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INITIAL TAYLOR-MACLAURIN COEFFICIENT BOUNDS AND THE FEKETE-SZEGÖ PROBLEM FOR SUBCLASSES OF m-FOLD SYMMETRIC ANALYTIC BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, we introduce two new subclasses of the m-fold symmetric, analytic and bi-univalent function class Σ_m defined in the open unit disk $\mathcal{D}_1 := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. These two subclasses are denoted by $\mathbf{S}_{\Sigma_m}(\alpha)$ and $\mathbf{S}^*_{\Sigma_m}(\beta)$. For the functions f belong to both of these subclasses, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$. Also, we obtain estimate on the Fekete-Szegö functional $|a_{2m+1} - ka^2_{m+1}|$, $k \in \mathbb{R}$. It is interesting to see that the geometrical similarities in these two subclasses also reflects in their coefficient estimates. Further, we pointed out interconnection of these results with some of the earlier known results.

Keywords: Analytic function, univalent function, bi-univalent function, coefficient bound, m-fold symmetric function, Fekete-Szegö functional.

AMS Subject Classification: 30C45, 30C50.

1. Introduction and Preliminaries

Univalent function theory (UFT) is one of the fascinating branch of the geometric function theory (GFT) in complex analysis. In around last 100 years, due to the famous Bieberbach conjecture (1916), researchers in this field have been accelerated the study of an interrelationship between geometric and analytical properties of analytic univalent functions, meromorphic univalent functions, m-fold symmetric univalent functions, multivalent functions, etc.

Further, in 1967, Lewin [12] extended this theory to bi-univalent functions and the research of this field has been accelerated extremities due to the groundbreaking research paper of Srivastava et al. [26], that revived the concept of bi-univalent functions. By

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motivation from it, many researchers have obtained estimates on initial coefficients for functions in the various subclasses of bi-univalent functions. For example, see [1, 4, 5, 8, 10, 14, 16, 17, 18, 21, 23, 31], etc. and some of the references used in them. Also, using Faber polynomial, many authors found the estimate on a_n by fixing n (e.g. see [13] and some references in it). But still there is a lot of scope to study the analytic bi-univalent functions by involving various polynomial functions and derivative or integral operators.

Let

$$\mathcal{A} = \{f : \mathcal{D}_1 \to \mathbb{C} : f \text{ is analytic in } \mathcal{D}_1,\ f(0) = 0 \text{ and } f^{'}(0) = 1\}$$

be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions f univalent in $\mathcal{D}_1$. In light of the Koebe one quarter theorem (see [7]), every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, (z \in \mathcal{D}_1)$$

and

$$f(f^{-1}(w)) = w,\ (|w| < r_0(f), r_0(f) \geq 1/4).$$

In fact, the analytic extension of f^{-1} to $\mathcal{D}_1$ is

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

Let

$$\Sigma = \{f \in \mathcal{A} : \text{both } f \text{ and } f^{-1} \text{ are univalent in } \mathcal{D}_1\}$$

denote the class of analytic bi-univalent functions in $\mathcal{D}_1$.

Lewin [12] proved that $|a_2| < 1.51$ for $f \in \Sigma$, after which, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ and at one stage Goodman [9] claimed that $|a_n| \leq 1$ may be true for every $n \in \mathbb{N}$ and $f \in \Sigma$. However, afterwards Netanyahu [15] proved that $max|a_2| = \frac{4}{3}$, Styer and Wright [28] showed that there exist functions in Σ for which $|a_2| > \frac{4}{3}$ and Tan [29] proved that $|a_2| \leq 1.485$ for $f \in \Sigma$.

A function which has the form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathcal{D}_1;\ m \in \mathbb{N} \equiv \{1, 2, 3, \cdots\}) \tag{3}$$

is said to be m-fold symmetric (see [11, 20]). Each function $h(z)$ defined as:

$$[h(z)]^m = f(z^m) \quad \text{or} \quad h(z) = [f(z^m)]^{\frac{1}{m}} \qquad (f \in \mathcal{S};\ z \in \mathcal{D}_1;\ m \in \mathbb{N})$$

is univalent and maps the unit disk $\mathcal{D}_1$ into a m-fold symmetric region.

