TWMS J. App. and Eng. Math. V.14, N.1, 2024, pp. 322-335

QUASI-BLOCK TOEPLITZ MATRIX IN MATLAB

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ABSTRACT. In this paper we try to approximate any properties of quasi-block Toeplitz matrix (QBT), by means of a finite number of parameters. A quasi-block Toeplitz (QBT) matrix is a semi-infinite block matrix of the kind $F = T(F) + E$ where $T(F) = (F_{j-k})_{j,k\in\mathbb{Z}}$, that F_k are $m \times m$ matrices such that $\sum_{i\in\mathbb{Z}} |F_i|$ has bounded entries, and $E = (e_{i,j})_{i,j\in\mathbb{Z}^+}$ is a compact correction. Also, we should say the norms $\| F \|_w = \sum_{i\in\mathbb{Z}} \| F_i \|$ and $\| E \|_2$ are finite. QBT-matrices are done with any given precision. The norm $\| F \|_{\mathcal{QBT}} = \alpha \| F \|_w + \| E \|_2$, is for $\alpha = (1+\sqrt{5})/2$. These matrices are a Banach algebra with the standard arithmetic operations. We try to analysis some structures and computational properties for arithmetic operations of QBT matrices with some MATLAB commands.

Keywords: Quasi-Block Toeplitz matrix, Banach algebra, Matlab.

AMS Subject Classification: 65F30, 60B20.

1. INTRODUCTION AND PRELIMINARIES

Some properties of the r-Toeplitz matrix $A_n = [a_{ij}]$ $(a_{i+r,j+r} = a_{ij},\ \ i,j = 1,2,\ldots,n-r)$ was obtained in [9, 19]. A Toeplitz matrix is an r-Toeplitz matrix when $r = 1$.
Nowadays, the matrices one encounters in applications are often not Toeplitz matrices but block Toeplitz matrices (as matrices with Toeplitz block). A block Toeplitz matrix is different from a matrix with Toeplitz blocks. For instance, if A is Toeplitz and B is any matrix, then $A \otimes B$ is a block Toeplitz matrix whereas $B \otimes A$ is a block matrix with Toeplitz blocks. Here $\otimes$ is the Kronecker product.
Many results on Toeplitz matrices can be generalized to block Toeplitz matrices, although this is usually a hard work [8].
The interest of the study of block Toeplitz matrices appears to be very important not only from a theoretical point of view in linear algebra or numerical analysis, e.g., but also in applications such as ranging from imaging to Markov chains, sound propagation problem, queuing models, finance to the solution of PDEs, Yule-Walker equations, and classic Levinson recursion algorithm [16].

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§ Manuscript received: November 09, 2021; accepted: February 24, 2022.
TWMS Journal of Applied and Engineering Mathematics, Vol.14, No.1

In [1], a description is provided of finitely representable QT matrices, together with the analysis of the computational properties of their arithmetic then an implementation is shown of QT matrices in the form of a MATLAB toolbox, called `cqt-toolbox`, for "computing with quasi-Toeplitz matrices".

This motivated us to sum some information for block Toeplitz matrices in this way.

Random walks in the quarter plain are modelled by a Markov chain where the set of states is formed by the pairs (i, j) for $i, j \in \mathbb{Z}^+$, and the probability transition matrix is block Toeplitz with Toeplitz block except for the (block) entries in the upper left corner, see [14, 17]. In these situations, the blocks can be written in the form $T(F) + E$, where $T(F) = (F_{j-i})$ is the Toeplitz matrix associated with the sequence $F = \{F_i\}_{i\in\mathbb{Z}}$, while E is a matrix having only a finite number of nonzero entries containing the information concerning the boundary conditions.

The numerical treatment of problems and the computation of interesting quantities related to these models are done by truncating the size to a finite large value, by solving the finite problem obtained this way and using this finite solution to approximate part of the solution of the infinite problem, that is, the steady-state vector. This computation can be carried out by solving a suitable quadratic matrix equations whose coefficients are given by the blocks of the transition probability matrix.

