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NOVEL CONCEPTS IN VAGUE INCIDENCE GRAPHS WITH APPLICATION

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Abstract. Fuzzy graph (FG) models enjoy the ubiquity of being in natural and human-made structures, namely dynamic process in physical, biological, and social systems. As a result of inconsistent and indeterminate information inherent in real-life problems, which are often uncertain, it is highly difficult for an expert to model those problems based on an FG. Vague incidence graph (VIG) can deal with the uncertainty associated with the inconsistent and determinate information of any real-world problem, where FGs may fail to reveal satisfactory results. Also, VIGs are outstandingly practical tools for analyzing different computer science domains such as networking, clustering, capturing the image, and also other issues such as medical sciences, and traffic planning. Hence, in this research, we introduce new operations on VIGs, namely, maximal product, rejection, and residue product with several examples. Likewise, some results related to operations have been described.

Keywords: Vague set (VS), vague incidence graph, maximal product, rejection.

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1. Introduction

Vague sets are denoted as a higher-order fuzzy sets which develop the solution procedure which are more complex to obtain the more accurate results than fuzzy sets but not affecting the complexity on computation time/volume and memory space. FG-models are beneficial mathematical tools for addressing the combinatorial problems in various fields involving research, optimization, algebra, computing, environmental science and topology. Fuzzy graphical models are obviously better than graphical models because of the natural existence of vagueness and ambiguity. In 1965 [30], the fuzzy set theory was first proposed by Zadeh. Fuzzy set theory is a very powerful mathematical tool for solving approximate reasoning related problems. By presenting the VS notion through changing the value of an element in a set with a sub-interval of $[0, 1]$, Gau and Buehrer [11] introduced and structured the vague set theory. Specifically, a true membership

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function of $t_v(x)$ and false membership function of $f_v(x)$ are used to define the boundaries of the membership degree. An immediate result of a rise of popularity of fuzzy set theory initiated by Rosenfeld [18] who introduced the concept of a fuzzy graph. The concept of a domination in an FG was introduced by Somasundaram [26]. Many researchers, notably Talebi and Rashmanlou [29], studied new application of the concept of domination in vague graphs (VGs). Akram et al. [1, 2, 3] described several concepts and results of FGs. Samanta et al. [27, 28] represented fuzzy competition graphs, and some remarks on bipolar fuzzy graphs. Borzooei et al. [4, 5, 6, 7] investigated new concepts of VGs. Rashmanlou et al. [19, 20, 21, 22, 23, 24, 25] analyzed new results in vague graphs. The incidence graphs can generally be represented as a triple (V, E, I), where V is a finite set of vertices and E is a finite set of edges, and $I \subseteq V \times E$ is an incidence function which indicates its end vertices for each edge whether the edge is directed (1) or not (0). If an edge e is directed, then, the first element and the second element of $I(e)$ denote the origin vertex and the destination vertex, respectively. The origins of the concept of an incidence graph is often attributed to Brualdi and Massey [8] who have been dealing with the incidence and incidence chromatic number. The fuzzification of the incidence graphs was proposed by Dinesh [9, 10] and among later contributions, a special place belongs to the works by Mordeson and his colleagues, notably Mathew, Mordeson and Malik [12], or Mordeson and Mathew [13, 14, 15, 16].

A VIG is referred to as a generalized structure of an FG that delivers more exactness, adaptability, and compatibility to a system when matched with systems running on FGs. Also, a VIG is able to concentrate on determining the uncertainity coupled with the inconsistent and indeterminate information of any real-world problem, where FGs may not lead to adequate results. Hence, in this paper, we introduced new operations on VIG, namely, maximal product, rejection, and residue product with several examples.

2. Preliminaries

A graph is a pair of $G = (V, E)$ which satisfies $E \subseteq V \times V$. The elements of $V(G)$ and $E(G)$ are the nodes and edges of the graph G, respectively.

An FG is of the from $G = (\sigma, \mu)$ which is a pair of mapping $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ as defined as $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$, $\forall x, y \in V$ and μ is a symmetric fuzzy relation on σ and $\wedge$ denotes the minimum.

A (VS) A is a pair (t_A, f_A) on the set V where t_A and f_A are taken as real valued functions which can be defined on $V \to [0, 1]$, so that $t_A(x) + f_A(x) \leq 1$, for all x belonging to V. The interval $[t_A(x), 1 - f_A(x)]$ is known as the vague value of x in A. $t_A(x)$, in this definition, is taken for the degree of membership as the lower bound when x in A and $f_A(x)$ is the lower bound for the membership negative of x in A.

Definition 2.1. [17] *A pair of $G = (A, B)$ is said to be a VG on a crisp graph $G = (V, E)$, where $A = (t_A, f_A)$ is a VS on V and $B = (t_B, f_B)$ is a VS on $E \subseteq V \times V$ so that $t_B(xy) \leq \min(t_A(x), t_A(y))$ and $f_B(xy) \geq \max(f_A(x), f_A(y))$, for each edge of $xy \in E$. A VG G is called strong if $t_B(xy) = \min(t_A(x), t_A(y))$ and $f_B(xy) = \max(f_A(x), f_A(y))$, for each edge of $xy \in E$. A VG G is called complete if $t_B(xy) = \min(t_A(x), t_A(y))$ and $f_B(xy) = \max(f_A(x), f_A(y))$, for every vertex of $x, y \in V$*

Definition 2.2. [5] *A vague graph G is said to be connected if $t_B^{\infty}(m_i m_j) > 0$, $f_B^{\infty}(m_i m_j) < 1$, for all $m_i, m_j \in V$. Also, we have:*

$$t_B^{\infty}(mn) = \sup\{t_B(mn_1) \wedge t_B(n_1 n_2) \wedge t_B(n_2 n_3) \wedge \cdots \wedge t_B(n_{k-1} n) | m, n_1, n_2, \cdots, n_{k-1}, n \in V\},$$

and

$$f_B^\infty(mn) = \inf\{f_B(mn_1) \vee f_B(n_1n_2) \vee f_B(n_2n_3) \vee \cdots \vee f_B(n_{k-1}n) | m, n_1, n_2, \cdots, n_{k-1}, n \in V\}.$$

Definition 2.3. [8] *Let $G = (V, E)$ be a graph. Then, $G^* = (V, E, I)$ is called an incidence graph, so that $I \subseteq V \times E$. If $V = \{m, n\}$, $E = \{mn\}$ and $I = \{(m, mn)\}$, then, (V, E, I) is an incidence graph even though $(n, mn) \notin I$. The pair (m, mn) is called an incidence pair or simply a pair. If $(m, mn), (n, mn), (n, nz), (z, nz) \in I$, then, mn or nz are called adjacent edges.*

Definition 2.4. [10] *Let $G^* = (V, E, I)$ be an incidence graph and σ be a fuzzy subset of V and μ a fuzzy subset of E. Let ϕ be a fuzzy subset of I. If $\phi(v, xy) \leq \sigma(v) \wedge \mu(x, y)$ for all $v \in V$ and $xy \in E$, then, ϕ is called a fuzzy incidence of graph G^* and $G = (\sigma, \mu, \phi)$ is called an FIG of G^*.*

Example 2.1. *Consider a VG G so that $V = \{x, y, z\}$ and $E = \{xy, xz, yz\}$, as shown in Figure 1. By a simple calculation, it is easy to see that G is a vague graph.*

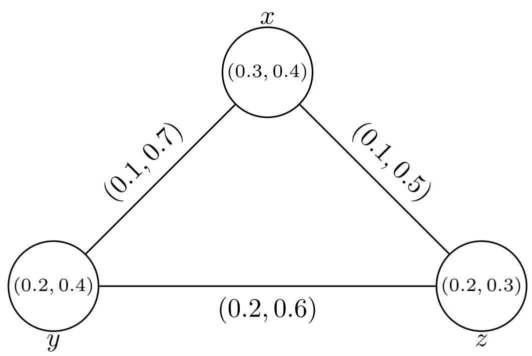


FIGURE 1. Vague graph of G

3. Vague incidence graph

A natural follow-up to the concept of VG, outlined in the previous section, is the concept of a VIG which will now be presented with its main properties. Also, we will discuss a very important concept of domination on VIG.