Let $\mathcal{S}_m$ denote the class of all m-fold symmetric analytic univalent functions in $\mathcal{D}_1$, which are represented by the series expansion (3). Moreover for $m = 1$, these functions reduces to members of class $\mathcal{S} \equiv \mathcal{S}_1$ and are said to be *1*-fold symmetric analytic univalent functions.

For each $m \in \mathbb{N}$, every bi-univalent function generates an m-fold symmetric analytic bi-univalent function. Srivastava et al. [27] proved that, for the function f as given in (3), the extension of the inverse function f^{-1} to $\mathcal{D}_1$ is given by:

$$\begin{aligned} g(w) = w &- a_{m+1} w^{m+1} + \left[(m+1)\, a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &- \left[\frac{1}{2}(m+1)(3m+2)\, a_{m+1}^3 - (3m+2)\, a_{m+1} a_{2m+1} + a_{3m+1}\right] w^{3m+1} \\ &+ \cdots . \end{aligned} \tag{4}$$

Observe that for $m = 1$, this equation (4) reduces to the equation (2). Hence the bi-univalent function class Σ can be generalized to the m-fold symmetric analytic bi-univalent function class Σ_m. For examples of the m-fold symmetric analytic bi-univalent functions and their corresponding inverse functions, see the work of Srivastava et al. [27]. Also see [2, 6, 13, 22, 24, 25, 30] etc. for coefficient problems of some new subclasses of Σ_m.

In order to derive our main results, we need the following lemma [19].

Lemma 1.1. [19] *If $\gamma \in \mathcal{P}$, the class of Carathéodary functions which are analytic in $\mathcal{D}_1$ with $\Re(\gamma(z)) > 0$ for all $z \in \mathcal{D}_1$ have the form*

$$\gamma(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (z \in \mathcal{D}_1);$$

then $|c_n| \leq 2$ for each $n \in \mathbb{N}$.

We use the m-fold symmetric function $\gamma \in \mathcal{P}$ (see [20]) of the form

$$\gamma(z) = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots, \quad (z \in \mathcal{D}_1).$$

In the present paper, we obtain estimates on the initial Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ and also on the Fekete-Szegö functional $|a_{2m+1} - k a_{m+1}^2|$, $(k \in \mathbb{R})$ for functions belong to the new subclasses $\mathbf{S}_{\Sigma_m}(\alpha)$ and $\mathbf{S}^*_{\Sigma_m}(\beta)$ of the class Σ_m. Also, we have mentioned the connections with some of the earlier known subclasses of the class Σ.

2. Main Results

Definition 2.1. *A function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathbf{S}_{\Sigma_m}(\alpha)$ if the following conditions are satisfied:*

$$\left| arg\left(\frac{z^3 f''(z)}{(f(z))^2} + 1 \right) \right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{D}_1)$$

and

$$\left| arg\left(\frac{w^3 g''(w)}{(g(w))^2} + 1 \right) \right| < \frac{\alpha\pi}{2}, \qquad (w \in \mathcal{D}_1);$$

where $0 < \alpha \leq 1$ and the function g is given by (4).

In particular, observe the following examples:

1. For the identity function $f(z) = z$ and its inverse function $g(w) = w$ defined in the unit disk $\mathcal{D}_1$, we have
$$\left| arg\left(\frac{z^3 f''(z)}{(f(z))^2} + 1 \right) \right| = \left| arg\left(\frac{w^3 g''(w)}{(g(w))^2} + 1 \right) \right| = 0 < \frac{\alpha\pi}{2}, \quad (0 < \alpha \leq 1).$$
2. For the *1*-fold symmetric analytic bi-univalent function $f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots$ and its inverse function $g(w) = \frac{w}{1+w} = w - w^2 + w^3 - \cdots$ defined in the unit disk $\mathcal{D}_1$, we have
$$\left(\frac{z^3 f''(z)}{(f(z))^2} + 1 \right) = \phi(z) = 1 + 2z + 2z^2 + \cdots$$
and
$$\left(\frac{w^3 g''(w)}{(g(w))^2} + 1 \right) = \psi(w) = 1 - 2w + 2w^2 - \cdots.$$

