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DIVISOR GRAPHS WITH FOUR TRIANGLES

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ABSTRACT. In this paper, we investigate divisor graphs with four triangles and establish a forbidden subgraph characterization for all divisor graphs containing four triangles.

Keywords: Divisor graph, divisor labeling, divisor digraph.

AMS Subject Classification: 05C12, 05C20, 05C78.

1. Introduction

In 2000, [11] defined the divisor graphs for a finite nonempty set of integers. A divisor graph G is an ordered pair (V, E) where $V \subset Z$ and for all $u, v \in V, u \neq v, uv \in E(G)$ if and only if $u|v$ or $v|u$.

In 2001, Chartrand et al. [2] defined the divisor graphs for a finite nonempty set S of positive integers. The divisor graph $G(S)$ of S has S as its vertex set and two distinct vertices i and j of $G(S)$ are adjacent if either i divides j or j divides i. A graph G is a divisor graph if G is isomorphic to $G(S)$ for some finite, nonempty set S of positive integers. Hence if G is a divisor graph, then there exists a function $f : V(G) \to N$, called a divisor labeling of G, such that $G \cong G[f(V(G))]$.

A labeling $f : V(G) \to N$ is called a divisor labeling if $u \neq v$ where $u, v \in V(G), uv \in E(G)$ if and only if $f(u) \mid f(v)$ or $f(v) \mid f(u)$. If a graph G possess divisor labeling, then G is called divisor graph. Every graph does not possess divisor labeling. See [1, 3, 6, 7, 8, 9] for more information on divisor graphs. For a dynamic survey on graph labelings such as graceful labelings, prime labelings and magic labelings, we refer to Gallian [4]. For recent results on graceful labelings, one can refer [10].

Let S be nonempty set of positive integers. The divisor digraph $D(S)$ of S has the vertex set S and (i, j) is an arc of $D(S)$ if i divides j. Thus $G(S)$ is the underlying graph of $D(S)$.

The degree $deg\ v$ of a vertex v in a digraph D is the sum of its indegree and outdegree, that is, $deg\ v = id\ v + od\ v$. A vertex v is an end vertex if $deg\ v = 1$.

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For a vertex u of D, let $N^+(u) = \{x|(u,x) \in E(D)\}$ and $N^-(u) = \{x|(x,u) \in E(D)\}$. u is called a transmitter (respectively, receiver) $N^-(u) = \emptyset$ (respectively $N^+(u) = \emptyset$).

A vertex u of D is a transitive vertex, if outdegree of u and indegree of u are both greater than zero and for every $x \in N^-(u)$ and $y \in N^+(u)$, necessarily $(x, y) \in E(D)$.

Let G be a divisor graph and G be isomorphic to $G(S)$. The orientation given by $D(S)$ is called the divisor orientation of G.

An orientation D of a graph G in which every vertex is a transmitter, a receiver, or a transitive vertex is called a divisor orientation of G. If G is a divisor graph and f is a divisor labeling of G, then $D[f(V(G))]$ is called the orientation of G induced by f.

For $S = \{2, 4, 6, 8, 18\}$, the divisor graph $G(S)$ and divisor digraph $D(S)$ are shown in Figure 1

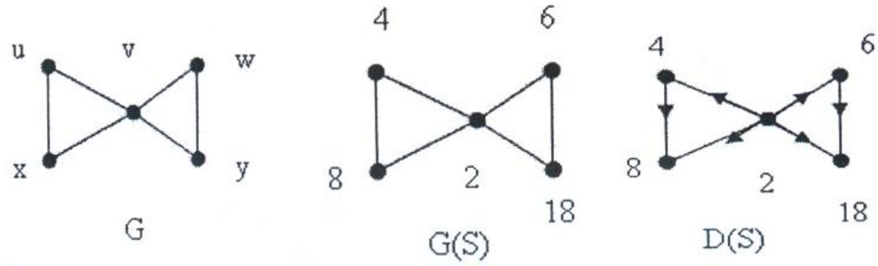


FIGURE 1. A divisor graph and a divisor digraph

The graph $G = 2P_2 + K_1$ of Figure 1 is a divisor graph and the function $f : V(G) \to N$ defined by $f(v) = 2, f(u) = 4, f(w) = 6, f(x) = 8, f(y) = 18$ is a divisor labeling. The vertex v is a transmitter, the vertices y and x are receivers and the vertices u and w are transitive vertices. The orientation given by $D(S)$ is the divisor orientation of G.

Also, the graph $G = K_4 - e$ is a divisor graph and the graph $K_3 \times K_2$ is not a divisor graph ([5] and [2]).

In [2], it is proved that the graph G' of Figure 2 is not a divisor graph.

FIGURE 2. The graph G' is not a divisor graph

It is shown in [11] that $K_n, K_{1,n}, C_{2n}, P_n, K_{m,n}$ are divisor graphs. Also it is shown that the odd cycles C_{2n+1} for all $n > 1$ are not divisor graphs and any graph with an induced subgraph which is an odd cycle of length greater than or equal to 5 is not a divisor graph.

It is known in [2] that no divisor graph contains an induced odd cycle of length 5 or more and every bipartite graph is a divisor graph.

It is known in [5] that a triangle-free graph G is a divisor graph if and only if G is bipartite. However there are divisor graphs that contain triangles. A forbidden subgraph characterization for all divisor graphs containing at most three triangles is given in [5].

Lemma 1.1. [11] *Every induced subgraph of a divisor graph is a divisor graph.*

Theorem 1.1. [2] *A graph G is a divisor graph if and only if G has a divisor orientation.*

Theorem 1.2. [5] *If G is a connected graph that contains at most three triangles and no other induced odd cycles, then G is a divisor graph if and only if G does not contain any of the graphs in Figure 3 as an induced subgraph, where each dashed line represents an edge that may or may not be present.*