Definition 3.1. *$\psi = (A, B, C)$ is called a VIG of underlying crisp incidence graph $G^* = (V, E, I)$ if:*

$$\begin{aligned}
A &= \{(t_A(v), f_A(v)) | v \in V\}, \\
B &= \{(t_B(xy), f_B(xy)) | xy \in E\}, \\
C &= \{(t_C(v, xy), f_C(v, xy)) | (v, xy) \in I\},
\end{aligned}$$

so that

$$\begin{aligned}
&t_B(xy) \leq t_A(x) \wedge t_A(y), \\
&f_B(xy) \geq f_A(x) \vee f_A(y), \\
&t_C(v, xy) \leq t_A(v) \wedge t_B(xy), \\
&f_C(v, xy) \geq f_A(v) \vee f_B(xy), \ v \in V, \ xy \in E,
\end{aligned}$$

and

$$0 \le t_A(v) + f_A(v) \le 1,$$
$$0 \le t_B(xy) + f_B(xy) \le 1,$$
$$0 \le t_C(v, xy) + f_C(v, xy) \le 1.$$

A vague incidence graph ψ is called strong if $t_B(xy) = t_A(x) \wedge t_A(y)$, $f_B(xy) = f_A(x) \vee f_A(y)$, $t_C(v, xy) = t_A(v) \wedge t_B(xy)$, and $f_C(v, xy) = f_A(v) \vee f_B(xy)$, for all $xy \in E$.

Example 3.1. *Consider an incidence graph $G^* = (V, E, I)$ so that $V = \{m, n, k, t\}$, $E = \{mn, nk, kt, mt\}$ and $I = \{(m, mn), (n, nm), (n, nk), (k, kn), (n, nt), (t, tn), (k, kt), (t, tk), (t, tm), (m, mt)\}$ as shown in Figure 2.*

It is easy to show that $G = (A, B, C)$ is a VIG of G^* as shown in Figure 3 where

$$A = \left\{ \frac{m}{(0.1, 0.2)}, \frac{n}{(0.3, 0.5)}, \frac{k}{(0.2, 0.6)}, \frac{t}{(0.3, 0.7)} \right\}$$

$$B = \left\{ \frac{mn}{(0.1, 0.6)}, \frac{nk}{(0.2, 0.7)}, \frac{nt}{(0.3, 0.7)}, \frac{mt}{(0.1, 0.8)}, \frac{kt}{(0.2, 0.7)} \right\}$$

$$C = \left\{ \frac{(m, mn)}{(0.1, 0.7)}, \frac{(n, nm)}{(0.1, 0.6)}, \frac{(n, nk)}{(0.2, 0.8)}, \frac{(k, kn)}{(0.1, 0.8)}, \frac{(n, nt)}{(0.2, 0.7)}, \frac{(t, tn)}{(0.2, 0.8)}, \frac{(k, kt)}{(0.1, 0.7)}, \frac{(t, tk)}{(0.2, 0.8)}, \frac{(t, tm)}{(0.1, 0.8)}, \frac{(m, mt)}{(0.1, 0.9)} \right\}.$$

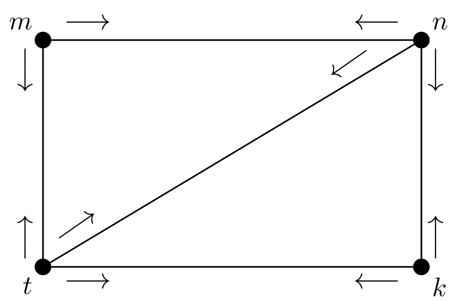


FIGURE 2. Incidence of graph G^*

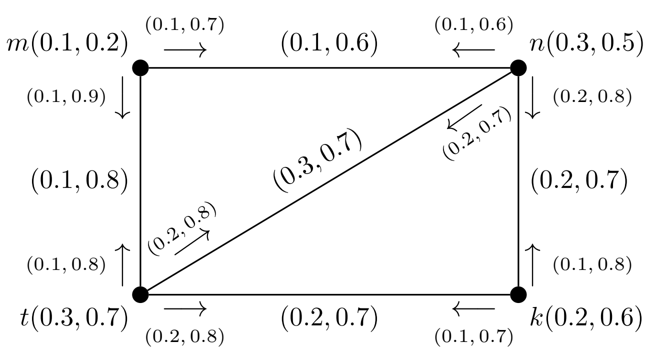


FIGURE 3. VIG ψ.

Definition 3.2. *If $\psi = (A, B, C)$ is a VIG, then, $H = (A', B', C')$ is a VI-subgraph of ψ whenever $A' \subset A$, $B' \subset B$ and $C' \subset C$. A vague incidence graph ψ is called complete if $t_B(xy) = t_A(x) \wedge t_A(y)$, $f_B(xy) = f_A(x) \vee f_A(y)$, $t_C(v, xy) = t_A(v) \wedge t_B(xy)$, and $f_C(v, xy) = f_A(v) \vee f_B(xy)$, for all $x, y \in V$.*

Definition 3.3. *Let $G_1 = (A_1, B_1, C_1)$ and $G_2 = (A_2, B_2, C_2)$ be two vague incidence graph with underlying crisp incidence of graph $G_1^* = (V_1, E_1, I_1)$ and $G_2^* = (V_2, E_2, I_2)$, respectively. $G_1 * G_2 = (A, B, C)$ is called maximal vague incidence graph $G^* = (V, E, I)$ where $V = V_1 \times V_2$ and $E_1 = \{(m_1, n_1)(m_2, n_2) | m_1 = m_2,\ n_1 n_2 \in E_2$ or $n_1 = n_2,\ m_1 m_2 \in E_1\}$.*