Clearly, both ϕ and ψ are members of the Carathéodary class $\mathcal{P}$ and hence their real parts are positive. Which implies that for some α with $0 < \alpha \le 1$, we have

$$|arg\,(\phi)| < \frac{\alpha\pi}{2} \quad \text{and} \quad |arg\,(\psi)| < \frac{\alpha\pi}{2}.$$

Theorem 2.1. *Let the function $f \in \Sigma_m$ given by (3) be in the class $\mathbf{S}_{\Sigma_m}(\alpha)$ where $0 < \alpha \le 1$. Then,*

$$|a_{m+1}| \le \frac{2\alpha}{m\,(m+1)}, \tag{5}$$

$$|a_{2m+1}| \le \frac{\alpha\,(2-\alpha)}{m\,(2m-1)} \tag{6}$$

and for some $k \in \mathbb{R}$,

$$\left|a_{2m+1} - k a_{m+1}^2\right| \le \frac{\alpha}{m\,(2m-1)} + \begin{cases} 2T + \frac{T}{m} - 2G(k) & ; \quad G(k) \le 2T \\ \frac{T}{m} - 2T & ; \quad 2T \le G(k) \le \frac{T}{m} \\ 2G(k) - 2T - \frac{T}{m} & ; \quad G(k) \ge \frac{T}{m}, \end{cases} \tag{7}$$

where $G(k) := \frac{2k\alpha^2}{m^2(m+1)^2}$ and $T := \frac{\alpha(\alpha-1)}{(2m+1)(2m-1)}$.

Proof. It follows from Definition 2.1 that

$$\frac{z^3 f''(z)}{(f(z))^2} + 1 = [s(z)]^\alpha \tag{8}$$

and

$$\frac{w^3 g''(w)}{(g(w))^2} + 1 = [t(w)]^\alpha\,, \tag{9}$$

where $s(z), t(w) \in \mathcal{P}$ have the series expansions:

$$s(z) = 1 + s_m z^m + s_{2m} z^{2m} + s_{3m} z^{3m} + \cdots\,,\ (z \in \mathcal{D}_1) \tag{10}$$

and

$$t(w) = 1 + t_m w^m + t_{2m} w^{2m} + t_{3m} w^{3m} + \cdots\,,\ (w \in \mathcal{D}_1). \tag{11}$$

Hence we have

$$[s(z)]^\alpha = 1 + \alpha s_m z^m + \left[\alpha s_{2m} + \frac{\alpha\,(\alpha-1)}{2} s_m^2\right] z^{2m} + \cdots$$

and

$$[t(w)]^\alpha = 1 + \alpha t_m w^m + \left[\alpha t_{2m} + \frac{\alpha\,(\alpha-1)}{2} t_m^2\right] w^{2m} + \cdots.$$

Also using (3) and (4), we get

$$\left[\frac{z^3 f''(z)}{(f(z))^2} + 1\right] = 1 + m\,(m+1)\,a_{m+1} z^m + \\ 2m\left[(2m+1)\,a_{2m+1} - (m+1)\,a_{m+1}^2\right] z^{2m} + \cdots$$

and

$$\left[\frac{w^3 g''(w)}{(g(w))^2} + 1\right] = 1 - m\,(m+1)\,a_{m+1} w^m + \\ \left[4m^2\,(m+1)\,a_{m+1}^2 - 2m\,(2m+1)\,a_{2m+1}\right] w^{2m} + \cdots.$$

Now equating the coefficients in (8) and (9), we obtain

$$m\,(m+1)\,a_{m+1} = \alpha s_m, \tag{12}$$

$$2m\left[(2m+1)\,a_{2m+1} - (m+1)\,a_{m+1}^2\right] = \left[\alpha s_{2m} + \frac{\alpha\,(\alpha-1)}{2}s_m^2\right], \tag{13}$$

$$-m\,(m+1)\,a_{m+1} = \alpha t_m, \tag{14}$$

$$\left[4m^2\,(m+1)\,a_{m+1}^2 - 2m\,(2m+1)\,a_{2m+1}\right] = \left[\alpha t_{2m} + \frac{\alpha\,(\alpha-1)}{2}t_m^2\right]. \tag{15}$$