The class QT of semi-infinite quasi-Toeplitz (QT) matrices that has been introduced in [1, 2, 3, 4, 5], forms a Banach algebra if endowed with a suitable norm, and enables the implementation of an approximate matrix arithmetic. This set is formed by matrices of the kind $A = T(a) + E$ where, in general, $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i$ is a Laurent series such that $\| a \|_w := \sum_{i\in\mathbb{Z}} |a_i|$ is finite, E is a compact correction. The norm which makes the class QT a Banach algebra is defined by $\| A \|_{\mathcal{QT}} := \alpha \| a \|_w + \| E \|_2, \quad \alpha = (1 + \sqrt{5})/2$.

In particular, each element of this class can be approximated with the sum of a banded Toeplitz $T(\tilde{a})$ plus a matrix $\tilde{E}$ with finite support, at any arbitrary precision. This kind of analysis motivated us to introduce and analyze a block extension of the concept of QT matrices. In fact, in this paper we introduce and analyze the class QBT of quasi block Toeplitz matrices formed by matrices which are the sum of a block-Toeplitz matrix and a compact correction where the acronym cqbt stands for "computing with quasi-block Toeplitz matrices".

In this paper, we analyze the representation of QBT matrices with the finite floating point representation of real numbers by means of a finite number of parameters. We did this work by investigating of some computational issues of a matrix arithmetic in this class and providing an effective implementation of the class of finitely representable QBT matrices, with the related matrix arithmetic by MATLAB software.

In fact, [1] does not deal with block Toeplitz matrices, we extend the results in [1] for QBT matrices, that still makes the set QBT a Banach algebra

$$\| F \|_{\mathcal{QBT}} = \alpha \| F \|_w + \| E \|_2, \quad \alpha = (1 + \sqrt{5})/2.$$

Here, F is a matrix-valued function in the Wiener class specify that $F(z) = \sum_{i\in\mathbb{Z}} z^i F_i$, and $T(F) = (F_{j-k})_{j,k\in\mathbb{Z}}$, $\{F_k\}_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients of F, then

$$\| F \|_w = \sum_{i=-\infty}^{\infty} \| F_i \| < \infty,$$

and E is a compact correction, see [1].

The rest of the paper is organized as follows:

Some definitions and theoretical results about QBT matrices, together with the norm

QBT are provided in Section 2.
Some cqbt commands are shown in Section 3. The definition and the analysis of the arithmetic operations in the algebra of finitely representable QT matrices are explained in Section 4. Then in the first subsections, we deal with addition, multiplication, and inversion.
Some results of the class of finitely representable QBT matrices, with the related matrix arithmetic by MATLAB software are expanded in Section 5.

2. The class of QBT matrices

The starting point of our derivation of a class of QBT matrices will be to indicate with a function in the Wiener class $\mathcal{W}$ formed by $F(w) = \sum_{k=-\infty}^{\infty} e^{kwi} F_k$ where $w \in \mathbb{R},\ F_k \in \mathbb{C}^{N\times N}$ with $k \in \mathbb{Z}$, and i denotes the imaginary unit.
The matrix-valued function $F : \mathbb{R} \to \mathbb{C}^{N\times N}$ is continuous and $2\pi-$periodic, and $\sum_{k=-\infty}^{k=\infty} |[F_k]_{r,s}| < \infty,\ \ 1 \le r \le N,\ 1 \le s \le N$, that $\{F_k\}_{k=-\infty}^{\infty}$ are the sequence of Fourier coefficients of F:

$$F_k = \frac{1}{2\pi}\int_0^{2\pi} F(w)e^{-ikw}dw. \tag{1}$$

The block analogue of Theorems 1.14 and 1.15 of [6] is Theorem 6.6 (Gohberg and M.G. Krein) of [6]. Gohberg and Krein say that the operator induced by $T(F)$ is invertible if and only if $F(w)$ has a canonical right Wiener-Hopf factorization.
Here,

$$T(F) = \begin{pmatrix} F_0 & F_{-1} & F_{-2} & \dots & \dots \\ F_1 & F_0 & F_{-1} & \ddots & \vdots \\ F_2 & F_1 & F_0 & F_{-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$

Gohberg and Feldman were the first to prove that the finite section method is applicable to $T(F)$ if and only if both $T(F)$ and $T(\tilde{F})$ are invertible, see Theorem VIII.5.3 of [10].
Equivalently, the finite section method is applicable to $T(F)$ if and only if $F(w)$ has a canonical right and a canonical left Wiener-Hopf factorization, [7].
$\tilde{F}(w) := F(1/w)$ that $(w = e^{i\theta} \in \mathbb{T})$, $\mathbb{T}$ stands for the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$.
Also, the formula of Proposition 1.12 of [8] and Theorem 2.1 of [1] remains true in the matrix case, namely:

$$T(FG) = T(F)T(G) + H(F)H(\tilde{G}),$$

here, $F,\ G$ are the matrix functions and $H(F),\ H(\tilde{G})$ are the Hankel operators.

