) = (\varphi_2 \circ \varphi_1, \psi_2 \circ \psi_1)$.

2.2. **Bornological structures.** In this subsection, we recall the concept of bornology in the framework of crisp and fuzzy mathematical structures. We remind the definitions of a bornology and an M-valued bornology, respectively.

In order to apply the concept of boundedness which is crucial in the theory of metric spaces, to the case of a general topological space S.-T. Hu introduced the notion of a bornology as follows.

Definition 2.5. [7] Let X be a nonempty classical set. Then a bornology $\mathcal{B}$ on X is a family of subsets of X (i.e., $\mathcal{B} \subseteq 2^X$) which satisfies the following axioms:

(1B) $\{x\} \in \mathcal{B}$, for all $x \in X$.

(2B) If $U \subseteq V$ and $V \in \mathcal{B}$, then $U \in \mathcal{B}$.

(3B) If $U, V \in \mathcal{B}$, then $U \cup V \in \mathcal{B}$.

The pair $(X, \mathcal{B})$ is called a bornological space and the sets belonging to $\mathcal{B}$ are viewed as bounded in this space.

Given bornological spaces $(X_1, \mathcal{B}^1)$ and $(X_2, \mathcal{B}^2)$, a function $\varphi : X_1 \to X_2$ is called bounded if $\varphi(U) \in \mathcal{B}^2$, for every $U \in \mathcal{B}^1$.

Important examples of bornological spaces $(X, \mathcal{B})$ are:

(1) a metric space and the family of its bounded subsets.

(2) a topological space and the family of its relatively compact subsets.

(3) a uniform space and the family of its totally bounded subsets.

Remark 2.1. [9] (1) Given a set X, the largest topology and bornology on X (both w.r.t the natural set inclusion order) is given by the powerset 2^X. The smallest topology and bornology on X are quite different. The former is given by a 2-element set $\{X, \emptyset\}$, and the later is given by the family of all finite subsets of X.

(2) A topology on a set X is a subframe of the powerset 2^X, and a bornology on X is a lattice ideal of 2^X which additionally satisfies the condition (1B) of Definition 2.9.

By considering the fuzzy analogue of a classical bornological structure in case when the structure itself is fuzzy but the sets are ordinary, Sostak and Uljane defined the concept of an M-valued bornology. This crisp-fuzzy approach of bornology is described by the following manner.

Definition 2.6. [15] An M-valued bornology on a set X is a mapping $\mathcal{B} : 2^X \to M$ which satisfies the following conditions:
(MB1) $\mathcal{B}(\{x\}) = 1_M$, for all $x \in X$.
(MB2) If $U \subseteq V$, then $\mathcal{B}(V) \leq \mathcal{B}(U)$.
(MB3) $\mathcal{B}(U \cup V) \geq \mathcal{B}(U) \wedge \mathcal{B}(V)$ for all $U, V \in 2^X$.
The pair $(X, \mathcal{B})$ is called an M-valued bornological space and the value $\mathcal{B}(U)$ is interpreted as the degree of boundedness of the set U in the space $(X, \mathcal{B})$.

Given two M-valued bornological spaces $(X_1, \mathcal{B}^1)$ and $(X_2, \mathcal{B}^2)$, a function $\varphi : X_1 \to X_2$ is called bounded if $\mathcal{B}^1(U) \leq \mathcal{B}^2(\varphi(U))$ for each $U \in 2^X$.

3. Boundedness for Soft Sets

In this section, we intend to describe the concept of a bornology in the framework of the soft mathematical structures as parameterized extensions of the bornologies given in Subsection 2.2, respectively. By this way, we hope to shed light on the description of a bounded soft set and the parameterized boundedness degree of a soft set. In addition, we demonstrate some properties and discuss some relations of the presented concepts.

3.1. **Soft bornological spaces.**

In this subsection, we present the concept of a soft bornology which is the parameterized extension of the classical bornology. Hence this gives rise to us describing the boundedness for a soft set in such spaces.