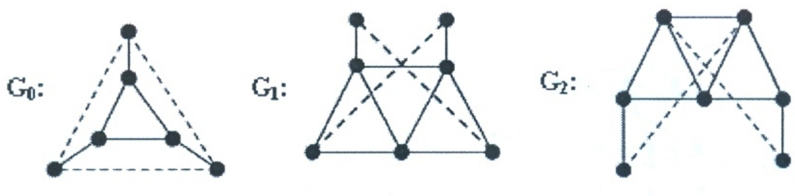


Figure 3. Non divisor graphs

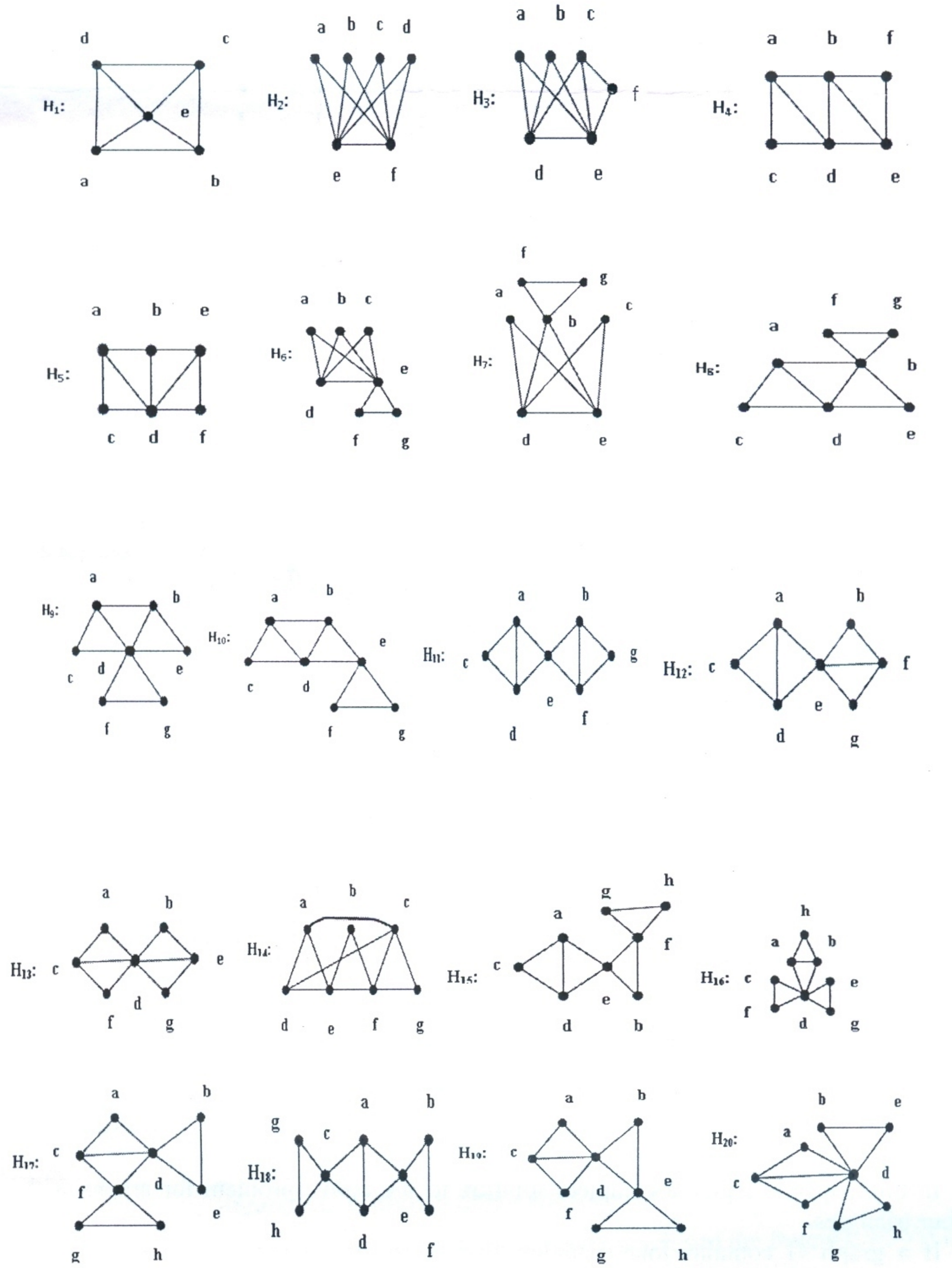
In [5] Gera et al. raised the following question, 'Which graphs with four or more triangles and no other induced odd cycles are divisor graphs?'

In this paper, we give a complete solution to the above problem for graphs with at most four triangles.

If a graph G contains four triangles, then it is more complicated to determine whether G is a divisor graph. In order to determine all forbidden subgraphs for divisor graphs with exactly four triangles, we first present preliminary results.

2. Main Results

Lemma 2.1. *Each graph in Figure 1 is a divisor graph.*


H1: d c e a b
H2: a b c d e f
H3: a b c f d e
H4: a b f c d e
H5: a b e c d f
H6: a b c e d f g
H7: f a g b c d e
H8: f g a b c d e
H9: a b c d e f g
H10: a b e c d f g
H11: a b c g d e f
H12: a b c f d e g
H13: a b c e f d g
H14: a b c d e f g
H15: g h a f c e d b
H16: h a b c e f d g
H17: a b c f d e g h
H18: a b g c h e d f
H19: a b c d f e g h
H20: b e a d c f h g