Definition 2.1. *The semi-infinite block matrix F is quasi-block Toeplitz matrix (QBT) if it can be written in the form*

$$F = T(F) + E$$

where $F(w) = \sum_{k=-\infty}^{\infty} e^{kwi} F_k$ is in the Wiener class, and $E = (e_{i,j})_{i,j\in\mathbb{Z}^+}$ is compact.

If E has finite support, i.e., if only a finite number of entries is nonzero. In this case the nonzero entries of E will stay in a sufficiently large leading principal submatrix (in the top left corner). These nonzero entries will affect a finite number of blocks of $T(F)$ so that $\mathbf{F}$ is block Toeplitz except in the blocks intersecting the leading principal submatrix.

In particular, the blocks can be written in the form $T(F) + E$, where $T(F) = (F_{j-i})$ is the Toeplitz matrix associated with the sequence
$F = \{F_i\}_{i\in\mathbb{Z}}$, $\| F \|_w = \sum_{i=-\infty}^{\infty} \| F_i \| < \infty$, while E is a matrix having only a finite number of nonzero entries containing the information concerning the boundary conditions.
Without loss of generality, we can extend the norm $\| \,.\, \|_{\mathcal{QT}}$ and form the Banach algebra for the class of QBT matrices that shows by $\| \,.\, \|_{\mathcal{QBT}}$.
Here, we have

$$\| F \|_{\mathcal{QBT}} = \alpha \| F \|_w + \| E \|_2, \quad \alpha = (1+\sqrt{5})/2,$$

and

$$\| FG \|_{\mathcal{QBT}} \leq \| F \|_{\mathcal{QBT}} \| G \|_{\mathcal{QBT}} .$$

This norm tries to solve the difficult to compute numerically by the l^2 norm and to complete the linear space of QBT matrices, similar $\| \,.\, \|_{\mathcal{QT}}$ of QT matrices.

Lemma 2.1. *Let $F = T(F) + E \in \mathcal{QBT}$ and $\epsilon > 0$. Then, we can have the negative integers n_-, n_+, n_r, n_c such that the matrix $\hat{F} = T(\hat{F}) + \hat{E}$ and*

$$\hat{E}_{i,j} = \begin{cases} E_{i,j} & if \quad 1 \leq i \leq n_r \\ 0 & otherwise. \end{cases} \tag{3}$$

Also, we can show $\| F - \hat{F} \|_{\mathcal{QBT}} \leq \| F \|_{\mathcal{QBT}} .\epsilon$.

Proof. From $F \in \mathcal{QBT}$, we have $\| F \|_w = \sum_{i=-\infty}^{\infty} \| F_i \| < \infty$, then there exist n_-, n_+ such that

$$\| F - \hat{F} \|_w = \sum_{k<-n_-} |F_k| + \sum_{k>-n_+} |F_k| \leq \epsilon \| F \|_{\mathcal{QBT}} /\alpha. \tag{4}$$

E is a compact operator that may be block form or not. If it is block form, we can delete these partitions and use the process of Lemma 2.4 of [1] for continuing to prove. ☐

Remark 2.1. *If matrix E is block form we can have $F = T(F) + E \in \mathcal{QBT}$, ordinary. Otherwise, we can set these partitions for continuing to prove.*
These commands do in MATLAB by

```
>>mat2cell(.)
```

and

```
>>cell2mat(.)
```

It goes without saying that the representation of semi-infinite quasi Toeplitz matrices and floating point numbers makes result that it possible to draw an analogy between semi-infinite quasi block Toeplitz matrices and floating point numbers. Where

$$fl(a) = a + \varepsilon, \quad |\varepsilon| \leq |a|.\epsilon,$$

that a is the real number in floating point format $fl(a)$ and ϵ is the so-called unit roundoff. Analogously to the operator "fl(.)" for floating point format and $\mathcal{QT}(.)$ for quasi Toeplitz matrices, we introduce a "truncation" operator $\mathcal{QBT}(.)$ that works on the block Toeplitz and on the compact correction, see Lemma 2.1. Here,