From (12) and (14), we find

$$s_m = -t_m \tag{16}$$

and

$$2m^2\,(m+1)^2\,a_{m+1}^2 = \alpha^2\left(s_m^2 + t_m^2\right). \tag{17}$$

Which, on applying Lemma 1.1 yields

$$|a_{m+1}| \leq \frac{2\alpha}{m\,(m+1)}. \tag{18}$$

On the other hand, by adding (13) and (15), we get

$$2m\,(m+1)\,(2m-1)\,a_{m+1}^2 = \alpha\,(s_{2m} + t_{2m}) + \frac{\alpha\,(\alpha-1)}{2}\left(s_m^2 + t_m^2\right).$$

Which, on simplifying yields

$$a_{m+1}^2 = \frac{2\alpha\,(s_{2m} + t_{2m}) + \alpha\,(\alpha-1)\left(s_m^2 + t_m^2\right)}{4m\,(m+1)\,(2m-1)}. \tag{19}$$

This, in light of Lemma 1.1, gives

$$\begin{aligned} |a_{m+1}|^2 &\leq \frac{2\alpha\,|s_{2m} + t_{2m}| + \alpha\,|\alpha-1|\left|s_m^2 + t_m^2\right|}{4m\,(m+1)\,(2m-1)} \\ &\leq \frac{2\alpha\,(|s_{2m}| + |t_{2m}|) + \alpha\,(1-\alpha)\left(|s_m|^2 + |t_m|^2\right)}{4m\,(m+1)\,(2m-1)} \\ &\leq \frac{2\alpha + 2\alpha\,(1-\alpha)}{m\,(m+1)\,(2m-1)} = \frac{2\alpha\,(2-\alpha)}{m\,(m+1)\,(2m-1)}. \end{aligned} \tag{20}$$

Equation (18) and (20) together shows that

$$|a_{m+1}| \leq min\left\{\frac{2\alpha}{m\,(m+1)}, \sqrt{\frac{2\alpha\,(2-\alpha)}{m\,(m+1)\,(2m-1)}}\right\} = \frac{2\alpha}{m\,(m+1)}.$$

Next, multiplying equation (13) by $2m$ and then adding equation (15), we get

$$2m\,(2m+1)\,(2m-1)\,a_{2m+1} = \alpha\,(2ms_{2m} + t_{2m}) + \frac{\alpha\,(\alpha-1)}{2}\left(2ms_m^2 + t_m^2\right).$$

This, on simplifying yields

$$a_{2m+1} = \frac{2\alpha\,(2ms_{2m} + t_{2m}) + \alpha\,(\alpha-1)\left(2ms_m^2 + t_m^2\right)}{4m\,(2m+1)\,(2m-1)}, \tag{21}$$

which implies that

$$|a_{2m+1}| \leq \frac{2\alpha\,|2ms_{2m} + t_{2m}| + \alpha\,|\alpha-1|\left|2ms_m^2 + t_m^2\right|}{4m\,(2m+1)\,(2m-1)}.$$

This, in light of Lemma 1.1, gives

$$|a_{2m+1}| \leq \frac{\alpha + \alpha(1-\alpha)}{m(2m-1)} = \frac{\alpha(2-\alpha)}{m(2m-1)}.$$

Further, for the Fekete-Szegö problem with $k \in \mathbb{R}$, from (17) and (21) we have

$$a_{2m+1} - k a_{m+1}^2 = \frac{2\alpha(2ms_{2m} + t_{2m}) + \alpha(\alpha-1)\left(2ms_m^2 + t_m^2\right)}{4m(2m+1)(2m-1)} - k\left[\frac{\alpha^2\left(s_m^2 + t_m^2\right)}{2m^2(m+1)^2}\right]$$

$$= \frac{\left[\begin{array}{c} 2m(m+1)^2\alpha(2ms_{2m} + t_{2m}) \\ +m(m+1)^2\alpha(\alpha-1)\left(2ms_m^2 + t_m^2\right) \\ -2k(2m+1)(2m-1)\alpha^2\left(s_m^2 + t_m^2\right) \end{array}\right]}{4m^2(m+1)^2(2m+1)(2m-1)},$$