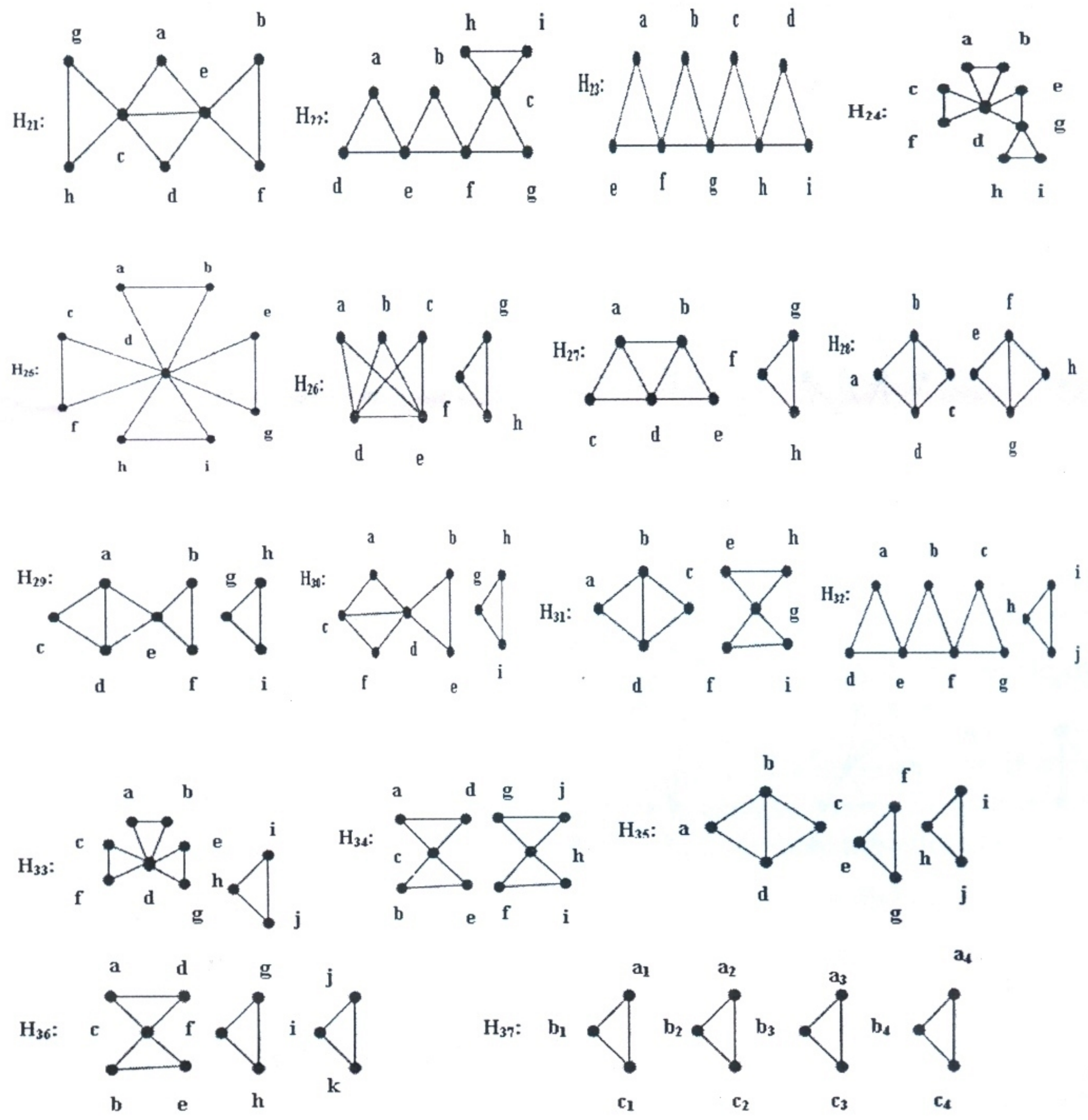


FIGURE 1. The graphs $H_i (1 \leq i \leq 37)$

The graph $H_1 (= W_5)$ is a divisor graph [11].

For each $2 \leq i \leq 25$, the graph H_i has an orientation, in which every vertex is a transmitter, a receiver, or a transitive vertex, as shown in Figure 2. Thus for each $2 \leq i \leq 25$ the graph H_i has a divisor orientation.

It follows by Theorem 1.1, that each graph $H_i (2 \leq i \leq 25)$ is a divisor graph.

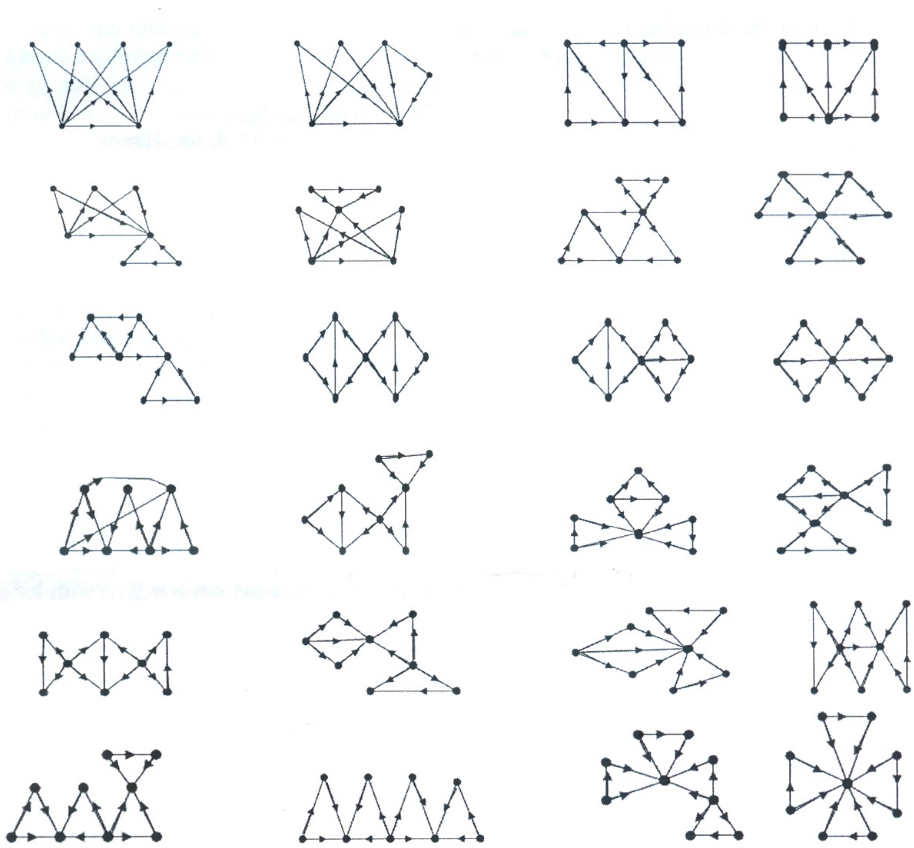


FIGURE 2. Orientations of H_i for $2 \leq i \leq 25$

For each $26 \leq i \leq 33$ except for $i = 28, 31$ the graph H_i has two components, one component is K_3 and the another component has divisor orientation as shown in Figure 3.

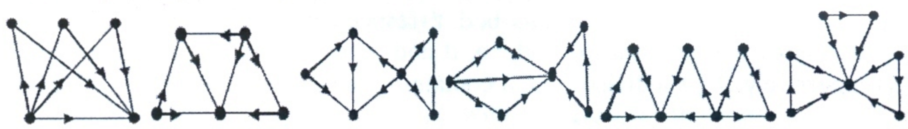


FIGURE 3. Orientation of a component H_i for $26 \leq i \leq 33$ except for $i = 28, 31$

It follows by Theorem 1.1 that each graph H_i (for $26 \leq i \leq 33$ except for $i = 28, 31$) is a divisor graph. Since the graph $K_4 - e, 2P_2 + K_1$ and K_3 are divisor graphs, each graph H_i (for $i = 28, 31$ and for $34 \leq i \leq 37$) is a divisor graph. Thus each graph $H_i (1 \leq i \leq 37)$ in Figure 1 is a divisor graph.

Each of the parts of the following lemma shows that the certain graphs that contain one of the graphs $H_i (1 \leq i \leq 37)$ in Figure 1 are divisor graphs. We omit the routine proofs of these lemma.