$$\mathcal{QBT}(F) = T(\hat{F}) + \hat{E} = F + \varepsilon, \quad \| \varepsilon \| \leq \| F \|_{\mathcal{QBT}} .\epsilon, \tag{5}$$

where ε is some prescribed tolerance set a priori, $\mathcal{QBT}(F)$ is a finite support of QBT-matrix $F = T(F) + E$ by the sum of a banded block Toeplitz matrix $T(\hat{F})$ and a semi-infinite matrix $\hat{E}$.

3. cqbt commands

$\mathcal{QBT}(F)$ is a class of finitely representable quasi-block Toeplitz matrices, where, the lengths of the representations are not constant and can vary in order to guarantee a uniform bound to the relative error in norm. Namely in this case, they are unlike of floating point numbers.

The cqbt-commands collect some commands for operating with finitely representable quasi block Toeplitz matrices. The block Toeplitz part is stored into two block vectors containing the coefficients of the symbol with non positive and with non negative indices of Fourier coefficients, respectively. The compact correction is represented in terms of two matrices $\hat{U} \in \mathbb{R}^{n_r \times k}$ and $\hat{V} \in \mathbb{R}^{n_c \times k}$ such that $\hat{E}(1 : n_r, 1 : n_c) = \hat{U}\hat{V}^T$ coincides with the upper left corner of the correction.

The following MATLAB commands help us to build a block Toeplitz matrix:

See Appendix A.

We define a new finitely representable $\mathcal{QBT}$ matrix by the cqbt function:

```
>> F=cqbt(neg, pos, E);
```

In the above command, the block vectors pos and neg contain the sequence of Fourier coefficients of F with non positive and non negative indices, respectively, and E is a finite section of the correction representing its nonzero part. Here, E represents a compact correction in the upper left corner, this result can be easily generalized to the lower right corner. Handling two separate corrections is convenient as long as they do not overlap.

4. Arithmetic operations

For the floating point operations where $\odot$ is any basilar arithmetic operation (sum, subtraction, multiplication, and division), we have

$$fl(a \odot b) = a \odot b + \varepsilon, \quad |\varepsilon| \leq (a \odot b).\epsilon.$$

For the matrix arithmetic in the set of finitely representable QT matrices, we have

$$\mathcal{QT}(A \odot B) + \varepsilon, \quad \| \varepsilon \| \leq \epsilon \| A \odot B \|_{QT},$$

for any pair of finitely representable $A, B \in \mathcal{QT}$ and $\odot \in \{+, -, *, /, \ \backslash\}$.

Now for the block matrix arithmetic in the set of finitely representable QBT matrices, we define

$$\mathcal{QBT}(A \odot B) + \varepsilon, \quad \| \varepsilon \| \leq \epsilon \| A \odot B \|_{QBT},$$

for any pair of finitely representable $A, B \in \mathcal{QBT}$.

We need to apply the $\mathcal{QBT}(.)$ operator, for an arithmetic operation between finitely representable QBT matrices might not be finitely representable and for it optimizes the memory usage. Since, this method minimizes the number of parameters required to store the data up to the required accuracy, similar this work is done for QT matrices in [1].

This arithmetic operations can be represented only the nonzero sections of infinite objects by usual operators $+, -, *, /, \ \backslash$ in the cqbt-toolbox by MATLAB. This work is done on operations between vectors and matrices that might be of non compatible sizes, e.g., sum of vectors with different lengths.

4.1. **Addition.** Some properties of sequences of block Toeplitz matrices generated by continuous matrix-valued functions is proven by asymptotically equivalent sequences of Matrices [12].
For two finitely representable QBT matrices $F = T(F) + E_F$ and $G = T(G) + E_G$ the block matrix $H = F + G$ is defined by $H(w) = F(w) + G(w)$ and $E_H = E_F + E_G$. The factorization $E_H = U_H V_H$ is given by

$$U_H = [U_F, U_G], \quad V_H = [V_F, V_G]. \tag{6}$$

$\hat{U}_b oldH$ and $\hat{V}_H$ show a lower number of columns U_H and V_H such that $\| E_H - \hat{U}_H \hat{V}_H^T \|_2$ is sufficiently small. Then, we have

$$\mathcal{QBT}(F + G) = F + G + \varepsilon, \quad \| \varepsilon \|_{QT} \leq \epsilon \| F + G \|_{QBT} .$$