which implies that

$$\left|a_{2m+1} - k a_{m+1}^2\right| \leq \frac{\left[\begin{array}{c} 2m(m+1)^2\alpha\left|2ms_{2m} + t_{2m}\right| \\ +\left|\begin{array}{c} m(m+1)^2\alpha(\alpha-1)\left(2ms_m^2 + t_m^2\right) \\ -2k(2m+1)(2m-1)\alpha^2\left(s_m^2 + t_m^2\right) \end{array}\right| \end{array}\right]}{4m^2(m+1)^2(2m+1)(2m-1)}$$

$$\leq \frac{\left[\begin{array}{c} 4m(m+1)^2\alpha(2m+1) \\ +\left|\begin{array}{c} 2m^2(m+1)^2\alpha(\alpha-1) \\ -2k(2m+1)(2m-1)\alpha^2 \end{array}\right| |s_m|^2 \end{array}\right]}{4m^2(m+1)^2(2m+1)(2m-1)}$$

$$+ \frac{\left|m(m+1)^2\alpha(\alpha-1) - 2k(2m+1)(2m-1)\alpha^2\right| |t_m|^2}{4m^2(m+1)^2(2m+1)(2m-1)}$$

$$\leq \frac{\left[\begin{array}{c} m(m+1)^2\alpha(2m+1) \\ +\left|\begin{array}{c} 2m^2(m+1)^2\alpha(\alpha-1) \\ -2k(2m+1)(2m-1)\alpha^2 \end{array}\right| \end{array}\right]}{m^2(m+1)^2(2m+1)(2m-1)}$$

$$+ \frac{\left|m(m+1)^2\alpha(\alpha-1) - 2k(2m+1)(2m-1)\alpha^2\right|}{m^2(m+1)^2(2m+1)(2m-1)}.$$

Hence we get

$$\left|a_{2m+1} - k a_{m+1}^2\right| \leq \frac{\alpha}{m(2m-1)} + \left|\frac{2\alpha(\alpha-1)}{(2m+1)(2m-1)} - \frac{2k\alpha^2}{m^2(m+1)^2}\right|$$

$$+ \left|\frac{\alpha(\alpha-1)}{m(2m+1)(2m-1)} - \frac{2k\alpha^2}{m^2(m+1)^2}\right|.$$

By putting $G(k) := \frac{2k\alpha^2}{m^2(m+1)^2}$ and $T := \frac{\alpha(\alpha-1)}{(2m+1)(2m-1)}$, we have

$$\left|a_{2m+1} - ka_{m+1}^2\right| \leq \frac{\alpha}{m\,(2m-1)} + |2T - G(k)| + \left|\frac{T}{m} - G(k)\right|.$$

(Note that since T is non-positive, $2T \leq \frac{T}{m}$.) This proves the desired estimate (7). Finally for sharpness, the m-fold symmetric analytic bi-univalent function

$$F_1(z) = z + \frac{2\alpha}{m\,(m+1)} z^{m+1} + \frac{\alpha\,(2-\alpha)}{m\,(2m-1)} z^{2m+1} + \cdots$$

along with the corresponding inverse function given by

$$F_1{}^{-1}(w) = w - \frac{2\alpha}{m\,(m+1)} w^{m+1} + \left[\frac{4\alpha^2}{m^2\,(m+1)} - \frac{\alpha\,(2-\alpha)}{m\,(2m-1)}\right] w^{2m+1} + \cdots$$

proves the sharpness of all the three results. In particular, the *1*-fold symmetric analytic bi-univalent function $f_1(z)$ provide sharp bounds $|a_2| \leq \alpha$, $|a_3| \leq \alpha\,(2-\alpha)$ and $|a_3 - ka_2^2| \leq \alpha + \left|\frac{2\alpha(\alpha-1)}{3} - \frac{k\alpha^2}{2}\right| + \left|\frac{\alpha(\alpha-1)}{3} - \frac{k\alpha^2}{2}\right|$, $(k \in \mathbb{R})$ for the function class $\mathbf{S}_\Sigma(\alpha)$ where