Lemma 2.2. *Let G be a bipartite graph with*

(1) $S = \{v_2, v_4, v_5\}$ *(respectively,* $S^* = \{v_1, v_2, v_3, v_4\}$*)* $\subseteq V(G)$ *such that* $\langle S\rangle = P_3 : v_4, v_5, v_2$ *(respectively,* $\langle S^*\rangle = C_4 : v_1, v_2, v_3, v_4, v_1$*). Define* D_1 *(respectively,* D_1^**) from* G *by adding two (respectively, one) new vertices* v_1, v_3 *(respectively, vertex* v_5*) such that each vertex* v_1, v_3 *is adjacent with* v_2, v_4, v_5 *(respectively, the vertex* v_5 *is adjacent with every vertex in* S^**).*

(2) $S = \{v_5, v_6\}$ *(respectively,* $S^* = \{v_1, v_2, v_3, v_4, v_6\}$*)* $\subseteq V(G)$ *such that* $v_5v_6 \in E(G)$ *(respectively,* $\langle S^*\rangle = K_{1,4}$ *with its central vertex* v_6*). Define* D_2 *(respectively,* D_2^**) from* G *by adding four (respectively, one) new vertices* v_1, v_2, v_3, v_4 *(respectively, vertex* v_5*) such that each vertex* v_1, v_2, v_3, v_4 *is adjacent with both* v_5, v_6 *(respectively, the vertex* v_5 *is adjacent with every vertex in* S^**).*

(3) $S = \{v_4, v_5, v_6\}$ *(respectively,* $S^* = \{v_1, v_2, v_3, v_5\}$*)* $\subseteq V(G)$ *such that* $\langle S\rangle = P_3 : v_4, v_5, v_6$ *(respectively,* $\langle S^*\rangle = K_{1,3}$ *with its central vertex* v_5*). Define* D_3 *(respectively* D_3^**) from* G *by adding three (respectively, two) new vertices* v_1, v_2, v_3 *(respectively* v_4, v_5*) such that each vertex* v_1, v_2 *is adjacent with both* v_4, v_5 *and the vertex* c *is adjacent with* v_4, v_5, v_6 *(respectively, the vertex* v_4 *is adjacent with* v_1, v_2, v_3, v_5 *and the vertex* v_6 *is adjacent with both* v_3, v_5*).*

(4) $S = \{v_2, v_3, v_4, v_6\} \subseteq V(G)$ *such that* $\langle S\rangle = P_4 : v_3, v_4, v_2, v_6$*. Define* D_4 *from* G *by adding two new vertices* v_1, v_5 *such that the vertex* a *is adjacent with* v_2, v_3, v_4 *and the vertex* v_5 *is adjacent with* v_2, v_4, v_6*.*

(5) $S = \{v_2, v_3, v_4, v_6\}$ *(respectively,* $S^* = \{v_1, v_4, v_5\}$*)* $\subseteq V(G)$ *such that* $\langle S\rangle = K_{1,3}$ *with its central vertex* d *(respectively,* $\langle S^*\rangle = P_3 : v_1, v_4, v_5$*). Define* D_5 *(respectively,* D_5^**) from* G *by adding two (respectively, three) new vertices* v_1, v_5 *(respectively,* v_2, v_3, v_6*) such that the vertex* v_1 *is adjacent with* v_2, v_3, v_4 *and the vertex* v_5 *is adjacent with* v_2, v_4, v_6 *(respectively, the vertex* v_2 *is adjacent to* v_3, v_4, v_5 *and the vertex* v_3 *is adjacent with both* v_1, v_4 *and the vertex* v_6 *is adjacent with both* v_4, v_5*).*

(6) $S = \{v_4, v_5, v_7\}$ *(respectively,* $S^* = \{v_1, v_2, v_3, v_5, v_7\}$*)* $\subseteq V(G)$ *such that* $\langle S\rangle = P_3 : v_4, v_5, v_7$*. (respectively,* $\langle S^*\rangle = K_{1,4}$ *with its central vertex* v_5*). Define* D_6 *(respectively,* D_6^**) from* G *by adding four (respectively, two) new vertices* v_1, v_2, v_3, v_6 *(respectively,* v_4, v_6*) such that each vertex* v_1, v_2, v_3 *is adjacent with both* v_4, v_5 *and the vertex* v_6 *is adjacent with both* v_5, v_7 *(respectively, the vertex* v_4 *is adjacent with* v_1, v_2, v_3, v_5 *and the vertex* v_6 *is adjacent with both* v_5, v_7*).*

(7) $S = \{v_1, v_2, v_3, v_5, v_7\} \subseteq V(G)$ *such that* $\langle\{v_1, v_2, v_3, v_5\}\rangle = K_{1,3}$ *with its central vertex* v_5 *and* $v_2v_7 \in E(G)$*. Define* D_7 *from* G *by adding two new vertices* v_4, v_6 *such that the vertex* v_4 *is adjacent with* v_1, v_2, v_3, v_5 *and the vertex* v_6 *is adjacent with both* v_2, v_7*.*

(8) $S = \{v_2, v_3, v_4, v_7\} \subseteq V(G)$ *such that* $\langle S\rangle = P_4 : v_3, v_4, v_2, v_7$*. Define* D_8 *from* G *by adding three new vertices* v_1, v_5, v_6 *such that the vertex* v_1 *is adjacent with* v_2, v_3, v_4 *and the vertex* v_5 *is adjacent with both* v_2, v_4 *and the vertex* v_6 *is adjacent with both* v_2, v_7*.*

(9) $S = \{v_2, v_3, v_4, v_7\} \subseteq V(G)$ *such that* $\langle S\rangle = K_{1,3}$ *with its central vertex* v_4*. Define* D_9 *from* G *by adding three new vertices* v_1, v_5, v_6 *such that the vertex* v_1 *is adjacent with* v_2, v_3, v_4 *and the vertex* v_5 *is adjacent with both* v_1, v_4 *and the vertex* v_6 *is adjacent with both* v_4, v_7*.*

(10) $S = \{v_1, v_4, v_5, v_7\} \subseteq V(G)$ *such that* $\langle S\rangle = P_4 : v_1, v_4, v_5, v_7$*. Define* D_{10} *from* G *by adding three new vertices* v_2, v_3, v_6 *such that the vertex* v_2 *is adjacent with the vertex* v_1, v_4, v_5 *and the vertex* v_3 *is adjacent with both the vertices* v_1, v_4 *and the vertex* v_6 *is adjacent with both the vertices* v_7, v_5*.*

(11) $S = \{v_3, v_4, v_5, v_6, v_7\} \subseteq V(G)$ *such that* $\langle S\rangle = P_5 : v_3, v_4, v_5, v_6, v_7$. *Define* D_{11} *from* G *by adding two new vertices* v_1, v_2 *such that the vertex* v_1 *is adjacent with* v_3, v_4, v_5 *and the vertex* v_2 *is adjacent with* v_5, v_6, v_7.