Vague vertex set of A and vague relation of B and vague incidence of C in maximal product $G_1 * G_2 = (A, B, C)$ are defined as: $A = A_1 * A_2$

$$1) \begin{cases} (t_{A_1} * t_{A_2})(m, n) = \max\{t_{A_1}(m), t_{A_2}(n)\} \\ (f_{A_1} * f_{A_2})(m, n) = \min\{f_{A_1}(m), f_{A_2}(n)\}, \\ \quad \textit{for all } (m, n) \in V = V_1 \times V_2 \end{cases}$$

$$2) \begin{cases} (t_{B_1} * t_{B_2})((m_1, n_1)(m_2, n_2)) = \max\{t_{A_1}(m_1), t_{B_2}(n_1 n_2)\} \\ (f_{B_1} * f_{B_2})((m_1, n_1)(m_2, n_2)) = \min\{f_{A_1}(m_1), f_{B_2}(n_1 n_2)\}, \\ \quad m_1 = m_2,\ n_1 n_2 \in E_2, \end{cases}$$

$$3) \begin{cases} (t_{B_1} * t_{B_2})((m_1, n_1)(m_2, n_2)) = \max\{t_{A_2}(n_1), t_{B_1}(m_1 m_2)\} \\ (f_{B_1} * f_{B_2})((m_1, n_1)(m_2, n_2)) = \min\{f_{A_2}(n_1), f_{B_1}(m_1 m_2)\}, \\ \quad m_1 m_2 \in E_1,\ n_1 = n_2, \end{cases}$$

$$4) \begin{cases} (t_{C_1} * t_{C_2})((m_1, m_1 n_1)(m_2, m_2 n_2)) = \max\{t_{A_1}(m_1), t_{B_2}(n_1 n_2)\} \\ (f_{C_1} * f_{C_2})((m_1, m_1 n_1)(m_2, m_2 n_2)) = \min\{f_{A_1}(m_1), f_{B_2}(n_1 n_2)\}, \\ \quad m_1 = m_2,\ n_1 n_2 \in E_2, \end{cases}$$

$$5) \begin{cases} (t_{C_1} * t_{C_2})((m_1, m_1 n_1)(m_2, m_2 n_2)) = \max\{t_{A_2}(n_1), t_{B_1}(m_1 m_2)\} \\ (f_{C_1} * f_{C_2})((m_1, m_1 n_1)(m_2, m_2 n_2)) = \min\{f_{A_2}(n_1), f_{B_1}(m_1 m_2)\}, \\ \quad m_1 m_2 \in E_1,\ n_1 = n_2. \end{cases}$$

Example 3.2. *Consider two VIG of G_1 and G_2 as shown in Figure 4. Their maximal product of $G_1 * G_2$ is shown in Figure 5.*

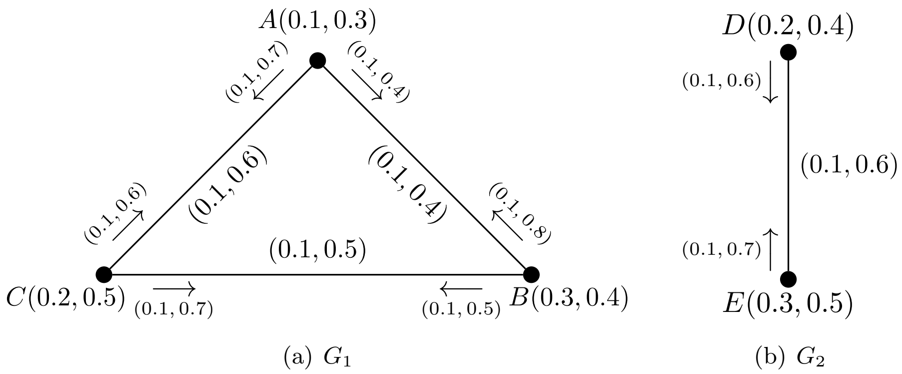


FIGURE 4. VIG G_1 and G_2.

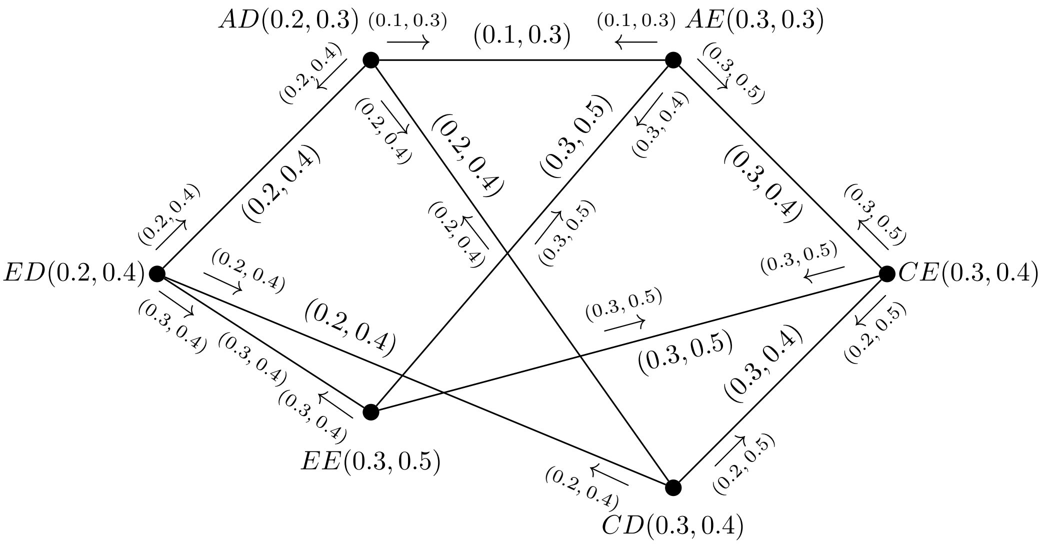


FIGURE 5. $G_1 * G_2$ maximal product of G_1 and G_2.

Theorem 3.1. *The maximal product of two VIG of G_1 and G_2 is a VIG, too.*

Proof. Let $G_1 = (A_1, B_1, C_1)$ and $G_2 = (A_2, B_2, C_2)$ be two VIG and $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. Then, by Definition 3.3, we have two cases.
(i) $m_1 = n_1 = m$

$$\begin{aligned}(t_{B_1} * t_{B_2})((m, m_2)(m, n_2)) &= \max\{t_{A_1}(m), t_{B_2}(m_2n_2)\}\\ &\leq \max\{t_{A_1}(m), \min\{t_{A_2}(m_2), t_{A_2}(n_2)\}\}\\ &= \min\{\max\{t_{A_1}(m), t_{A_2}(m_2)\}, \max\{t_{A_1}(m), t_{A_2}(n_2)\}\}\\ &= \min\{(t_{A_1} * t_{A_2})(m, m_2), (t_{A_1} * t_{A_2})(m, n_2)\},\end{aligned}$$

$$\begin{aligned}(f_{B_1} * f_{B_2})((m, m_2)(m, n_2)) &= \min\{f_{A_1}(m), f_{B_2}(m_2n_2)\}\\ &\geq \min\{f_{A_1}(m), \max\{f_{A_2}(m_2), f_{A_2}(n_2)\}\}\\ &= \max\{\min\{f_{A_1}(m), f_{A_2}(m_2)\}, \min\{f_{A_1}(m), f_{A_2}(n_2)\}\}\\ &= \max\{(f_{A_1} * f_{A_2})(m, m_2), (f_{A_1} * f_{A_2})(m, n_2)\},\end{aligned}$$

$$\begin{aligned}(t_{C_1} * t_{C_2})((m, m_1n_1)(m, n_2m_2)) &= \max\{t_{A_1}(m), t_{B_2}(m_2n_2)\}\\ &\leq \max\{t_{A_1}(m), \min\{t_{A_2}(m_2), t_{A_2}(n_2)\}\}\\ &= \min\{\max\{t_{A_1}(m), t_{A_2}(m_2)\}, \max\{t_{A_1}(m), t_{A_2}(n_2)\}\}\\ &= \min\{(t_{A_1} * t_{A_2})(m_1, m_1m_2), (t_{A_1} * t_{A_2})(m_2, m_1n_2)\},\end{aligned}$$

$$\begin{aligned}(f_{C_1} * f_{C_2})((m, m_1n_1)(m, n_2m_2)) &= \min\{f_{A_1}(m), f_{B_2}(m_2n_2)\}\\ &\geq \min\{f_{A_1}(m), \max\{f_{A_2}(m_2), f_{A_2}(n_2)\}\}\\ &= \max\{\min\{f_{A_1}(m), f_{A_2}(m_2)\}, \min\{f_{A_1}(m), f_{A_2}(n_2)\}\}\\ &= \max\{(f_{A_1} * f_{A_2})(m_1, m_1m_2), (f_{A_1} * f_{A_2})(m_2, m_1n_2)\}.\end{aligned}$$