Here, ε is shown the local error of the addition. The original QBT matrices F and G are represented by approximations

$$\hat{F} = F + \varepsilon_F, \quad \hat{G} = G + \varepsilon_G. \tag{7}$$

So, the computed matrix is

$$\mathcal{QBT}(\hat{F} + \hat{G}) - (F + G) = \varepsilon_F + \varepsilon_G + \varepsilon,$$

where $\varepsilon_F + \varepsilon_G$ is the inherent error, ε is the local error, and the sum of the local error and the inherent error is the global error.

4.2. **Multiplication.** As we know from Proposition 1.12 of [8] and Theorem 2.1 of [1], multiplication remains true in the matrix case, namely:

$$H = FG = T(FG) - H(F)H(\tilde{G}) = T(H) + E_H, \tag{8}$$

where $H(w) = F(w)G(w)$, F, G are the matrix functions and $H(F)$, $H(\tilde{G})$ are the Hankel operators.
Then, we have

$$H = FG = (T(F) + E_F)(T(G) + E_G) = T(F)T(G) + E_F T(G) + E_G T(F) + E_F E_G,$$

From asymptotic result about the product of block Toeplitz matrices that was given in Theorem 6.2. of [12],

$$H = T(FG) + E_H$$

that inherent error is

$$E_H = E_F T(G) + E_G T(F) + E_F E_G.$$

The local error is defined by

$$\mathcal{QBT}(H) - \mathcal{QBT}(FG) = \varepsilon,$$

that $\| \varepsilon \|_{QT} \leq \epsilon \| FG \|_{QBT}$.
The global error is

$$\mathcal{QBT}(\hat{F}\hat{G}) - FG = \varepsilon_F G + \varepsilon_G F + \varepsilon_F \varepsilon_G + \varepsilon.$$

without loss of generality, neglecting the quadratic part $\varepsilon_F \varepsilon_G$.

4.3. **Inversion.** A famous theorem by Gohberg and Krein says that the operator induced by $T(F)$ is invertible if and only if $F(w)$ has a canonical right Wiener-Hopf factorization; see Theorem VIII.5.3 of [10] or [11].
Gohberg and Feldman were the first to prove that the finite section method is applicable to $T(F)$ if and only if both $T(F)$ and $T(\tilde{F})$ are invertible.
They prove that if

$$m_1 \leq m_2 \leq \ldots \leq m_n$$

will be the right partial indices of the matrix function F, then $T(F)$ is invertible if and only if all its right partial indices are zero. Also, from Theorems 6.5 and 6.6 of [6], we have

$$m_1 + \ldots + m_n = wind(det\ T(F), 0).$$

Equivalently, the finite section method is applicable to $T(F)$ if and only if $F(w)$ has a canonical right and a canonical left Wiener-Hopf factorization, here $\tilde{F}(w) := F(1/w)$.
For interested readers, some similar topics like the inversion of the Toeplitz two-level matrices in [15, 18] is discussed.

4.3.1. *Truncated block Toeplitz matrices.* Let $F = T(F) + E_F \in \mathcal{QBT}$ be a finitely representable QBT matrix such that the symbol $F(w) = \sum_{k=-n}^{n} e^{kwi} F_k$.
For $F \in L^\infty_{n\times n}$, we have

$$T_n(F) = P_n T(F) P_n | Im\ P_n = \begin{pmatrix} F_0 & F_{-1} & F_{-2} & \ldots & F_{1-n} \\ F_1 & F_0 & F_{-1} & \ddots & \vdots \\ F_2 & F_1 & F_0 & \ddots & F_{-2} \\ \vdots & \ddots & \ddots & \ddots & F_{-1} \\ F_{n-1} & \ldots & F_2 & F_1 & F_0 \end{pmatrix},$$

here, the projection P_n on l^2_n by

$$P_n : \{x_1, x_2, x_3, \ldots\} \mapsto \{x_1, \ldots, x_n, 0, 0, \ldots\}$$

$x_k \in C^N$.
Note that $T_n(F)$ is actually an $nN \times nN$ matrix. The operator W_n is defined on l^2_n by