Definition 3.1. A mapping $\mathcal{B} : E \to 2^{2^X}$ is called a soft bornology on a set X with respect to the parameter set E, if for each $e \in E$, $\mathcal{B}(e) := \mathcal{B}_e \in 2^{2^X}$ is a classical bornology on X. Hence, a soft bornology may be thought as a parameterized family of classical bornologies defined by Hu.

The soft bornology is denoted by $\mathcal{B}(X, E)$ and the triple $(X, \mathcal{B}, E)$ is called a soft bornological space.

Remark 3.1. *A mapping $\mathcal{B} : E \to 2^{2^X}$ is a soft bornology on a set X if and only if $\mathcal{B}(e) : 2^X \to 2$ is a classical bornology on X, for any $e \in E$.*

A soft bornology $\mathcal{B}(X, E)$ is said to be coarser than a soft bornology $\mathcal{B}^*(X, E)$ if for all $e \in E$, $\mathcal{B}(e) \geq \mathcal{B}^*(e)$.

Example 3.1. (1) Let $E = \{*\}$ and let $(\mathcal{B}, \{*\})$ be a soft set on 2^X; i.e., $\mathcal{B} : \{*\} \to 2^{2^X}$. If in this case, $\mathcal{B}(*)$ is a crisp bornology on X, then $\mathcal{B}$ is a soft bornology on X with respect to $\{*\}$.

(2) Let $E = \{0, 1\}$ and let $(\mathcal{B}, \{0, 1\})$ be a soft set on 2^X; i.e., $\mathcal{B} : \{0, 1\} \to 2^{2^X}$. If in this case, $\mathcal{B}(0)$ and $\mathcal{B}(1)$ are crisp bornologies on X, then $\mathcal{B}$ is a soft bornology on X with respect to $\{0, 1\}$.

(3) Let $E = [0, 1] = I$ and let $(\mathcal{B}, I)$ be a soft set on 2^X; i.e., $\mathcal{B} : I \to 2^{2^X}$. If $\mathcal{B}(\alpha)$ is a crisp bornology on X for all $\alpha \in I$, then $\mathcal{B}$ is a soft bornology on X with respect to I.

Example 3.2. Let $E = \{e_1, e_2, e_3\}$ be the parameter set and the mapping $\mathcal{B} : E \to \mathcal{P}(\mathbb{R})$ be defined by follows: $\mathcal{B}(e_1) = \{U \subseteq \mathbb{R} \mid U \text{ is finite}\}$, $\mathcal{B}(e_2) = \{U \subseteq \mathbb{R} \mid U \subseteq (-r, r),\ \exists r > 0\}$ and $\mathcal{B}(e_3) = \mathcal{P}(\mathbb{R})$. Since $\mathcal{B}(e)$ is a bornology for each $e \in E$, then the mapping $\mathcal{B}$ is a soft bornology on $\mathbb{R}$ with respect to E.

Remark 3.2. *It is noted that a soft bornology on a set X is actually a parameterized family of ideals on X.*

Definition 3.2. Let $(X, \mathcal{B}, E)$ be a soft bornological space and let (F, E, X) be a soft set. Then (F, E, X) is called a bounded soft set if $F(e) \in \mathcal{B}(e)$, for all $e \in E$.

Remark 3.3. *Notice that each soft point is a bounded soft set. Also it is obviously seen that being a bounded soft set is hereditary and closed under finite unions.*

Example 3.3. Let $E = \{e_1, e_2, e_3\}$ be the parameter set and the soft sets F, G be defined as follows: $F(e_1) = \{0, 1\}, F(e_2) = [0, 1], F(e_3) = \mathbb{Z}$ and $G(e_1) = \mathbb{N}, G(e_2) = (2, 3), G(e_3) = \emptyset$. Then F is a bounded soft set in the soft bornological space described in Example 3.2, but G is not bounded in the same space. If one defines $\mathcal{B}(e_1) = \{U \subseteq \mathbb{R} \mid U \text{ is countable}\}$, then in this case G will be soft bounded, too.