$$f_1(z) = z + \alpha z^2 + \alpha\,(2-\alpha)\,z^3 + \cdots$$

along with its inverse function

$$f_1{}^{-1}(w) = w - \alpha w^2 + \alpha\,(3\alpha - 2)\,w^3 + \cdots .$$

Hence the proof. $\square$

Definition 2.2. *A function $f \in \Sigma_m$ given by* (3) *is said to be in the class $\mathbf{S}^*_{\Sigma_m}(\beta)$ if the following conditions are satisfied:*

$$\Re\left[\frac{z^3 f''(z)}{(f(z))^2} + 1\right] > \beta, \qquad (z \in \mathcal{D}_1)$$

and

$$\Re\left[\frac{w^3 g''(w)}{(g(w))^2} + 1\right] > \beta, \qquad (w \in \mathcal{D}_1);$$

where $0 \leq \beta < 1$ and the function g is given by (4).

In particular, observe the following examples:

1. For the identity function $f(z) = z$ and its inverse function $g(w) = w$ defined in the unit disk $\mathcal{D}_1$, we have
$$\Re\left[\frac{z^3 f''(z)}{(f(z))^2} + 1\right] = \Re\left[\frac{w^3 g''(w)}{(g(w))^2} + 1\right] = 1 > \beta, \quad (0 \leq \beta < 1).$$
2. For the *1*-fold symmetric analytic bi-univalent function $f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots$ and its inverse function $g(w) = \frac{w}{1+w} = w - w^2 + w^3 - \cdots$ defined in the unit disk $\mathcal{D}_1$, we have
$$\left(\frac{z^3 f''(z)}{(f(z))^2} + 1\right) = \phi(z) = 1 + 2z + 2z^2 + \cdots$$

and

$$\left(\frac{w^3 g''(w)}{(g(w))^2}+1\right)=\psi(w)=1-2w+2w^2-\cdots.$$

Clearly, both ϕ and ψ are members of the Carathéodary class $\mathcal{P}$ and hence their real parts are positive. Which implies that for some β with $0\leq\beta<1$, we have

$$\Re(\phi)>\beta \quad \text{and} \quad \Re(\psi)>\beta.$$

Theorem 2.2. *Let the function $f\in\Sigma_m$ given by* (3) *be in the class $\mathbf{S}^*_{\Sigma_m}(\beta)$ where $0\leq\beta<1$. Then,*

$$|a_{m+1}|\leq\frac{2(1-\beta)}{m(m+1)}, \tag{22}$$

$$|a_{2m+1}|\leq\frac{(1-\beta)}{m(2m-1)} \tag{23}$$

and for some $k\in\mathbb{R}$,

$$\left|a_{2m+1}-ka_{m+1}^2\right|\leq\left|\frac{(1-\beta)}{m(2m-1)}-\frac{4k(1-\beta)^2}{m^2(m+1)^2}\right|. \tag{24}$$

Proof. It follows from Definition 2.2 that

$$\frac{z^3 f''(z)}{(f(z))^2}+1=\beta+(1-\beta)\ s(z) \tag{25}$$

and

$$\frac{w^3 g''(w)}{(g(w))^2}+1=\beta+(1-\beta)\ t(w), \tag{26}$$

where $s(z),t(w)\in\mathcal{P}$ are as given in equation (10) and (11). So that we have

$$\beta+(1-\beta)\ s(z)=1+(1-\beta)\,s_m z^m+(1-\beta)\,s_{2m}z^{2m}+\cdots$$

and

$$\beta+(1-\beta)\ t(w)=1+(1-\beta)\,t_m w^m+(1-\beta)\,t_{2m}w^{2m}+\cdots.$$

Now equating the coefficients in (25) and (26), we obtain

$$m(m+1)\,a_{m+1}=(1-\beta)\,s_m, \tag{27}$$

$$2m\left[(2m+1)\,a_{2m+1}-(m+1)\,a_{m+1}^2\right]=(1-\beta)\,s_{2m}, \tag{28}$$

$$-m(m+1)\,a_{m+1}=(1-\beta)\,t_m, \tag{29}$$

$$\left[4m^2(m+1)\,a_{m+1}^2-2m(2m+1)\,a_{2m+1}\right]=(1-\beta)\,t_{2m}. \tag{30}$$