(12) $S = \{v_3, v_4, v_5, v_6\}$ *(respectively,* $S^* = \{v_2, v_3, v_4, v_5, v_7\}$*)* $\subseteq V(G)$ *such that* $\langle S\rangle = P_4 : v_3, v_4, v_5, v_6$ *(respectively,* $\langle\{v_2, v_4, v_5, v_7\}\rangle = K_{1,3}$ *with its central vertex* v_5 *and* $v_3v_4 \in E(G)$*). Define* D_{12} *(respectively,* D^*_{12}*) from* G *by adding three (respectively, two) new vertices* v_1, v_2, v_7 *(respectively,* v_1, v_6*) such that the vertex* v_1 *is adjacent with* v_3, v_4, v_5 *and each vertex* v_2, v_7 *is adjacent with both* v_5, v_6 *(respectively, the vertex* v_1 *is adjacent with* v_3, v_4, v_5 *and the vertex* v_6 *is adjacent with* v_2, v_5, v_7*).*

(13) $S = \{v_3, v_4, v_5\}$ *(respectively,* $S^* = \{v_2, v_3, v_4, v_6\}$*) [respectively,* $S^{**} = \{v_1, v_2, v_4, v_6, v_7\}$*]* $\subseteq V(G)$ *such that* $\langle S\rangle = P_3 : v_3, v_4, v_5$ *(respectively,* $\langle S^*\rangle = K_{1,3}$ *with its central vertex* v_4*) [respectively,* $\langle S^{**}\rangle = K_{1,4}$ *with its central vertex* v_4*]. Define* D_{13} *(respectively,* D^*_{13}*) [respectively,* D^{**}_{13}*] from* G *by adding four (respectively, three) [respectively, two] new vertices* v_1, v_2, v_6, v_7 *(respectively,* v_1, v_5, v_6*) [respectively,* v_3, v_5*] such that each vertex* v_1, v_6 *is adjacent with both* v_3, v_4 *and each vertex* v_2, v_7 *is adjacent with both* v_4, v_5 *(respectively, each vertex* v_1, v_6 *is adjacent with both* v_3, v_4 *and the vertex* v_5 *is adjacent with* v_2, v_4, v_7*) [respectively, the vertex* v_3 *is adjacent with* v_1, v_4, v_6 *and vertex* v_5 *is adjacent with* v_2, v_4, v_7*].*

(14) $S = \{v_3, v_4, v_5, v_6\} \subseteq V(G)$ *such that* $\langle S\rangle = C_4 : v_3, v_4, v_5, v_6, v_3$. *Define* D_{14} *from* G *by adding three new vertices* v_1, v_2, v_7 *such that the vertex* v_1 *is adjacent with* v_3, v_4, v_5 *and the vertex* v_2 *is adjacent with both* v_5, v_6 *and the vertex* v_7 *is adjacent with both* v_3, v_6.

(15) $S = \{v_3, v_4, v_5, v_6, v_8\} \subseteq V(G)$ *such that* $\langle S\rangle = P_5 : v_3, v_4, v_5, v_6, v_8$. *Define* D_{15} *from* G *by adding three new vertices* v_1, v_2, v_7 *such that the vertex* v_1 *is adjacent with* v_3, v_4, v_5 *and the vertex* v_2 *is adjacent with both* v_5, v_6 *and the vertex* v_7 *is adjacent with both* v_6, v_7.

(16) $S = \{v_2, v_4, v_6, v_7, v_8\} \subseteq V(G)$ *such that* $\langle\{v_2, v_4, v_6, v_7\}\rangle = K_{1,3}$ *with its central vertex* v_4 *and* $v_2v_8 \in E(G)$. *Define* D_{16} *from* G *by adding three new vertices* v_1, v_3, v_5 *such that the vertex* v_1 *is adjacent with* v_2, v_4, v_8 *and the vertex* v_5 *is adjacent with both* v_4, v_7 *and the vertex* v_3 *is adjacent with both* v_4, v_6.

(17) $S = \{v_1, v_4, v_5, v_6, v_8\} \subseteq V(G)$ *such that* $\langle\{v_1, v_4, v_5, v_6\}\rangle = K_{1,3}$ *with its central vertex* v_4 *and* $v_6v_8 \in E(G)$. *Define* D_{17} *from* G *by adding three new vertices* v_2, v_3, v_7 *such that the vertex* v_2 *is adjacent with both* v_4, v_5 *and the vertex* v_3 *is adjacent with* v_1, v_4, v_6 *and the vertex* v_7 *is adjacent with both* v_6, v_8.

(18) $S = \{v_3, v_4, v_5, v_6, v_8\} \subseteq V(G)$ *such that* $\langle S\rangle = P_5 : v_8, v_3, v_4, v_5, v_6$. *Define* D_{18} *from* G *by adding three new vertices* v_1, v_2, v_7 *such that the vertex* v_1 *is adjacent with* v_3, v_4, v_5 *and the vertex* v_7 *is adjacent with both* v_3, v_8 *and the vertex* v_2 *is adjacent with both* v_5, v_6.

(19) $S = \{v_3, v_4, v_5, v_8\}$ *(respectively,* $S^* = \{v_1, v_4, v_5, v_6, v_8\}$*)* $\subseteq V(G)$ *such that* $\langle S\rangle = P_4 : v_3, v_4, v_5, v_8$ *(respectively,* $\langle\{a, v_4, v_5, v_6\}\rangle = K_{1,3}$ *with its central vertex* v_4 *and* $v_5v_8 \in E(G)$*). Define* D_{19} *(respectively,* D^*_{19}*) from* G *by adding four (respectively, three) new vertices* v_1, v_2, v_6, v_7 *(respectively,* v_2, v_3, v_7*) such that each vertex* v_1, v_6 *is adjacent with both* v_3, v_4 *and the vertex* v_2 *is adjacent with both* v_4, v_5 *and the vertex* v_7 *is adjacent to both* v_5, v_8 *(respectively, the vertex* v_2 *is adjacent with both* v_4, v_5 *and the vertex* v_3 *is adjacent with* v_1, v_4, v_6 *and the vertex* v_7 *is adjacent with both* v_5, v_8*).*

(20) $S = \{v_3, v_4, v_5, v_8\}$ *(respectively,* $S^* = \{v_1, v_4, v_5, v_6, v_8\}$*)* $\subseteq V(G)$ *such that* $\langle S\rangle = K_{1,3}$ *with its central vertex* v_4 *(respectively,* $\langle S^*\rangle = K_{1,4}$ *with its central vertex* v_4*). Define* D_{20} *(respectively,* D^*_{20}*) from* G *by adding four (respectively, three) new*

vertices v_1, v_2, v_6, v_7 (respectively, v_2, v_3, v_7) such that each vertex v_1, v_6 is adjacent with both v_3, v_4 and the vertex v_2 is adjacent with both v_4, v_5 and the vertex v_7 is adjacent with both v_4, v_8 (respectively, the vertex v_2 is adjacent with both v_4, v_5 and the vertex v_3 is adjacent with v_1, v_4, v_6 and the vertex v_7 is adjacent with both v_4, v_8).