Definition 3.3. Let $(X, \mathcal{B}, E)$ be a soft bornological space and $(\mathcal{D}, E)$ be a soft set on 2^X. If for each $e \in E$, a subcolloection $\mathcal{D}(e)$ of $\mathcal{B}(e)$ is a base for $\mathcal{B}(e)$; i.e., each elements of $\mathcal{B}(e)$ is a subset of an element of $\mathcal{D}(e)$, then $\mathcal{D}(X, E)$ is called a soft bornology base of $\mathcal{B}(X, E)$.

Example 3.4. Let $E = \{e_1, e_2, e_3\}$ be the parameter set and let us consider the soft bornology $\mathcal{B}$ which is described in Example 3.2. Then the mapping $\mathcal{D} : E \to \mathcal{P}(\mathbb{R})$ defined by follows: $\mathcal{D}(e_1) = \{U \subseteq \mathbb{R} \mid U \text{ is countable}\}$, $\mathcal{D}(e_2) = \{U \subseteq \mathbb{R} \mid U \subseteq [a, b], a, b \in \mathbb{R}\}$ and $\mathcal{D}(e_3) = \mathcal{P}(\mathbb{R})$, is a soft bornology base of $\mathcal{B}$.

Theorem 3.1. *Let $(\mathcal{D}, E)$ be a soft set on 2^X which satisfies he following conditions for all $e \in E$,*

(1) $\forall x \in X, \{x\} \in \mathcal{D}(e)$ *(or equivalently,* $X = \bigcup_{D \in \mathcal{D}(e)} D$*)*

(2) *If* $U \in \mathcal{D}(e)$ *and* $V \subseteq U$*, then* $V \in \mathcal{D}(e)$*.*

Then the mapping $D : E \to 2^{2^X}$ generates a soft bornology $\langle \mathcal{D} \rangle$ on X, which has the base $\mathcal{D}(X, E)$.

Proof. Since a soft bornology is a parameterized family of the classical bornologies, the claim is clear to verify. □

Definition 3.4. Let $(X, \mathcal{B}, E)$ be a soft bornological space, $E_0 \subseteq E$ and $X_0 \subseteq X$. Observe the mapping $\mathcal{B}_0 : E_0 \to 2^{2^{X_0}}$ which is defined by $\mathcal{B}_0(e) = \{X_0 \cap \mathcal{B}(e) \mid \mathcal{B}(e) \subseteq 2^X\}$ for each $e \in E_0$, is a classical bornology on X_0. Then $\mathcal{B}_0(X_0, E_0)$ is called the relative soft bornology.

Definition 3.5. Let $(X, \mathcal{B}^1, E_1)$ and $(Y, \mathcal{B}^2, E_2)$ be two soft bornological spaces. Then the soft mapping $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ is called soft bounded if the function $\varphi : (X, \mathcal{B}^1(e)) \to (Y, \mathcal{B}^2(\psi(e)))$ is bounded between classical bornological spaces, for all $e \in E$. In other words,

$(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ is soft bounded if $\varphi^{\Rightarrow}(\mathcal{B}^1(e)) \leq \mathcal{B}^2(\psi(e))$ for each $e \in E$, where $\varphi^{\Rightarrow} := (\varphi^{\to})^{\to}$ is the Zadeh image operator.

Proposition 3.1. *Let $(X, \mathcal{B}^1, E_1), (Y, \mathcal{B}^2, E_2)$ and $(Z, \mathcal{B}^3, E_3)$ be the soft bornological spaces. If $(\varphi_1, \psi_1) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ and $(\varphi_2, \psi_2) : (Y, \mathcal{B}^2, E_2) \to (Z, \mathcal{B}^3, E_3)$ are soft bounded functions, then their composition $(\varphi_2, \psi_2) \circ (\varphi_1, \psi_1) : (X, \mathcal{B}^1, E_1) \to (Z, \mathcal{B}^3, E_3)$ is soft bounded, too.*