(ii) If $m_2 = n_2 = z$

$$\begin{aligned}(t_{B_1} * t_{B_2})((m_1, z)(n_1, z)) &= \max\{t_{B_1}(m_1 n_1), t_{A_2}(z)\}\\ &\leq \max\{\min\{t_{A_1}(m_1), t_{A_1}(n_1)\}, t_{A_2}(z)\}\\ &= \min\{\max\{t_{A_1}(m_1), t_{A_2}(z)\}, \max\{t_{A_1}(n_1), t_{A_2}(z)\}\}\\ &= \min\{(t_{A_1} * t_{A_2})(m_1, z), (t_{A_1} * t_{A_2})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(f_{B_1} * f_{B_2})((m_1, z)(n_1, z)) &= \min\{f_{B_1}(m_1 n_1), f_{A_2}(z)\}\\ &\geq \min\{\max\{f_{A_1}(m_1), f_{A_1}(n_1)\}, f_{A_2}(z)\}\\ &= \max\{\min\{f_{A_1}(m_1), f_{A_2}(z)\}, \min\{f_{A_1}(n_1), f_{A_2}(z)\}\}\\ &= \max\{(f_{A_1} * f_{A_2})(m_1, z), (f_{A_1} * f_{A_2})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(t_{C_1} * t_{C_2})((m_1, m_1 n_1)(m_2, m_2 n_2)) &= \max\{t_{B_1}(m_1 n_1), t_{A_2}(z)\}\\ &\leq \max\{\min\{t_{A_1}(m_1), t_{A_1}(n_1)\}, t_{A_2}(z)\}\\ &= \min\{\max\{t_{A_1}(m_1), t_{A_2}(z)\}, \max\{t_{A_1}(n_1), t_{A_2}(z)\}\}\\ &= \min\{(t_{A_1} * t_{A_2})(m_1, z), (t_{A_1} * t_{A_2})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(f_{C_1} * f_{C_2})((m_1, m_1 n_1)(m_2, m_2 n_2)) &= \min\{f_{B_1}(m_1 n_1), f_{A_2}(z)\}\\ &\geq \min\{\max\{f_{A_1}(m_1), f_{A_1}(n_1)\}, f_{A_2}(z)\}\\ &= \max\{\min\{f_{A_1}(m_1), f_{A_2}(z)\}, \min\{f_{A_1}(n_1), f_{A_2}(z)\}\}\\ &= \max\{(f_{A_1} * f_{A_2})(m_1, z), (f_{A_1} * f_{A_2})(n_1, z)\}.\end{aligned}$$

$\square$

Theorem 3.2. *The maximal product of two strong VIGs of G_1 and G_2 is a strong VIG.*

Proof. Let $G_1 = (A_1, B_1, C_1)$ and $G_2 = (A_2, B_2, C_2)$ be two strong VIGs and $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. Then, by Theorem 3.1, $G_1 * G_2$ is a VIG. Now, we have two cases:

Case (i). If $m_1 = n_1 = m$

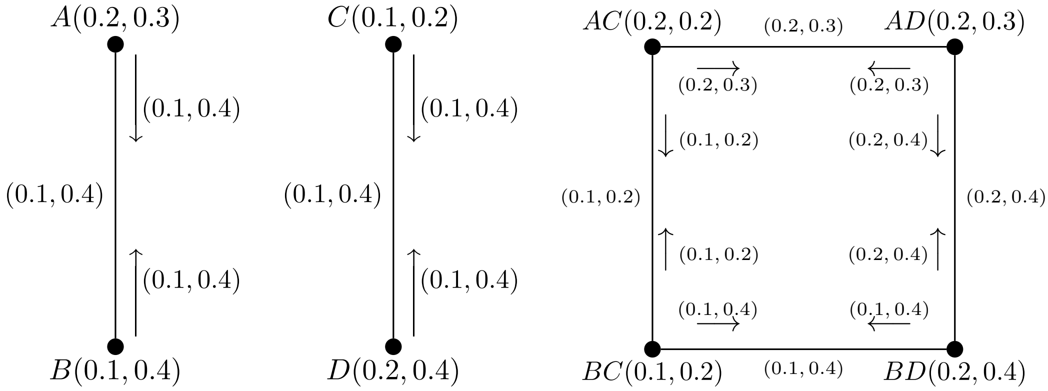
$$\begin{aligned}
(t_{B_1} * t_{B_2})((m_1, m_2)(m, n_2)) &= \max\{t_{A_1}(m), t_{B_2}(m_2n_2)\}\\
&= \max\{t_{A_1}(m), \min\{t_{A_2}(m_2), t_{A_2}(n_2)\}\}\\
&= \min\{\max\{t_{A_1}(m), t_{A_2}(m_2)\}, \max\{t_{A_1}(m), t_{A_2}(n_2)\}\}\\
&= \min\{(t_{A_1} * t_{A_2})(m, m_2), (t_{A_1} * t_{A_2})(m, n_2)\},\\
(f_{B_1} * f_{B_2})((m, m_2)(m, n_2)) &= \min\{f_{A_1}(m), f_{B_2}(m_2n_2)\}\\
&= \min\{f_{A_1}(m), \max\{f_{A_2}(m_2), f_{A_2}(n_2)\}\}\\
&= \max\{\min\{f_{A_1}(m), f_{A_2}(m_2)\}, \min\{f_{A_1}(m), f_{A_2}(n_2)\}\}\\
&= \max\{(f_{A_1} * f_{A_2})(m, m_2), (f_{A_1} * f_{A_2})(m, n_2)\},\\
(t_{C_1} * t_{C_2})((m, m_1n_1)(m, n_2m_2)) &= \max\{t_{A_1}(m), t_{B_2}(m_2n_2)\}\\
&= \max\{t_{A_1}(m), \min\{t_{A_2}(m_2), t_{A_2}(n_2)\}\}\\
&= \min\{\max\{t_{A_1}(m), t_{A_2}(m_2)\}, \max\{t_{A_1}(m), t_{A_2}(n_2)\}\}\\
&= \min\{(t_{A_1} * t_{A_2})(m_1, m_1m_2), (t_{A_1} * t_{A_2})(m_2, m_1n_2)\},\\
(f_{C_1} * f_{C_2})((m, m_1n_1)(m, n_2m_2)) &= \min\{f_{A_1}(m), f_{B_2}(m_2n_2)\}\\
&= \min\{f_{A_1}(m), \max\{f_{A_2}(m_2), f_{A_2}(n_2)\}\}\\
&= \max\{\min\{f_{A_1}(m), f_{A_2}(m_2)\}, \min\{f_{A_1}(m), f_{A_2}(n_2)\}\}\\
&= \max\{(f_{A_1} * f_{A_2})(m_1, m_1m_2), (f_{A_1} * f_{A_2})(m_2, m_1n_2)\}.
\end{aligned}$$