(21) $S = \{v_3, v_5, v_6, v_8\} \subseteq V(G)$ *such that* $\langle S\rangle = P_4 : v_8, v_3, v_5, v_6$. *Define D_{21} from G by adding four new vertices v_1, v_2, v_4, v_7 such that each vertex v_1, v_4 is adjacent with both v_3, v_5 and the vertex v_7 is adjacent with both v_3, v_8 and the vertex v_2 is adjacent with both v_5, v_6.*

(22) $S = \{v_3, v_4, v_5, v_6, v_9\} \subseteq V(G)$ *such that* $\langle S\rangle = P_5 : v_4, v_5, v_6, v_3, v_9$. *Define D_{22} from G by adding four new vertices v_1, v_2, v_7, v_8 such that the vertex v_1 is adjacent with both v_4, v_5 and the vertex v_2 is adjacent with both v_5, v_6 and the vertex v_7 is adjacent with both v_3, v_6 and the vertex v_8 is adjacent with both v_3, v_9.*

(23) $S = \{v_5, v_6, v_7, v_8, v_9\} \subseteq V(G)$ *such that* $\langle S\rangle = P_5 : v_5, v_6, v_7, v_8, v_9$. *Define D_{23} from G by adding four new vertices v_1, v_2, v_3, v_4 such that the vertex v_1 is adjacent with both v_5, v_6 and the vertex v_2 is adjacent with both v_6, v_7 and the vertex v_3 is adjacent with both v_7, v_8 and the vertex v_4 is adjacent with both v_8, v_9.*

(24) $S = \{v_2, v_4, v_6, v_7, v_9\} \subseteq V(G)$ *such that* $\langle\{v_2, v_4, v_6, v_7\}\rangle = K_{1,3}$ *with its central vertex v_4 and $v_7v_9 \in E(G)$. Define D_{24} from G by adding four new vertices v_1, v_3, v_5, v_8 such that the vertex v_1 is adjacent with both v_2, v_4 and the vertex v_3 is adjacent with both v_6, v_d and the vertex v_5 is adjacent with both v_4, v_7 and the vertex v_8 is adjacent with both v_7, v_9.*

(25) $S = \{v_2, v_4, v_6, v_7, v_9\} \subseteq V(G)$ *such that* $\langle S\rangle = K_{1,4}$ *with its central vertex v_4. Define D_{25} from G by adding four new vertices v_1, v_3, v_5, v_8 such that the vertex v_1 is adjacent with both v_2, v_4 and the vertex v_3 is adjacent with both v_4, v_6 and the vertex v_5 is adjacent with both v_4, v_7 and the vertex v_8 is adjacent with both v_4, v_9.*

(26) $S = \{v_4, v_5, v_7, v_8\}$ *(respectively,* $S^* = \{v_1, v_2, v_3, v_5, v_7, v_8\}$*)* $\subseteq V(G)$ *such that $v_4v_5, v_7v_8 \in E(G)$ (respectively, $\langle\{v_1, v_2, v_3, v_5\}\rangle = K_{1,3}$ with its central vertex v_5 and $v_7v_8 \in E(G)$). Define D_{26} (respectively, D^*_{26}) from G by adding four (respectively, two) new vertices v_1, v_2, v_3, v_6 (respectively, d,f) such that each vertex v_1, v_2, v_3 is adjacent with both v_4, v_5 and the vertex v_6 is adjacent with both v_7, v_8. (respectively, the vertex v_4 is adjacent with v_1, v_2, v_3, v_5 and the vertex v_6 is adjacent with both v_7, v_8).*

(27) $S = \{v_2, v_3, v_4, v_7, v_8\} \subseteq V(G)$ *such that* $\langle\{v_2, v_3, v_4\}\rangle = P_3 : v_3, v_4, v_2$ *and $v_7v_8 \in E(G)$. Define D_{27} from G by adding three new vertices v_1, v_5, v_6 such that the vertex v_1 is adjacent with v_2, v_3, v_4 and the vertex v_5 is adjacent with both v_2, v_4 and the vertex v_6 is adjacent with both v_7, v_8.*

(28) $S = \{v_2, v_4, v_6, v_7\}$ *(respectively,* $S^* = \{v_2, v_4, v_5, v_7, v_8\}$*) [respectively,* $S^{**} = \{v_1, v_3, v_4, v_5, v_7, v_8\}$*]* $\subseteq V(G)$ *such that $v_2v_4, v_6v_7 \in E(G)$ (respectively, $\langle\{v_5, v_7, v_8\}\rangle = P_3 : v_5, v_7, v_8$ and $v_2v_4 \in E(G)$) [respectively, $\langle\{v_1, v_3, v_4\}\rangle = P_3 : v_1, v_4, v_3$ and $\langle\{v_5, v_7, v_8\}\rangle = P_3 : v_5, v_7, v_8$]. Define D_{28} (respectively, D^*_{28}) [respectively, D^*_{28}] from G by adding four (respectiely, three) [respectively, two] new vertices v_1, v_3, v_5, v_8 (respectively, v_1, v_3, v_6) [respectively, v_2, v_6] such that each vertex v_1, v_3 is adjacent with both v_2, v_4 and each vertex v_5, v_8 is adjacent with both v_6, v_7 (respectively, each vertex v_1, v_3 is adjacent with both v_2, v_4 and the vertex v_6 is adjacent with v_5, v_7, v_8) [respecitvely, the vertex v_2 is adjacent with v_1, v_3, v_4 and the vertex v_6 is adjacent with v_5, v_7, v_8].*

(29) $S = \{v_3, v_4, v_5, v_6, v_8, v_9\} \subseteq V(G)$ *such that* $\langle\{v_3, v_4, v_5, v_6\}\rangle = P_4 : v_3, v_4, v_5, v_6$ *and $v_8v_9 \in E(G)$. Define D_{29} from G by adding three new vertices v_1, v_2, v_7 such*

that the vertex v_1 *is adjacent with* v_3, v_4, v_5 *and the vertex* v_2 *is adjacent with both* v_5, v_6 *and the vertex* v_7 *is adjacent with both* v_8, v_9.