Proof. Since the soft mapping (φ_1, ψ_1) is soft bounded, then for each $e_1 \in E_1$, $\varphi_1 : (X, \mathcal{B}^1(e_1)) \to (Y, \mathcal{B}^2(\psi_1(e_1)))$ is bounded. And since the soft mapping (φ_2, ψ_2) is soft bounded, then for each $e_2 \in E_2$, $\varphi_2 : (Y, \mathcal{B}^2(e_2)) \to (Z, \mathcal{B}^3(\psi_2(e_2)))$ is bounded. So, the composition $\varphi_2 \circ \varphi_1 : (X, \mathcal{B}^1(e_1)) \to (Z, \mathcal{B}^3(\psi_2(\psi_1(e_1))))$ is bounded for each $e_1 \in E_1$. Hence the composition of soft mappings $(\varphi_2, \psi_2) \circ (\varphi_1, \psi_1)$ is soft bounded. □

Besides since the identity function $id_X : (X, \mathcal{B}(e)) \to (X, \mathcal{B}(e))$ is bounded for each $e \in E$, then by Definition 3.5, we conclude that the identity soft mapping $(id_X, id_E) : (X, \mathcal{B}, E) \to (X, \mathcal{B}, E)$ is soft bounded. As a result, soft bornological spaces and soft bounded mapings between them form a category which will be denoted by $SBOR$ and called the category of soft bornological spaces.

Definition 3.6. Let $(X, \mathcal{B}^1, E_1)$ and $(Y, \mathcal{B}^2, E_2)$ be two soft bornological spaces. Consider the triple $(X \times Y, \langle \mathcal{B}^1 \times \mathcal{B}^2 \rangle, E_1 \times E_2)$ where the mapping $\mathcal{B}^1 \times \mathcal{B}^2 : E_1 \times E_2 \to 2^{2^{X \times Y}}$ is defined as follows:

$$(\mathcal{B}^1 \times \mathcal{B}^2)(e_1, e_2) =: \mathcal{B}^1(e_1) \times \mathcal{B}^2(e_2) \in 2^{2^X} \times 2^{2^Y} \subseteq 2^{2^{X \times Y}}$$

where $\mathcal{B}^1(e_1)$ and $\mathcal{B}^2(e_2)$ are classical bornologies on X and Y, respectively.
The pairs of projections $p_1 : E_1 \times E_2 \to E_1, q_1^{\Rightarrow} : 2^{2^{X \times Y}} \to 2^{2^X}$ and $p_2 : E_1 \times E_2 \to E_2, q_2^{\Rightarrow} : 2^{2^{X \times Y}} \to 2^{2^Y}$ determine morphisms:
$(p_1, q_1^{\Rightarrow}) : (X \times Y, \mathcal{B}^1 \times \mathcal{B}^2, E_1 \times E_2) \to (X, \mathcal{B}^1, E_1)$
and
$(p_2, q_2^{\Rightarrow}) : (X \times Y, \mathcal{B}^1 \times \mathcal{B}^2, E_1 \times E_2) \to (Y, \mathcal{B}^2, E_2)$.

Definition 3.7. Let $\{(X_i, \mathcal{B}^i, E_i)\}_{i \in \Gamma}$ be a family of soft bornological spaces. Then the initial soft bornology on $X = \Pi X_i$ generated by the family of $\{(p_i, q_i^{\Rightarrow})\}_{i \in \Gamma}$ is called the product soft bornology on X.

According to this definition, products exist in the category of soft bornological spaces $SBOR$.

Definition 3.8. *Let $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ be a soft mapping between soft bornological spaces. If (φ, ψ) is bijective, soft bounded and its inverse $(\varphi, \psi)^{-1}$ is also soft bounded, then it is called bornological isomorphism. In this case, the soft bornological spaces $(X, \mathcal{B}^1, E_1)$ and $(Y, \mathcal{B}^2, E_2)$ are called bornological isomorphic spaces.*

3.2. **M-valued soft bornological spaces.** In this subsection, we present the parameterized extension of the M-valued bornology which we call M-valued soft bornology. Besides, we provide some of its basic properties.