$$W_n : \{x_1, x_2, x_3, \ldots\} \mapsto \{x_n, \ldots, x_1, 0, 0, \ldots\}$$

that

$$W_n T_n(F) W_n = T_n(\tilde{F}).$$

Proposition 2.12 of [8] holds in the matrix case without any changes. In particular,

$$T_n(FG) = T_n(F)T_n(G) + P_n H(F) H(\tilde{G}) P_n + W_n H(\tilde{F}) H(G) W_n \tag{9}$$

for every $F, G \in L^\infty_{n\times n}$. Also, we claim that

$$T(\tilde{F})\ \ is\ \ invertible \leftrightarrow T(F^{-1})\ \ is\ \ invertible, \tag{10}$$

if $T(\tilde{F})$ is invertible, then $T(F) - H(F)T^{-1}(\tilde{F})H(\tilde{F})$ is the inverse of $T(F^{-1})$, while if $T(F^{-1})$ is invertible, then $T(\tilde{F}^{-1}) - H(\tilde{F}^{-1})T^{-1}(F^{-1})H(F^{-1})$ is the inverse of $T(\tilde{F})$.

Corollary 4.1. *$\{T_n(F)\}$ is stable if and only if $T(F)$ and $T(\tilde{F})$ are invertible. In that case the operators*

$$K(F) := T^{-1}(F) - T(F^{-1}) \text{ and } K(\tilde{F}) := T^{-1}(\tilde{F}) - T(\tilde{F}^{-1})$$

are compact and

$$T_n^{-1}(F) = T_n(F^{-1}) + P_n K(F) + W_n K(\tilde{F}) W_n + C_n$$

where $\| C_n \| \to 0$ as $n \to \infty$.

A novel Newton method for canonical Wiener-Hopf and spectral factorization of matrix polynomials by the initial vector results from solving a block Toeplitz-like system, and the Jacobi matrix governing the Newton iteration has shown in [7].
Overall, numerical tests in [7], which were done on a usual laptop using Matlab with the standard machine precision 2^{-52} and which included highly perfidious examples, showed very good results for matrix polynomials.

Remark 4.1. *All the arithmetic operations can be done in $O(nNlog(nN))$ time relying on the fast Fourier transform (FFT). The only exception is the inversion, see* [1, 7].

5. Some results

In the following way, we try to show a new finitely representable $\mathcal{QBT}$ matrix by the cqbt function:

```
>> F=cqbt(neg, pos, E);
```

or

```
>> F=cqbt(neg, pos, U, V);
```

Example 5.1. *See Appendix B.*

Here, we have $F = T(F) + E$ that the vectors pos and neg contain the coefficients of the symbol $F(w)$ with non positive and non negative indices, respectively. E is a finite section of the correction representing its nonzero part and it is in the upper left corner. It is possible that the correction representing its nonzero part in the lower right corner. If the corrections overlap, then we switch to a single correction format, as in the semi-infinite case. This is done by storing it as an upper left correction and setting the lower right to the empty matrix. Also, in this way, we lost the sparsity and it to be convenient, the rank of the correction needs to stay small compared to the size of the matrix. Then in output page, we have:

```
CQBT Matrix of size 6 * 6

 Rank of top-left correction: 2
- Toeplitz part (leading 6 * 6 block):


 0.5573   0.7725   0.3390   0.2101   0.6289   0.1015
 0.0442   0.5573   0.1790   0.3390   0.9064   0.6289
 0.1079   0.1822   0.5573   0.7725   0.3390   0.2101
 0.9345   0.1079   0.0442   0.5573   0.1790   0.3390
 0.1932   0.8959   0.1079   0.1822   0.5573   0.7725
 0.4898   0.1932   0.9345   0.1079   0.0442   0.5573
```

```
 – Finite correction (top-left corner):

E =

     -1      1
     -3      3
```

Here, we show a small block Toeplitz matrix (F is a $3-block \times 3-block$ *matrix that every block is* 2×2*) for saving space. For example, the coefficients of the symbol* $F(w)$ *with non positive and non negative indices is shown by:*

```
>> pos{1}

ans =

     0.5573     0.7725
     0.0442     0.5573

>> neg{3}

ans =

     0.1932     0.8959
     0.4898     0.1932
```

The instructions symbol and correction for the $\mathcal{QBT}$ *matrix* F *can be expand easily. We have* $[U,V] = correction(F)$ *that*

```
>> U=[1;3];
>> V=[-1;1];
```

In particular, if the lengths of the corrections, compared to the dimension of the block matrices, are small then the support of the nonToeplitz component is split into two parts located in the upper left corner and in the lower right corner, respectively. So, we consider two separate corrections as long as they do not overlap. When, we represent finite quasi-block Toeplitz matrices by storing two additional matrices that represent the lower right correction in factorized form.
See the following steps:

1- Compute $\parallel F \parallel_{QBT}$.