(30) $S = \{v_3, v_4, v_5, v_8, v_9\}$ *(respectively,* $S^* = \{v_1, v_4, v_5, v_6, v_8, v_9\}$*)* $\subseteq V(G)$ *such that* $\langle\{v_3, v_4, v_5\}\rangle = P_3 : v_3, v_4, v_5$ *and* $hi \in E(G)$ *(respectively,* $\langle\{v_1, v_4, v_5, v_6\}\rangle = K_{1,3}$ *with its central vertex* v_4 *and* $v_8v_9 \in E(G)$*). Define* D_{30} *(respectively,* D^*_{30}*) from* G *by adding four (respectively, three) new vertices* v_1, v_2, v_6, v_7 *(respectively,* v_2, v_3, v_7*) such that each vertex* v_1, v_6 *is adjacent with both* v_3, v_4 *and the vertex* v_2 *is adjacent with both* v_4, v_5 *and the vertex* v_7 *is adjacent with both* v_8, v_9 *(respectively, the vertex* v_3 *is adjacent with* v_1, v_4, v_6 *and the vertex* v_2 *is adjacent with both* v_4, v_5 *and the vertex* v_7 *is adjacent with both* v_8, v_9*).*

(31) $S = \{v_2, v_4, v_7, v_8, v_9\}$ *(respectively,* $S^* = \{v_1, v_3, v_4, v_7, v_8, i\} \subseteq V(G)$ *such that* $\langle\{v_7, v_8, v_9\}\rangle = P_3 : v_8, v_7, v_9$ *and* $bd \in E(G)$ *(respectively,* $\langle\{a, c, d\}\rangle = P_3 : v_1, v_4, v_3$ *and* $\langle\{v_7, v_8, v_9\}\rangle = P_3 : v_8, v_7, v_9$*). Define* D_{31} *(respectively,* D^*_{31}*) from* G *by adding four (respectively, three) new vertices* v_1, v_3, v_5, f *(respectively,* v_2, v_4, v_5*) such that each vertex* v_1, v_3 *is adjacent with both* v_2, v_4 *and the vertex* v_5 *is adjacent with both* v_7, v_8 *and the vertex* f *is adjacent with both* v_7, v_9 *(respectively, the vertex* v_2 *is adjacent with* v_1, v_4, v_3 *and the vertex* v_5 *is adjacent with both* v_7, v_8 *and the vertex* v_6 *is adjacent with both* v_7, v_9*).*

(32) $S = \{v_4, v_5, v_6, v_7, v_9, v_{10}\} \subseteq V(G)$ *such that* $\langle\{v_4, v_5, v_6, v_7\}\rangle = P_4 : v_4, v_5, v_6, v_7$ *and* $v_9v_{10} \in E(G)$*. Define* D_{32} *from* G *by adding four new vertices* v_1, v_2, v_3, v_8 *such that the vertex* v_1 *is adjacent with both* v_4, v_5 *and the vertex* v_2 *is adjacent with both* v_5, v_6 *and the vertex* v_3 *is adjacent with both* v_6, v_7 *and the vertex* v_8 *is adjacent with both* v_9, v_{10}.

(33) $S = \{v_2, v_4, v_6, v_7, v_9, v_{10}\} \subseteq V(G)$ *such that* $\langle\{v_2, v_4, v_6, v_7\}\rangle = K_{1,3}$ *with its central vertex* v_4 *and* $v_9v_{10} \in E(G)$*. Define* D_{33} *from* G *by adding four new vertices* v_1, v_3, v_5, v_8 *such that the vertex* v_1 *is adjacent with both* v_2, v_4 *and the vertex* v_3 *is adjacent with both* v_4, v_6 *and the vertex* v_5 *is adjacent with both* v_4, v_7 *and the vertex* v_8 *is adjacent with both* v_9, v_{10}.

(34) $S = \{v_3, v_4, v_5, v_8, v_9, v_{10}\} \subseteq V(G)$ *such that* $\langle\{v_3, v_4, v_5\}\rangle = P_3 : v_4, v_3, v_5$ *and* $\langle\{v_8, v_9, v_{10}\}\rangle = P_3 : v_{10}, v_8, v_9$*. Define* D_{34} *from* G *by adding four new vertices* v_1, v_2, v_6, v_7 *such that the vertex* v_1 *is adjacent with both* v_3, v_4 *and the vertex* v_2 *is adjacent with both* v_3, v_5 *and the vertex* v_6 *is adjacent with both* v_8, v_9 *and the vertex* v_7 *is adjacent with both* v_8, v_{10}.

(35) $S = \{v_2, v_4, v_6, v_7, v_9, v_{10}\}$ *(respectively,* $S^* = \{v_1, v_3, v_4, v_6, v_7, v_9, v_{10}\}$*)* $\subseteq V(G)$ *such that* $v_2v_4, v_6v_7, v_9v_{10} \in E(G)$ *(respectively,* $\langle\{v_1, v_3, v_4\}\rangle = P_3 : v_1, v_4, v_3$ *and* $v_6v_7, v_9v_{10} \in E(G)$*). Define* D_{35} *(respectively* D^*_{35}*) from* G *by adding four (respectively, three) new vertices* v_1, v_3, v_5, v_8 *(respectively,* v_2, v_5, v_8*) such that each vertex* v_1, v_3 *is adjacent with both* v_2, v_4 *and the vertex* v_5 *is adjacent with both* v_6, v_7 *and the vertex* v_8 *is adjacent with both* v_9, v_{10} *(respectively, the vertex* v_2 *is adjacent with* v_1, v_3, v_5 *and the vertex* v_5 *is adjacent with both* v_6, v_7 *and the vertex* v_8 *is adjacent with both* v_9, v_{10}*).*

(36) $S = \{v_3, v_4, v_5, v_7, v_8, v_{10}, v_{11}\} \subseteq V(G)$ *such that* $< \{v_3, v_4, v_5\} >= P_3 : v_4, v_3, v_5$ *and* $v_7v_8, v_{10}v_{11} \in E(G)$*. Define* D_{36} *from* G *by adding four new vertices* v_1, v_2, v_6, v_9 *such that the vertex* v_1 *is adjacent with both* v_3, v_4 *and the vertex* v_2 *is adjacent with both* v_3, v_5 *and the vertex* v_6 *is adjacent with both* v_7, v_8 *and the vertex* v_9 *is adjacent with both* v_{10}, v_{11}.

(37) $S = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\} \subseteq V(G)$ *such that* $u_iv_i \in E(G)$ *for all* $i = 1, 2, 3, 4$*. Define* D_{37} *from* G *by adding four new vertices* a_1, a_2, a_3, a_4 *such that the vertex* a_i *is adjacent with both* u_i, v_i *for all* $i = 1, 2, 3, 4$.

Then D_i(*respectively* $(D_i^*), 1 \leq i \leq 37$ *is a divisor graph containing the graph* H_i *of Figure 1 as a subgraph.*

We are now prepared to determine all forbidden subgraphs for connected divisor graphs that contain exactly four triangles. We will only outline the proof of this result.