Definition 3.9. A mapping $\mathcal{B} : E \to M^{2^X}$ (where $\mathcal{B}(e) := \mathcal{B}_e \in M^{2^X}$, for all $e \in E$) is said to be an M-valued soft bornology on a set X with respect to the parameters of E, if the mappings $\mathcal{B}(e) : 2^X \to M$ are M-valued bornologies on X, for all $e \in E$. Hence it may be thought that each M-valued soft bornology is a parameterized family of M-valued bornologies.

An M-valued soft bornology on X with respect to the parameters of E, is denoted by $\mathcal{B}(X, E)$ and the triple $(X, \mathcal{B}, E)$ is called the M-valued soft bornological space.

Definition 3.10. Let $(X, \mathcal{B}, E)$ be an M-valued soft bornological space and (F, E, X) be a soft set. Then the value $\mathcal{B}_e(F(e))$ is interpreted as the degree of boundedness of the soft set (F, E, X) with respect to the parameter e.

Definition 3.11. Let $(X, \mathcal{B}^1, E_1)$ and $(Y, \mathcal{B}^2, E_2)$ be two M-valued soft bornological spaces and $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ be a fuzzifying soft mapping. Then (φ, ψ) is called M-valued soft bounded if $\varphi_e : (X, \mathcal{B}^1(e)) \to (Y, \mathcal{B}^2(\psi(e)))$ is bounded for all $e \in E_1$. This means that $\mathcal{B}^1_e(A) \leq \mathcal{B}^2_{\psi(e)}(\varphi(A))$, for all $A \in 2^X$ and for all $e \in E_1$.

Proposition 3.2. *Composition of two bounded soft mappings between M-valued soft bornological spaces is also soft bounded. And the identity soft mapping of M-valued soft bornological spaces is bounded, too.*

Proof. It is straightforward and therefore omitted. □

As a result of the above proposition M-valued soft bornological spaces and bounded maps between them form a category which will be denoted $SBOR(M)$ and called the category of M-valued soft bornological spaces. In the case when $M = 2$ is a two-pointed lattice, then the concept of a 2-valued soft bornological spaces is equivalent to the category of $SBOR$.

Proposition 3.3. *Let $\mathfrak{B}(X, M)$ be a family of all M-valued soft bornologies on the set X. Define an order " $\preceq$ " on this family as follows.*

$\mathcal{B}^1 \preceq \mathcal{B}^2 :\Leftrightarrow \mathcal{B}^1_e(A) \geq \mathcal{B}^2_e(A)$, *for each $e \in E, A \in 2^X$.*

Then $(\mathfrak{B}(X, M), \preceq)$ is a complete lattice.

Proof. Let the mappings $\mathcal{B}^\perp : E \to M^{2^X}$ and $\mathcal{B}^\top : E \to M^{2^X}$ be defined as follows, respectively. For each $e \in E$, $\mathcal{B}^\perp_e(A) = 1_M$, and $\mathcal{B}^\top_e(A) = 1_M$, if $|A| < \aleph_0$ and $= 0_M$, otherwise. Then $\mathcal{B}^\perp$ is the bottom element and $\mathcal{B}^\top$ is the top element of the $\mathfrak{B}(X, M)$. Given a family $\{\mathcal{B}^i : E \to M^{2^X}\}_{i \in I}$ of M-valued soft bornologies, we define its join as a mapping $\mathcal{B}^* : E \to M^{2^X}$ where $\mathcal{B}^*_e(A) = \bigwedge_{i \in I} \mathcal{B}^i_e(A)$ for each $e \in E$, where $\wedge$ is the infimum in the lattice M. The existence of the meets can be obtained similarly. □

Proposition 3.4. *Let $\mathcal{B} : E \to M^{2^X}$ be an M-valued soft bornology on X, then the mapping $\mathcal{B}^\beta : E \to 2^{2^X}$ which is defined by $\mathcal{B}^\beta_e = \{A \in 2^X \mid \mathcal{B}_e(A) \geq \beta\}$, is a soft bornology on X for all $\beta \in M$.*

Proof. Since $\mathcal{B} : E \to M^{2^X}$ is an M-valued soft bornology on X, then each $\mathcal{B}_e$ is an M-valued bornology on X. Hence this implies that $\mathcal{B}^\beta_e$ is a crisp bornology on X for each $e \in E$ and for each $\beta \in M$. As a result, this fact witnesses that the mapping $\mathcal{B}^\beta$ is a soft bornology on X for each $\beta \in M$. □