Case (ii). If $m_2 = n_2 = z$

$$\begin{aligned}
(t_{B_1} * t_{B_2})((m_1, z)(n_1, z)) &= \max\{t_{B_1}(m_1n_1), t_{A_2}(z)\}\\
&= \max\{\min\{t_{A_1}(m_1), t_{A_1}(n_1)\}, t_{A_2}(z)\}\\
&= \min\{\max\{t_{A_1}(m), t_{A_2}(z)\}, \max\{t_{A_1}(n_1), t_{A_2}(z)\}\}\\
&= \min\{(t_{A_1} * t_{A_2})(m_1, z), (t_{A_1} * t_{A_2})(n_1, z)\},\\
(f_{B_1} * f_{B_2})((m_1, z)(n_1, z)) &= \min\{f_{B_1}(m_1n_1), f_{A_2}(z)\}\\
&= \min\{\max\{f_{A_1}(m_1), f_{A_1}(n_1)\}, f_{A_2}(z)\}\\
&= \max\{\min\{f_{A_1}(m_1), f_{A_2}(z)\}, \min\{f_{A_1}(n_1), f_{A_2}(z)\}\}\\
&= \max\{(f_{A_1} * f_{A_2})(m_1, z), (f_{A_1} * f_{A_2})(n_1, z)\},\\
(t_{C_1} * t_{C_2})((m_1, m_1n_1)(m_2, m_2n_2)) &= \max\{t_{B_1}(m_1n_1), t_{A_2}(z)\}\\
&= \max\{\min\{t_{A_1}(m_1), t_{A_1}(n_1)\}, t_{A_2}(z)\}\\
&= \min\{\max\{t_{A_1}(m_1), t_{A_2}(z)\}, \max\{t_{A_1}(n_1), t_{A_2}(z)\}\}\\
&= \min\{(t_{A_1} * t_{A_2})(m_1, z), (t_{A_1} * t_{A_2})(n_1, z)\},\\
(f_{C_1} * f_{C_2})((m_1, m_1n_1)(m_2, m_2n_2)) &= \min\{f_{B_1}(m_1n_1), f_{A_2}(z)\}\\
&= \min\{\max\{f_{A_1}(m_1), f_{A_1}(n_1)\}, f_{A_2}(z)\}\\
&= \max\{\min\{f_{A_1}(m_1), f_{A_2}(z)\}, \min\{f_{A_1}(n_1), f_{A_2}(z)\}\}\\
&= \max\{(f_{A_1} * f_{A_2})(m_1, z), (f_{A_1} * f_{A_2})(n_1, z)\}.
\end{aligned}$$

Therefore, $G_1 * G_2$ is a strong VIG. □

Example 3.3. *Consider the strong VIGs of G_1 and G_2 as Figure 6.*



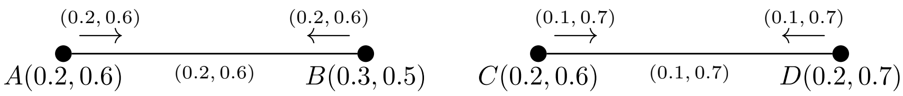
(a) G_1 (b) G_2 (c) $G_1 * G_2$

FIGURE 6. Vague incidence graphs of G_1, G_2 and $G_1 * G_2$.

It is easy to see that $G_1 * G_2$ is a strong VIG, too.

Remark 3.1. *If the maximal product of two VIGs of G_1 and G_2 is a strong, then, G_1 and G_2 need not to be strong, in general.*

Example 3.4. *Consider the VIGs G_1 and G_2 as Figure 7. We can see that the maximal product of two VIGs G_1 and G_2 is $G_1 * G_2$ in Figure 8.*



(a) G_1 (b) G_2

FIGURE 7. VIGs of G_1 and G_2.

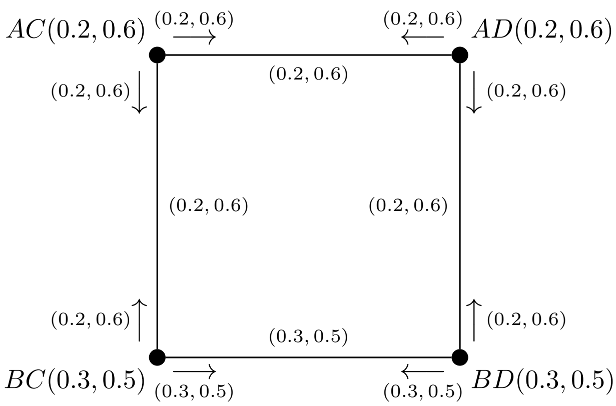


FIGURE 8. $G_1 * G_2$.

Clearly, G_1 and $G_1 * G_2$ are strong VIGs, but G_2 is not strong.

Remark 3.2. *The maximal product of two complete VIGs is not a complete vague incidence graph, in general. Because we do not include the case* $(m_1, m_2) \in E_1$ *and* $(n_1, n_2) \in E_2$ *in the definition of the maximal product of two vague incidence graphs.*

Remark 3.3. *The maximal product of two complete VIGs is a strong vague incidence graph.*

Example 3.5. *Consider the complete VIGs* G_1 *and* G_2 *as in Figure 6. A simple calculation concludes that* $G_1 * G_2$ *is a strong VIG.*

Theorem 3.3. *The maximal product of two connected VIGs is a connected vague incidence graph.*

Proof. Let G_1 and G_2 be two connected VIGs where $V_1 = \{m_1, m_2, \ldots, m_k\}$ and $V_2 = \{n_1, n_2, \ldots, n_k\}$. Then, $t^{\infty}_{B_1}(m_i m_j) > 0$, for all $m_i, m_j \in V_1$ and $t^{\infty}_{B_2}(n_i n_j) > 0$, for all $n_i n_j \in V_2$ or $f^{\infty}_{B_1}(m_i m_j) < 1$, for all $m_i, m_j \in V_1$ and $f^{\infty}_{B_2}(n_i n_j) < 1$, for all $n_i n_j \in V_2$. The maximal product of G_1 and G_2 can be taken as $G = (A, B)$. Now, consider the 'k' as the subgraphs of G with the vertex set of $\{(m_i, n_1), (m_i, n_2), \ldots, (m_i, n_s)\}$, for $i = 1, 2, \ldots, k$. Each of these subgraphs of G is connected, since the m_i's are the same and since G_2 is connected, each n_i is adjacent to at least one of the vertices in V_2. Also, since G_1 is connected, each x_i is adjacent to at least one of the vertices in V_1. Hence, there exists at least one edge between any pair of the above 'k' subgraphs. Thus, we have $t^{\infty}_{B}((m_i, n_j)(m_m, n_n)) > 0$ (or $f^{\infty}_{B}((m_i, n_j)(m_m, n_n)) < 1$), for all $((m_i, n_j)(m_m, n_n)) \in E$. Hence, G is a connected vague incidence graph. □