2- Obtain a truncated version $\hat{F}(w)$ of the symbol $F(w)$ by discarding the sequence of Fourier coefficients of F, such that $\parallel F - \hat{F} \parallel_w \leq \parallel F \parallel_{QBT} \cdot \frac{\epsilon}{2\alpha}$.

3- Compute a compressed version $\hat{E}$ (a truncation error bounded by $\parallel F \parallel_{QBT} \cdot \frac{\epsilon}{2\alpha}$) of the correction using the SVD and dropping negligible rows and columns.

The truncation of a QBT matrix $F = T(F) + E$ is described by the above steps. It is performed by some details of the operator $\mathcal{QBT}$ on a finitely generated QBT matrix. The QBT norm enables to recognize unbalanced representations and to completely drop the negligible part.

We set $\hat{\epsilon} = \frac{\epsilon}{4} \parallel F \parallel_{QBT}$,
if $min(\parallel F_{n-} \parallel, \parallel F_{n+} \parallel) < \hat{\epsilon}$
then by the following commands
$F(w) = F(w) - F_{n-}w^{n-}, \quad \hat{\epsilon} = \hat{\epsilon} - \parallel F_{n-} \parallel$,
or
$F(w) = F(w) - F_{n+}w^{n+}, \quad \hat{\epsilon} = \hat{\epsilon} - \parallel F_{n+} \parallel$,
the sequence of Fourier coefficients of F and by
$\hat{\epsilon} = \hat{\epsilon} - \parallel E \parallel$, the correction's support,
will be truncated to provide a specified threshold.

Example 5.2. *The analysis of a random walk on the semi-infinite strip $\{0, ..., m\} \times \mathbb{N}$ is considered. The random walk to be a Markov chain, and that movements are possible only to adjacent states; that is, from (i, j), one can reach only (i', j') with $|i - i'|, |j - j'| \leq 1$, with probabilities of moving up/down and left/right not depending on the current state. Then, the transition matrix F is an infinite quasi-Toeplitz-block-quasi-Toeplitz matrix of the form*

$$F = \begin{pmatrix} \hat{F}_0 & F_{-1} & & \\ F_1 & F_0 & F_{-1} & \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

If $min(\parallel F_{-2} \parallel, \parallel F_2 \parallel) < \hat{\epsilon}$ then we only have the transition probabilities are chosen in a way that gives the following matrices:

$$F_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0 & \frac{1}{5} & & \\ \frac{1}{10} & 0 & \frac{1}{5} & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad F_{-1} = \begin{pmatrix} 1 & \frac{1}{3} & & \\ \frac{1}{2} & 1 & \frac{1}{3} & \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

properly rescaled in order to make $F_{-1} + F_0 + F_1$ a row-stochastic matrix. The matrices F_i are non negative tridiagonal Toeplitz matrices with corrections to the elements in position $(1, 1)$ and (m, m), and satisfy $(F_{-1} + F_0 + F_1)e = e$, where e is the vector of all ones. The top and bottom corrections are chosen to ensure stochasticity on the first and last rows. Let we have

```
U = randn(corr-size, 3);
V = randn(corr-size, 3);
U = U / norm(U * V') / 5;
E=U*V';
```