Theorem 2.1. *Let* G *be a connected graph that contains exactly four triangles and no other induced odd cycles. Then* G *is a divisor graph if and only if* G *does not contain any of the graphs in the Figure 4 as an induced subgraph, where each dashed line represents an edge that may or may not be present.*

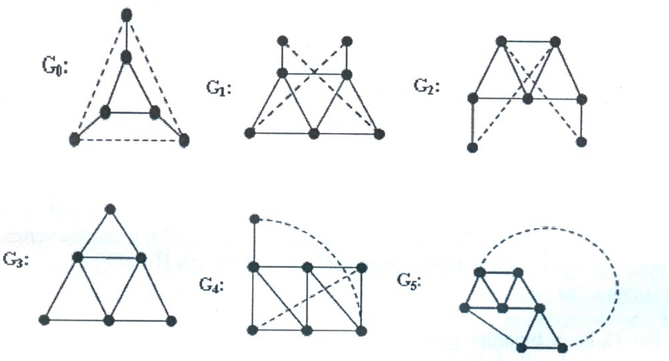


FIGURE 4. Non divisor graphs

Proof. Since each graph in Figure 4 is not a divisor graph, it follows by Lemma 1.1 that if G contains any of the graphs of Figure 4 as an induced subgraph, then G is not a divisor graph. For the converse, assume that G does not contain any of the graphs of Figure 4 as an induced subgraph. We show that G is a divisor graph. Since G contains exactly four triangles and no other induced odd cycles, it follows that G contains exactly one of the graphs $H_i (1 \leq i \leq 37)$ shown in Figure 1 as subgraph. Since each $H_i (1 \leq i \leq 37)$ is a divisor graph, by Lemma 1.1, we may assume that $G \neq H_i$. We consider these 37 cases.

Case 1. G contains H_1 as a subgraph.

If $|V(G) - V(H_1)| \leq 1$, then $G = D_1$ or $G = D_1^*$, since G has exactly four triangles. Thus G is a divisor graph by Lemma 1.1. Thus we may assume that $|V(G) - V(H_1)| \geq 2$. Since G does not contain G_0 as an induced subgraph, it follows that

(1) at least one of $deg\ v_1 = 3, deg\ v_2 = 3, deg\ v_5 = 4$ (and)
(2) at least one of $deg\ v_2 = 3, deg\ v_3 = 3, deg\ v_5 = 4$ (and)
(3) at least one of $deg\ v_1 = 3, deg\ v_4 = 3, deg\ v_5 = 4$ (and)
(4) at least one of $deg\ v_3 = 3, deg\ v_4 = 3, deg\ v_5 = 4$.

We have the following subcases

1.1. $deg\ v_1 = 3, deg\ v_3 = 3$
1.2. $deg\ v_2 = 3, deg\ v_4 = 3$
1.3. $deg\ v_5 = 4$.

Subcase 1.1. $deg\ v_1 = 3, deg\ v_3 = 3$.

$G - \{v_1, v_3\}$ is a bipartite graph and so $G = D_1$. Thus G is a divisor graph by Lemma 1.1. Similar proof holds for the subcase 1.2.

Subcase 1.3. $deg\ v_5 = 4$.

$G - \{v_5\}$ is a bipartite graph and so $G = D_1^*$. Thus G is a divisor graph by Lemma 2.2. If

(1) $deg\ v_1 > 3$ and $deg\ v_2 > 3$ and $deg\ v_5 > 4$ (or)
(2) $deg\ v_2 > 3$ and $deg\ v_3 > 3$ and $deg\ v_5 > 4$ (or)
(3) $deg\ v_1 > 3$ and $deg\ v_4 > 3$ and $deg\ v_5 > 4$ (or)
(4) $deg\ v_3 > 3$ and $deg\ v_4 > 3$ and $deg\ v_5 > 4$,

then G contains G_0 as an induced subgraph, which is impossible.

Case 2. G contains H_2 as a subgraph.

If $|V(G) - V(H_2)| \leq 1$, then $G = D_2$ or $G = D_2^*$, since G has exactly four triangles. Thus G is a divisor graph by Lemma 2.2. Thus we may assume that $|V(G) - V(H_2)| \geq 2$. Since G does not contain G_0 as an induced subgraph, it follows that

(1) at least one of $deg\ v_1 = 2, deg\ v_6 = 5, deg\ v_5 = 5$ (and)
(2) at least one of $deg\ v_2 = 2, deg\ v_6 = 5, deg\ v_5 = 5$ (and)
(3) at least one of $deg\ v_3 = 2, deg\ v_6 = 5, deg\ v_5 = 5$ (and)
(4) at least one of $deg\ v_4 = 2, deg\ v_6 = 5, deg\ v_5 = 5$.

We have the following subcases
2.1. $deg\ v_1 = 2, deg\ v_2 = 2, deg\ v_3 = 2, deg\ v_4 = 2$
2.2. $deg\ v_5 = 5$
2.3. $deg\ v_6 = 5$.

Subcase 2.1. $deg\ v_1 = 2, deg\ v_2 = 2, deg\ v_3 = 2, deg\ v_4 = 2$

$G - \{v_1, v_2, v_3, v_4\}$ is a bipartite graph and so $G = D_2$. Thus G is a divisor graph by Lemma 2.2.

Subcase 2.2. $deg\ v_5 = 5$.

$G - \{v_5\}$ is a bipartite graph and so $G = D_2^*$. Thus G is a divisor graph by Lemma 2.2. Proof is similar to the Subcase 2.3. If

(1) $deg\ v_1 > 2$ and $deg\ v_6 > 5$ and $deg\ v_5 > 5$ (or)
(2) $deg\ v_2 > 2$ and $deg\ v_6 > 5$ and $deg\ v_5 > 5$ (or)
(3) $deg\ v_3 > 2$ and $deg\ v_6 > 5$ and $deg\ v_5 > 5$ (or)
(4) $deg\ v_{>}2$ and $deg\ v_6 > 5$ and $deg\ v_5 > 5$,

then G contains G_0 as an induced subgraph, which is impossible.

It is verified that G is a divisor graph for the Case k: G contains H_k as a subgraph for $3 \leq k \leq 37$. □

We conclude this paper with the following forbidden subgraph characterization for connected divisor graphs that contain at most four triangles.

Corollary 2.1. *G is a connected graph that contains at most four triangles and no other induced odd cycles. Then G is a divisor graph if and only if G does not contain any of the graphs in Figure 4 as an induced subgraph, where each dashed line induced subgraph, where each dashed line respresents an edge that may or may not present.*

Proof. Combining Theorem 1.2 and Theorem 2.1, we get the forbidden subgraph characterization for connected divisor graphs that contain at most four triangles. □

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