Definition 3.4. *The rejection of* $G_1|G_2$ *of the two VIG* $G_1 = (A_1, B_1, C_1)$ *and* $G_2 = (A_2, B_2, C_2)$ *is defined as:*

$$1) \begin{cases} (t_{A_1}|t_{A_2})((m,n)) = \min\{t_{A_1}(m), t_{A_2}(n)\} \\ (f_{A_1}|f_{A_2})((m,n)) = \max\{f_{A_1}(m), f_{A_2}(n)\}, \ for\ all\ (m,n) \in V_1 \times V_2, \end{cases}$$

$$2) \begin{cases} (t_{B_1}|t_{B_2})((m,m_2)(m,n_2)) = \min\{t_{A_1}(m), t_{A_2}(m_2), t_{A_2}(n_2)\} \\ (f_{B_1}|f_{B_2})((m,m_2)(m,n_2)) = \max\{f_{A_1}(m), f_{A_2}(m_2), f_{A_2}(n_2)\} \\ for\ all\ m \in V_1\ and\ m_2 n_2 \notin E_2, \end{cases}$$

$$3) \begin{cases} (t_{B_1}|t_{B_2})((m_1,m)(n_1,m)) = \min\{t_{A_1}(m_1), t_{A_1}(n_1), t_{A_2}(m)\} \\ (f_{B_1}|f_{B_2})((m_1,m)(n_1,m)) = \max\{f_{A_1}(m_1), f_{A_1}(n_1), f_{A_2}(m)\} \\ for\ all\ m \in V_2\ and\ m_1 n_1 \notin E_1, \end{cases}$$

$$4) \begin{cases} (t_{B_1}|t_{B_2})((m_1,m_2)(n_1,n_2)) = \min\{t_{A_1}(m_1), t_{A_1}(n_1), t_{A_2}(m_2), t_{A_2}(n_2)\} \\ (f_{B_1}|f_{B_2})((m_1,m_2)(n_1,n_2)) = \max\{f_{A_1}(m_1), f_{A_1}(n_1), f_{A_2}(m_2), f_{A_2}(n_2)\} \\ for\ all\ m_1 n_1 \notin E_1\ and\ m_2 n_2 \notin E_2, \end{cases}$$

$$5) \begin{cases} (t_{C_1}|t_{C_2})((m,m_1m_2)(m,n_1n_2)) = \min\{t_{A_1}(m), t_{A_1}(m_1m_2), t_{A_2}(n_1n_2)\} \\ (f_{C_1}|f_{C_2})((m,m_1m_2)(m,n_1n_2)) = \max\{f_{A_1}(m), f_{A_1}(m_1m_2), f_{A_2}(n_1n_2)\} \\ for\ all\ m \in V_1\ and\ m_1 m_2 \notin E_1, \end{cases}$$

$$6) \begin{cases} (t_{C_1}|t_{C_2})((m_1m_2,m)(n_1n_2,m)) = \min\{t_{A_1}(m_1m_2), t_{A_2}(m), t_{A_2}(n_1n_2)\} \\ (f_{C_1}|f_{C_2})((m_1m_2,m)(n_1n_2,m)) = \max\{f_{A_1}(m_1m_2), f_{A_2}(m), f_{A_2}(n_1n_2)\} \\ for\ all\ m \in V_2\ and\ n_1 n_2 \notin E_2, \end{cases}$$

$$7) \begin{cases} (t_{C_1}|t_{C_2})((m_1,m_1m_2)(n_1,n_1n_2)) = \min\{t_{A_1}(m_1), t_{A_1}(m_1m_2), t_{A_2}(n_1), t_{A_2}(n_1n_2)\} \\ (f_{C_1}|f_{C_2})((m_1,m_1m_2)(n_1,n_1n_2)) = \max\{f_{A_1}(m_1), f_{A_1}(m_1m_2), f_{A_2}(n_1), f_{A_2}(n_1n_2)\} \\ for\ all\ m_1 m_2 \notin E_1\ and\ n_1 n_2 \notin E_2. \end{cases}$$

Proposition 3.1. *The rejection of the two VIGs is not a VIG in general.*

Example 3.6. *Consider two vague incidence graphs of G_1 and G_2 as Figure 9. It is easy to show that the rejection of G_1 and G_2 is not vague incidence graph.*

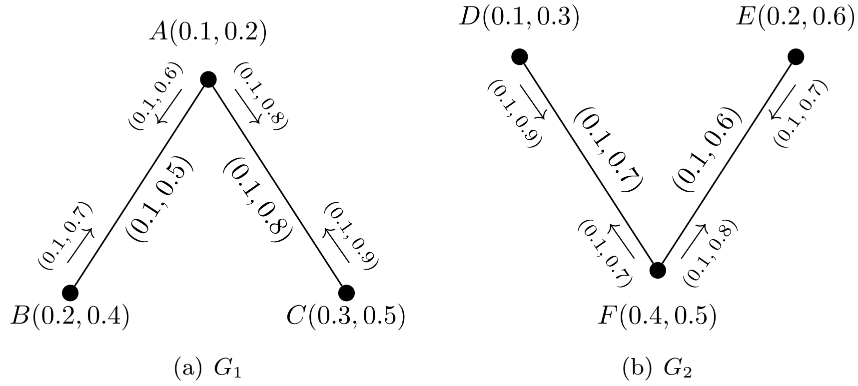


Figure 9. Vague incidence graph (VIG) of G_1 and G_2.

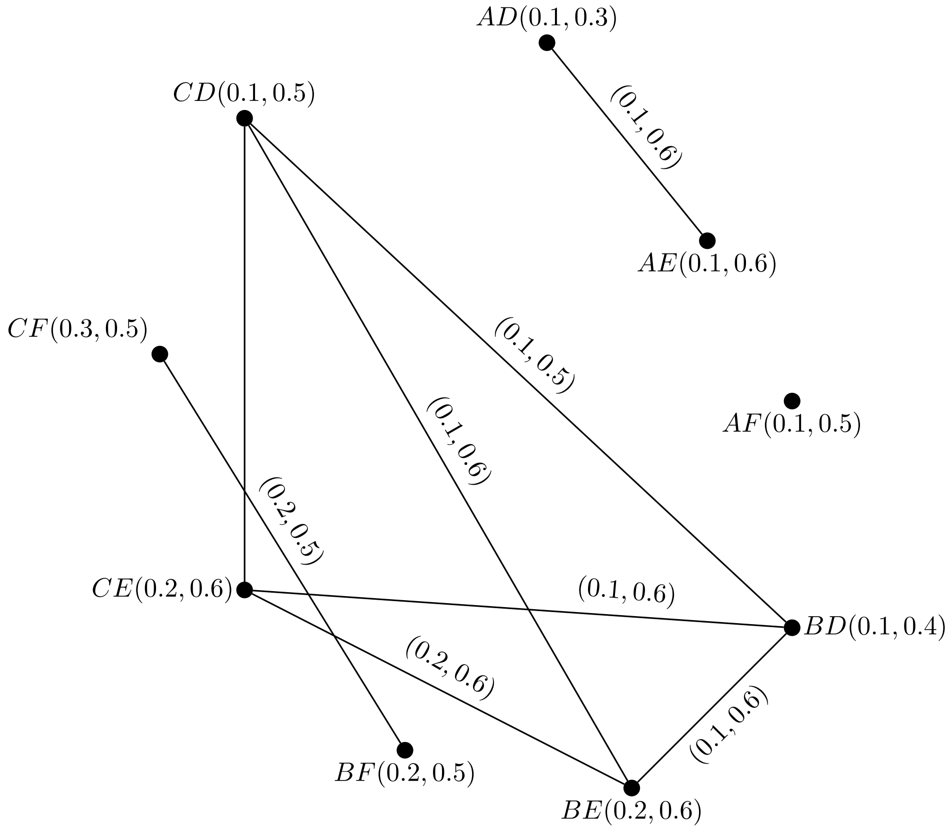


Figure 10. Rejection of two VIG.