From (27) and (29), we find

$$s_m=-t_m \tag{31}$$

and

$$2m^2(m+1)^2 a_{m+1}^2=(1-\beta)^2\left(s_m^2+t_m^2\right). \tag{32}$$

Which, on applying Lemma 1.1 yields

$$|a_{m+1}|\leq\frac{2(1-\beta)}{m(m+1)}. \tag{33}$$

On the other hand, by adding (28) and (30), we get

$$2m\,(m+1)\,(2m-1)\,a_{m+1}^2 = (1-\beta)\,(s_{2m}+t_{2m})\,. \tag{34}$$

This, on applying Lemma 1.1, gives

$$|a_{m+1}| \leq \sqrt{\frac{2\,(1-\beta)}{m\,(m+1)\,(2m-1)}}. \tag{35}$$

Equation (33) and (35) together shows that

$$|a_{m+1}| \leq min\left\{\frac{2\,(1-\beta)}{m\,(m+1)}, \sqrt{\frac{2\,(1-\beta)}{m\,(m+1)\,(2m-1)}}\right\} = \frac{2\,(1-\beta)}{m\,(m+1)}.$$

Next, multiplying equation (28) by $2m$ and then adding equation (30), we get

$$2m\,(2m+1)\,(2m-1)\,a_{2m+1} = (1-\beta)\,(2ms_{2m}+t_{2m})\,. \tag{36}$$

Again applying Lemma 1.1 for the coefficients s_{2m} and t_{2m}, we obtain

$$|a_{2m+1}| \leq \frac{(1-\beta)}{m\,(2m-1)}.$$

Further, for the Fekete-Szegö problem with $k \in \mathbb{R}$, from (32) and (36) we have

$$\begin{aligned} a_{2m+1} - ka_{m+1}^2 &= \frac{(1-\beta)\,(2ms_{2m}+t_{2m})}{2m\,(2m+1)\,(2m-1)} - k\left[\frac{(1-\beta)^2\left(s_m^2+t_m^2\right)}{2m^2\,(m+1)^2}\right] \\ &= \frac{\begin{bmatrix} m\,(m+1)^2\,(1-\beta)\,(2ms_{2m}+t_{2m}) \\ -k\,(2m+1)\,(2m-1)\,(1-\beta)^2\left(s_m^2+t_m^2\right)\end{bmatrix}}{2m^2\,(m+1)^2\,(2m+1)\,(2m-1)}. \end{aligned}$$

This, on applying Lemma 1.1, gives

$$\begin{aligned} \left|a_{2m+1} - ka_{m+1}^2\right| &\leq \frac{\left|m\,(m+1)^2\,(1-\beta) - 4k\,(2m-1)\,(1-\beta)^2\right|}{m^2\,(m+1)^2\,(2m-1)} \\ &= \left|\frac{(1-\beta)}{m\,(2m-1)} - \frac{4k\,(1-\beta)^2}{m^2\,(m+1)^2}\right|, \end{aligned}$$

which proves the desired estimate (24).
Finally for sharpness, the m-fold symmetric analytic bi-univalent function

$$F_2(z) = z + \frac{2\,(1-\beta)}{m\,(m+1)}z^{m+1} + \frac{(1-\beta)}{m\,(2m-1)}z^{2m+1} + \cdots$$

along with the corresponding inverse function given by

$$F_2{}^{-1}(w) = w - \frac{2\,(1-\beta)}{m\,(m+1)}w^{m+1} + \left[\frac{4\,(1-\beta)^2}{m^2\,(m+1)} - \frac{(1-\beta)}{m\,(2m-1)}\right]w^{2m+1} + \cdots$$

proves the sharpness of all the three results. In particular, the *1*-fold symmetric analytic bi-univalent function $f_2(z)$ provide sharp bounds $|a_2| \leq (1-\beta)$, $|a_3| \leq (1-\beta)$ and $|a_3 - ka_2^2| \leq \left|(1-\beta) - k\,(1-\beta)^2\right|$, $(k \in \mathbb{R})$ for the function class $\mathbf{S}_\Sigma^*(\beta)$ where

$$f_2(z) = z + (1-\beta)\,z^2 + (1-\beta)\,z^3 + \cdots$$

along with its inverse function

$$f_2{}^{-1}(w) = w - (1-\beta)\,w^2 + \Big[2\,(1-\beta)^2 - (1-\beta)\Big]\,w^3 + \cdots .$$

Hence the proof. $\square$

3. Connections with Earlier Known Results

For *1*-fold symmetric analytic bi-univalent functions (*i.e.* for $m = 1$), our Theorem 2.1 and Theorem 2.2 reduces to the following two Corollaries, respectively. These Corollaries relates with the recent results proved by Patil and Naik [16].