Further, since M is a completely distributive lattice, then the collection $\{\mathcal{B}^\beta_e \mid \beta \in M\}$ for a fixed parameter e is lower semi-continuous in the following sense

$\mathcal{B}^\beta_e = \bigcap\{\mathcal{B}^\gamma_e \mid \gamma \lhd \beta,\ \gamma \in M\}$ for any $\beta \in M$,

in particular $\mathcal{B}^{0_M}_e = 2^X$ as the intersection of the empty family.

Hence an M-valued soft bornology $\mathcal{B} : E \to M^{2^X}$ can be characterized by its lower semi-continuous decomposition into level soft bornologies:

$\mathcal{B}_e(A) = \sup\{\beta \in M \mid A \in \mathcal{B}^\beta_e\}$, for each $e \in E$.

This construction of restoration of an M-valued soft bornology from its level soft bornologies may also be generalized to the construction of an M-valued soft bornology from a given family of soft bornologies satisfies some conditions. We will consider the similar observation for the LM-valued soft bornologies in detail.

Theorem 3.2. *Each source $\{(\varphi, \psi)_i : (X, E) \to (Y_i, \mathcal{B}^i, E_i) \mid i \in \Gamma\}$ has a unique initial lift $\{(\varphi, \psi)_i : (X, \mathcal{B}, E) \to (Y_i, \mathcal{B}^i, E_i) \mid i \in \Gamma\}$ in the category $SBOR(M)$ of M-valued soft bornological spaces.*

Proof. Let $(\varphi, \psi) : (X, E) \to (Y, \mathcal{B}^2, E_2)$ be a soft mapping. Then the mapping $\mathcal{B} : E \to M^{2^X}$ defined by $\mathcal{B}_e(A) = \mathcal{B}^2_{\psi(e)}(\varphi(A))$ for each $A \in 2^X, e \in E$ is an M-valued soft bornology on X. The axioms are easily shown. Also, it is clear that $(\varphi, \psi) : (X, \mathcal{B}, E) \to (Y, \mathcal{B}^2, E_2)$ is bounded. In addition, the uniqueness is also be shown. Hence the source $\{(\varphi, \psi) : (X, E) \to (Y, \mathcal{B}^2, E_2)\}$ has a unique initial lift $(\varphi, \psi) : (X, \mathcal{B}, E) \to (Y, \mathcal{B}^2, E_2)$. Now let us consider the general case $\{(\varphi, \psi)_i : (X, E) \to (Y_i, \mathcal{B}^i, E_i) \mid i \in \Gamma\}$ and define the mapping $\mathcal{B} : E \to M^{2^X}$ by $\mathcal{B}_e(A) = \bigwedge_{i \in \Gamma} \mathcal{B}^i_{\psi_i(e)}(\varphi_i(A))$ for all $e \in E,\ A \in 2^X$. Then $\mathcal{B}$ is an M-valued bornology on X which makes all $(\varphi, \psi)_i : (X, \mathcal{B}, E) \to (Y_i, \mathcal{B}^i, E_i)$ bounded. □

Corollary 3.1. *Products exist in the category $SBOR(M)$. If the answer of the Problem 2 is positive, then one may conclude that co-products exist in the category $SBOR(M)$. Besides this gives rise to be topological of $SBOR(M)$.*

4. Conclusions

General bornological spaces play a key role in recent research of convergence structures on hyperspaces, in optimization theory and in the study of topologies on function spaces. Also to describe boundedness for sets in a space without having any distance function takes attention in several meaning. So, in order to make contribution to this field, we aimed to investigate boundedness for soft sets, and our idea gave birth to the parameterized extension of the bornology. We hope that this study will be helpful for researchers working on soft topology, soft metric and soft structures. For future work, we intend to generalize brnological structures to the fuzzy soft setting and also plan to observe relations between fuzzy soft boundedness and fuzzy soft compactnes.

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