Definition 3.5. *The residue product of* $G_1 \bullet G_2$ *of two VIG* $G_1 = (A_1, B_1, C_1)$ *and* $G_2 = (A_2, B_2, C_2)$ *is defined as:*

1)$(t_{A_1} \bullet t_{A_2})((m_1, m_2)) = \max\{t_{A_1}(m_1), t_{A_2}(m_2)\}$

$(f_{A_1} \bullet f_{A_2})((m_1, m_2)) = \min\{f_{A_1}(m_1), f_{A_2}(m_2)\},\ forall\ (m_1, m_2) \in V_1 \times V_2,$

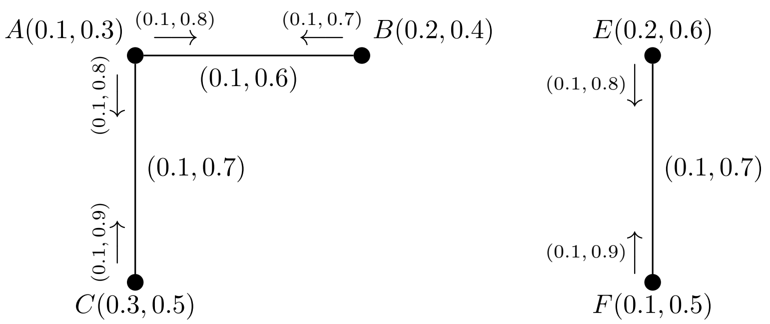
2)$(t_{B_1} \bullet t_{B_2})((m_1, m_2)(n_1, n_2)) = t_{B_1}(m_1n_1)$

$(f_{B_1} \bullet f_{B_2})((m_1, m_2)(n_1, n_2)) = f_{B_1}(m_1n_1),\ for\ all\ m_1n_1 \in E_1,\ m_2 \neq n_2,$

3)$(t_{C_1} \bullet t_{C_2})((m_1, m_1m_2)(n_1, n_1n_2)) = t_{C_1}(m_1n_1)$

$(f_{C_1} \bullet f_{C_2})((m_1, m_1m_2)(n_1, n_1n_2)) = f_{C_1}(m_1n_1),\ for\ all\ m_1n_1 \in E_1,\ m_2 \neq n_2,$

Example 3.7. *Consider VIGs of* G_1 *and* G_2 *in Figure 11. The residue product of* G_1 *and* G_2 $(G_1 * G_2)$ *is shown in Figure 12.*



(a) G_1 (b) G_2

FIGURE 11. VIGs of G_1 and G_2.

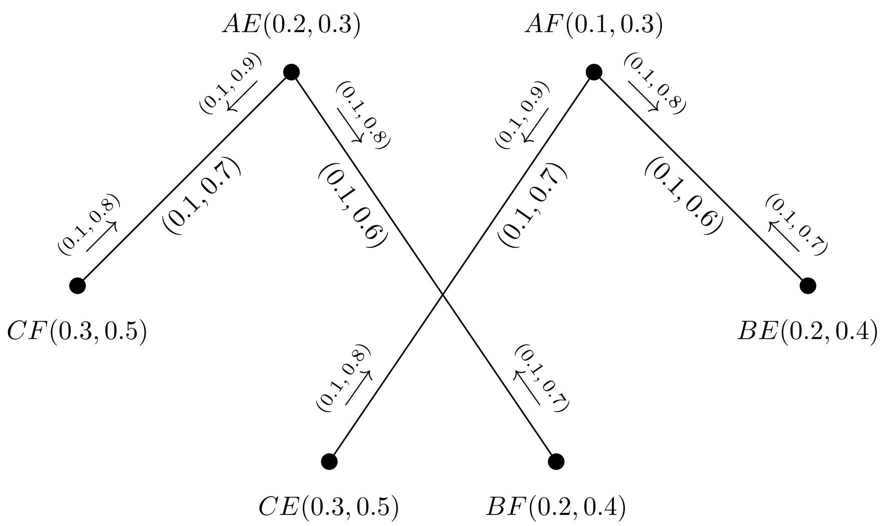


FIGURE 12. Residue product of VIGs.

Proposition 3.2. *The residue product of two VIG of* G_1 *and* G_2 *is a VIG.*

Proof. Let $G_1 = (A_1, B_1, C_1)$ and $G_2 = (A_2, B_2, C_2)$ be two VIG and $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. If $m_1 n_1 \in E_1$ and $m_2 \neq n_2$, then, we have

$$\begin{aligned}(t_{B_1} \bullet t_{B_2})((m_1, m_2)(n_1, n_2)) &= t_{B_1}(m_1 n_1) \leq \min\{t_{A_1}(m_1), t_{A_1}(n_1)\} \\ &\leq \max\{\min\{t_{A_1}(m_1), t_{A_1}(n_1)\}, \min\{t_{A_2}(m_2), t_{A_2}(n_2)\}\} \\ &= \min\{\max\{t_{A_1}(m_1), t_{A_1}(n_1)\}, \max\{t_{A_2}(m_2), t_{A_2}(n_2)\}\} \\ &= \min\{(t_{A_1} \bullet t_{A_2})(m_1, m_2), (t_{A_1} \bullet t_{A_2})(n_1, n_2)\},\end{aligned}$$

$$\begin{aligned}(f_{B_1} \bullet f_{B_2})((m_1, m_2)(n_1, n_2)) &= f_{B_1}(m_1 n_1) \geq \max\{f_{A_1}(m_1), f_{A_1}(n_1)\} \\ &\geq \min\{\max\{f_{A_1}(m_1), f_{A_1}(n_1)\}, \max\{f_{A_2}(m_2), f_{A_2}(n_2)\}\} \\ &= \max\{\min\{f_{A_1}(m_1), f_{A_1}(n_1)\}, \min\{f_{A_2}(m_2), f_{A_2}(n_2)\}\} \\ &= \max\{(f_{A_1} \bullet f_{A_2})(m_1, m_2), (f_{A_1} \bullet_{A_2})(n_1, n_2)\},\end{aligned}$$

$$\begin{aligned}(t_{C_1} \bullet t_{C_2})((m_1, m_1 m_2)(n_1, n_1 n_2)) &= t_{C_1}(m_1 n_1) \leq \min\{t_{C_1}(m_1), t_{C_1}(n_1)\} \\ &\leq \max\{\min\{t_{C_1}(m_1), t_{C_1}(n_1)\}, \min\{t_{C_2}(m_1 m_2), t_{C_2}(n_1 n_2)\}\} \\ &= \min\{\max\{t_{C_1}(m_1), t_{C_1}(n_1)\}, \max\{t_{C_2}(m_1 m_2), t_{C_2}(n_1 n_2)\}\} \\ &= \min\{(t_{C_1} \bullet t_{C_2})(m_1, n_1), (t_{C_1} \bullet t_{C_2})(m_1 m_2, n_1 n_2)\},\end{aligned}$$

$$\begin{aligned}(f_{C_1} \bullet f_{C_2})((m_1, m_1 m_2)(n_1, n_1 n_2)) &= f_{C_1}(m_1 n_1) \geq \max\{f_{C_1}(m_1), f_{C_1}(n_1)\} \\ &\geq \min\{\max\{f_{C_1}(m_1), f_{C_1}(n_1)\}, \max\{f_{C_2}(m_1 m_2), f_{C_2}(n_1 n_2)\}\} \\ &= \max\{\min\{f_{C_1}(m_1), f_{C_1}(n_1)\}, \min\{f_{C_2}(m_1 m_2), f_{C_2}(n_1 n_2)\}\} \\ &= \max\{(f_{C_1} \bullet f_{C_2})(m_1, n_1), (f_{C_1} \bullet f_{C_2})(m_1 m_2, n_1 n_2)\}.\end{aligned}$$

□

4. Applications

Today, the issue of serving the people is one of the most important issues that governments should take very seriously, because the greater the level of people's satisfaction, the better the development of that country, as well as the growth and prosperity of that country's economy will be.