Corollary 3.1. *Let $f \in \chi_\Sigma^\alpha$ where $0 < \alpha \le 1$ be of the form (1). Then*

$$|a_2| \le \alpha,$$

$$|a_3| \le \alpha\,(2-\alpha)\,,$$

and for some $k \in \mathbb{R}$,

$$\left|a_3 - k a_2^2\right| \le \begin{cases} (1-k)\,\alpha^2 & ; \quad k \le \frac{4}{3}\left(1 - \frac{1}{\alpha}\right) \\ \frac{\alpha(4-\alpha)}{3} & ; \quad \frac{4}{3}\left(1 - \frac{1}{\alpha}\right) \le k \le \frac{2}{3}\left(1 - \frac{1}{\alpha}\right) \\ 2\alpha + (k-1)\,\alpha^2 & ; \quad k \ge \frac{2}{3}\left(1 - \frac{1}{\alpha}\right). \end{cases}$$

Corollary 3.2. *Let $f \in \chi_\Sigma(\beta)$ where $0 \le \beta < 1$ be of the form (1). Then*

$$|a_2| \le (1-\beta)\,,$$

$$|a_3| \le (1-\beta)\,,$$

$$\left|a_3 - k a_2^2\right| \le \left|(1-\beta) - k\,(1-\beta)^2\right|, \quad (k \in \mathbb{R})\,.$$

Here the subclasses $\chi_\Sigma^\alpha \equiv \mathbf{S}_{\Sigma_1}(\alpha)$ and $\chi_\Sigma(\beta) \equiv \mathbf{S}^*_{\Sigma_1}(\beta)$ of $\Sigma \equiv \Sigma_1$ are defined in the following way (see [16]):

Definition 3.1. *A function $f \in \Sigma$ given by (1) is said to be in the class χ_Σ^α if the following conditions are satisfied:*

$$\left| arg \left\{ \frac{z^3 f''(z)}{(f(z))^2} + 1 \right\} \right| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{D}_1)$$

and

$$\left| arg \left\{ \frac{w^3 g''(w)}{(g(w))^2} + 1 \right\} \right| < \frac{\alpha\pi}{2} \qquad (w \in \mathcal{D}_1),$$

where $0 < \alpha \le 1$ and the function g is given by (2).

Definition 3.2. *A function $f \in \Sigma$ given by (1) is said to be in the class $\chi_\Sigma(\beta)$ if the following conditions are satisfied:*

$$\Re \left\{ \frac{z^3 f''(z)}{(f(z))^2} + 1 \right\} > \beta \qquad (z \in \mathcal{D}_1)$$

and

$$\Re \left\{ \frac{w^3 g''(w)}{(g(w))^2} + 1 \right\} > \beta \qquad (w \in \mathcal{D}_1),$$

where $0 \le \beta < 1$ and the function g is given by (2).

4. Conclusion

In the present paper, we have obtained sharp estimates on the first two Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ and also on the Fekete-Szegö functional $|a_{2m+1} - ka_{m+1}^2|$, $(k \in \mathbb{R})$ for functions belong to the two new subclasses $\mathbf{S}_{\Sigma_m}(\alpha)$ and $\mathbf{S}^*_{\Sigma_m}(\beta)$ of Σ_m. According to our main results, we can conclude that the geometrical similarities in the subclasses $\mathbf{S}_{\Sigma_m}(\alpha)$ and $\mathbf{S}^*_{\Sigma_m}(\beta)$ (of the m-fold symmetric analytic bi-univalent function class Σ_m defined in the open unit disk $\mathcal{D}_1$) also reflects in their initial Taylor-Maclaurin coefficient estimations, which assures the connection between analytic characterization and geometric behaviour of the functions belong to these subclasses.

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