in the top corrections. If $n = 3; corr - size = 2$; then $Sparse(F)$ will be

```
ans =

(1,1)    -0.0153    (2,1)    -0.0897    (4,1)    0.5000
(5,1)     0.1000    (1,2)     0.1872    (2,2)    0.0626
(3,2)     0.1000    (4,2)     0.5000    (5,2)    0.5000
(6,2)     0.1000    (2,3)     0.2000    (5,3)    0.5000
(6,3)     0.5000    (1,4)     1.0000    (2,4)    0.5000
(5,4)     0.1000    (7,4)     0.5000    (8,4)    0.1000
(1,5)     0.3333    (2,5)     1.0000    (3,5)    0.5000
(4,5)     0.2000    (6,5)     0.1000    (7,5)    0.5000
(8,5)     0.5000    (9,5)     0.1000    (2,6)    0.3333
(3,6)     1.0000    (5,6)     0.2000    (8,6)    0.5000
```

```
(9,6)      0.5000    (4,7)      1.0000    (5,7)      0.5000
(8,7)      0.1000    (4,8)      0.3333    (5,8)      1.0000
(6,8)      0.5000    (7,8)      0.2000    (9,8)      0.1000
(5,9)      0.3333    (6,9)      1.0000    (8,9)      0.2000
```

Conjecture. Hankel compression to store a low-rank approximation of $H(F)$ and $H(G)$ for computing the multiplication of two block Toeplitz matrices $T(F)$ and $T(G)$ in Equation (8) is possible by some strategies such as random sampling techniques and reblocking [13, 20]. Also, the computation of block matrix functions, norms, and extraction of submatrices will be investigated in future works.

6. Conclusions

In this paper, we have introduced a suitable norm for approximating any QBT matrix by means of a finitely representable matrix within a given relative error bound. Then, we have analyzed the class of quasi- block Toeplitz matrices by this norm. We try to expand some computational aspects of a block matrix arithmetic by Matlab toolbox.

Acknowledgement. The author thanks the authors whose work largely constitutes this sample work such as professor Dario A. Bini and his co-workers. The author would also like to thank the anonymous reviewers for their helpful comments.

Appendix A. Build a block Toeplitz matrix. This appendix provides a plain Matlab implementation of block Toeplitz matrix.

```
n=input(Please enter the number of blocks (n):);
N=input(Please enter the size of blocks (N):);
 for i=1:n
        c1=input(Please enter a vector of size (N):);
        r1=input(Please enter a vector of size (N):);
        c1(1)=r1(1);
        C{i}=toeplitz(c1,r1);

 end
 for i=1:n
        c2=input(Please enter a vector of size (N):);
        r2=input(Please enter a vector of size (N):);
        c2(1)=r2(1);
        R{i}=toeplitz(c2,r2);
 end
    C{1}=R{1};
 for j=1:n
         F{1,j}=R{j};
         F{j,1}=C{j};
 end
 for i=1:n
      for j=1:n
          if i==j
            F{i,j}=R{1}
          elseif i>j
              F{i,j}=C{i-j+1};
```

```
            else
                F{i,j}=R{j-i+1};
            end
        end
end
FF=cell2mat(F);
```

The following MATLAB commands help us to build the compact correction E:

```
E=input(please enter  the compact correction (E):);
n1=size(E);
n2=size(FF);
EE=zeros(n2);
for i=1:n1
    for j=1:n1
        EE(i,j)=E(i,j);
    end
end
```

Appendix B. This appendix provides a plain Matlab implementation for Example 5.1.

```
E=[-1,1;-3,3];
n=3;
N=2;
for i=1:n
c1=rand(1,N);
r1=rand(1,N);
c1(1)=r1(1);
C{i}=toeplitz(c1,r1);
end
for i=1:n
c2=rand(1,N);
r2=rand(1,N);
c2(1)=r2(1);
R{i}=toeplitz(c2,r2);
end
C{1}=R{1};
for j=1:n
F{1,j}=R{j};
F{j,1}=C{j};
end
for j=1:n
F{1,j}=R{j};pos{j}=F{1,j};
F{j,1}=C{j};neg{j}=F{j,1};
end
for i=1:n
        for j=1:n
            if i==j
              F{i,j}=pos{1};
            elseif i>j
```

```
                F{i,j}=neg{i-j+1};
            else
                F{i,j}=pos{j-i+1};
            end
        end
end

fprintf(CQBT Matrix of size %d x %d\n\n, n*N, n*N);
if size(E, 1) > 0
  fprintf(Rank of top-left correction:%d\n,size(E,2));
end
fprintf(Toeplitz part (leading %d x %d block):\n, ...
        n*N, n*N);
F=cell2mat(F)

fprintf('\n');
fprintf(\n-Finite correction(top-left corner):\n);
```

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