One of the most important services to the people is the establishment of banks and banking services. In fact, governments should set up the required banks according to the population of a city and how people have access to them, so that people can do their banking activities in the shortest possible time and in the shortest distance.

So in this application we are going to explain this further with the help of an ambiguous graph. Hence, suppose that A, B, and C are three branches of a Bank in a city. We consider these three branches and their relationship on a vague incidence graph. Let's consider that these branches are the vertex of graph G, and their relationship is shown on the edge of the graph. Now we consider special weight for each branch and name it "I" according to the amount of traffic of customers who have to go from one branch to another to do their banking activities during the day.

$$I = \{(A, AC), (C, CA), (C, CB), (B, BC), (B, BA), (A, AB), (D, DE), (E, ED)\}.$$

TABLE 1. The ability of bank staff to serve the people.

Name of banks	A	B	C	D	E
t_A	0.1	0.2	0.2	0.1	0.3
f_A	0.3	0.4	0.6	0.8	0.4

TABLE 2. The degree of influence of bank staff on each other.

Edges	AB	AC	BC	DE
t_B	0.1	0.1	0.2	0.1
f_B	0.4	0.7	0.8	0.9

TABLE 3. The amount of customer traffic from one bank branch to another branch.

Incidence edges	(A, AC)	(C, CA)	(C, CB)	(B, BC)
Weight	$(0.1, 0.8)$	$(0.1, 0.9)$	$(0.1, 0.8)$	$(0.1, 0.9)$
Incidence edge	(B, BA)	(A, AB)	(D, DE)	(E, ED)
Weight	$(0.1, 0.6)$	$(0.1, 0.5)$	$(0.1, 0.9)$	$(0.1, 0.9)$

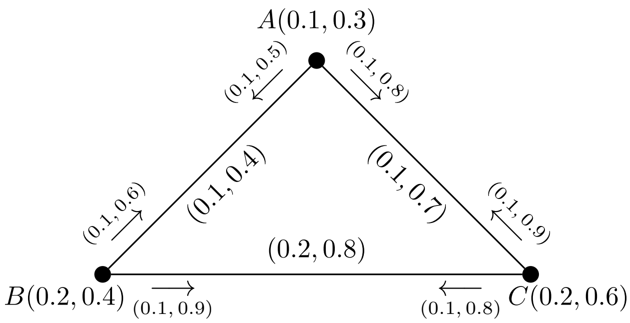


FIGURE 13. Vague incidence graph of G

The vertex B shows that the Bank B employees have only 20% of the power needed to do banking services to the public, and unfortunately do not have the 40% percentage knowledge needed to do so. Today, there is a lot of competition between banks to attract customers and create deposits in banks, and this has led to very weak friendly relations between branches. For example, the BC edge shows that there are only 20% of the friendships between these two branches of the bank, and unfortunately they have 80% of the differences. The incidence edge of (A, AC) indicates that only 10% of the customers go from Branch A to C and do about 80% of their banking activities in Branch A.

The other two bank branches are shown in Figure 14.

Then, we can consider the Maximal product of graph G and graph G' at follows.

As we can see in Figure 15, with the integration of branches, the ability of branch employees to respond to people and also the level of friendly relations between branch staff have increased sharply because with the integration of employees, information is exchanged between staff. This will make serving the people faster. For example, the vertex of $AD(0.1, 0.3)$ shows that the ability of the employees of this branch has reached 10% and they do not have only 30% of the necessary information. Also for (AE, CE) edge,

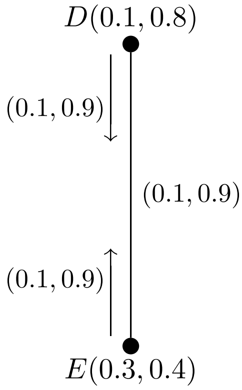


FIGURE 14. Vague incidence graph of G'

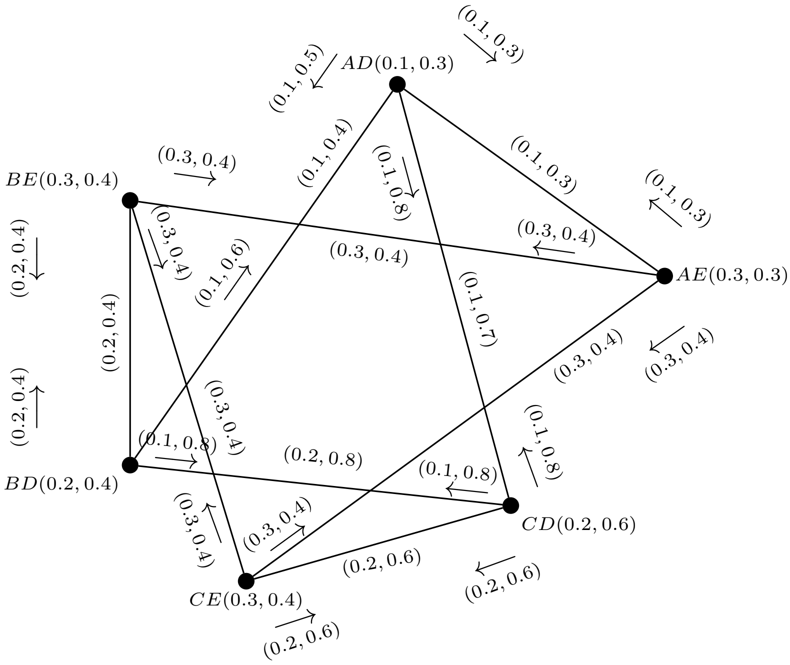


FIGURE 15. Maximal product of graph $G * G'$.

as we see, the staff effectiveness between these two banks has increased to 30%, which is very useful and valuable. Finally, in the case of the incidence edges, it should be noted that people can now more freely decide which branch to invest in, so the number of people moving to different branches has increased during the aggregation of branches, which is evident in Figure 15.

Therefore, governments should provide the necessary facilities to the banks so that they can be integrated, which will make serving the people faster, and the employees will be able to serve the people better by providing the necessary information. Finally, people can choose the best option by deciding on the facilities of each branch and their time will be saved.

5. Conclusion

It is well known that graphs are among the most ubiquitous models of both natural and human made structures. They can be used to model many types of relations and process dynamics in computer science and biological, social and physical systems. So, we have applied the concept of vague sets to vague incidence graph. We have discussed some operations on vague incidence graph with several examples. In our future work, we will define vague incidence soft graph, cubic vague incidence graph, bondage number and non-bondage number of vague incidence graph, and give some applications that will be useful in our daily life.

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