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A STOCHASTIC MAXIMUM PRINCIPLE FOR GENERAL MEAN-FIELD BACKWARD DOUBLY STOCHASTIC CONTROL

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Abstract. In this paper we study the optimal control problems of general Mckean-Vlasov for backward doubly stochastic differential equations (BDSDEs), in which the coefficients depend on the state of the solution process as well as of its law. We establish a stochastic maximum principle on the hypothesis that the control field is convex. For example, an example of a control problem is offered and solved using the primary result.

Keywords: Backward doubly stochastic differential equations. Optimal control. McKean-Vlasov differential equations. Probability measure. Derivative with respect to measure.

AMS Subject Classification: 93E20, 60H10.

1. Introduction

Nonlinear backward stochastic differential equations (BSDEs) were introduced in the first time by Pardoux and Peng [12] throughout that paper, authors investigated and explained the existense and uniquenes of the solution under the Lipschitz assumption and also given a probabilistic interpretation for the solution of a class of semi-linear parabolic partial differential equations (PDEs). After that, the authors [13] introduced a new kind of BSDEs called backward doubly, aiming to provide a probabilistic representation for a system of parabolic stochastic partial differential equations (SPDEs). Peng and Shi [14] introduced a another form of type of time-symmetric forward backward stochastic differential equation combining the theory of forward backward stochastic differential equations and backward doubly stochastic differential equations. Along this, other crucial results have been obtained by other researchers such as [18], [23], [16].

The maximum principle is one of the crucial methods opted in order to solve optimal control problem. Due to its wide applications in several fields such as economics, biology and finance, it attracted a large number of researchers. Kushner [9] was the first who studied the stochastic case. in the same way, Bensoussan [6] used the convexe perturbation method to derive the stochastic maximum principle in local form. Peng [15] proved the general maximum principle for the stochastic control system by using a second order

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variational equation and second order adjoint equation to overcome the difficulty appearing along with the nonconvex control domain and control entering the diffusion term. As a result, various results emerged for other stochastic control systems; for further information, readers of this article are advised to consult Agram et al [1], [3], Wu [19], Agram and Oksendal [2].

Thanks to Han et al [8], a huge contribution was made on stochastic optimal control for backward doubly stochastic system, authors investigated and obtained the necessary condition of optimality where the control domain is convex and the coefficients depend explicitly on the variable control, and from this result many results on controlled BDSDEs was obtained we refer to the works of [24], [21], [22], [17], [25].

The mean-field models were initially suggested to study the aggregate behavior of a large number of mutually interacting particles in diverse areas of statistical mechanics (e.g., derivation of Boltzmann or Vlasov equation in the kinetic gas theory), quantum mechanics and quantum chemistry (e.g., the density functional models or also Hartree and Hartree-Fock type models), economics, finance and game theory (N players stochastic differential games and the related problem of the existence of Nash equilibrium points, by letting n tends to infinity they derived in a periodic setting the mean filed limit equation), we refer the reader to [7], [11], [5],. . . etc.

The first version for general McKean–Vlasov stochastic optimal control refers to the works conducted by Buck et al [4]. In that paper authors studied an optimal control for forward stochastic system in which the coefficients of the system depend on both the state process as well as of its law, and the control domain is not necessary convex. they gave their results by using the derivatives with respect to probability measure. The current paper aims to apply the derivatives with respect to probability measure method for the sake of studying a class of general stochastic control problems in which the dynamics of the controlled system take the following backward -doubly systems of McKean -Vlasov type

$$\begin{cases} -dy^v(t) & = f\left(t, y^v(t), z^v(t), \mathbb{P}_{y^v(t)}, v(t)\right) dt - g(t, y^v(t), z^v(t), \mathbb{P}_{y^v(t)}, \\ & \quad v(t)) dB(t) - z^v(t)\, dW(t) \\ y(T) & = \eta, \end{cases}$$

where $\{W(t):\ 0 \leq t \leq T\}$ and $\{B(t):\ 0 \leq t \leq T\}$ be two mutually independent standard Brownian motion precesses defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and $\mathbb{P}_X$ denotes the law of the random variable X. The maps $f, g : [0,T] \times \mathbb{R} \times \mathbb{R} \times\ \mathbb{Q}_2(\mathbb{R}) \times \mathbb{U} \to \mathbb{R}$ are given deterministic functions, where $\mathbb{Q}_2(\mathbb{R})$ is the space of all probability measures μ on $\mathbb{R}$, endowed with $2-$Wasserstein matric. We note that the integral with respect to (B_t) is a "backward Itô integral" and the integral with respect to (W_t) is a standard forward integral, and the control variable $v = v(t)$ is a $\mathcal{F}_t-$adapted process with values in a convex set $\mathbb{U}$ of $\mathbb{R}$.

The cost functional to be minimized over the class of admissible controls is also of McKean-Vlasov type, which has the form

$$J(v(.)) = \mathbb{E}\left[\int_0^T l\left(t, y^v(t), z^v(t), \mathbb{P}_{y^v(t)}, v(t)\right) dt + \Phi\left(y^v(0), \mathbb{P}_{y^v(0)}\right)\right],$$

where

$$\begin{aligned} l &: [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{Q}_2(\mathbb{R}) \times \mathbb{U} \to \mathbb{R}, \\ \Phi &: \mathbb{R} \times \mathbb{Q}_2(\mathbb{R}) \to \mathbb{R}, \end{aligned}$$

are deterministic function.

The rest of the paper is organized as follows. In section 2, we formulate the problem, including the precise definition of the derivatives with respect to probability measure and give the notations and assumptions which are needed throughout this work. In section 3, we prove the stochastic maximum principle for our backward doubly stochastic control problem of general McKean-Vlasov. Finally in section 4, we discuss backward doubly stochastic LQ optimal control problem.

2. Assumptions and Problem Formulation

Let us consider a complete probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ on which is defined a one dimensional two mutually independent standard Brownian motions $W = \{W_t\}_{t\geq 0}, B = (B(t))_{t\geq 0}$ and let $T > 0$ be a given time horizon, we denote by $\mathcal{N}$ the class of $\mathbb{P}-$null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define $\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$ and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, where

$$\begin{aligned} \mathcal{F}_t^W &= \mathcal{N} \vee \sigma\{W(s):\ 0 \leq s \leq T\}, \\ \mathcal{F}_{t,T}^B &= \mathcal{N} \vee \sigma\{B(s) - B(t):\ t \leq s \leq T\}. \end{aligned}$$

Note that the collection $\{\mathcal{F}_t : t \in [0,T]\}$ is neither increasing nor decreasing, so it does not constitute a filtration. For a generic Euclidean space $\mathbb{X}$, we denote its inner product by $(.,.)$, its norm by $|.|$, and its Borel σ-field by $\mathcal{B}(\mathbb{X})$. Also, for any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}$ we denote

- $L^2(\mathcal{G};\mathbb{X})$ to be all $\mathbb{X}$ -valued, $\mathcal{G}$-measurable random variables ξ with $\|\xi\|_2 \triangleq \mathbb{E}\left[|\xi|^2\right]^{1/2} < \infty$.
- $\mathbb{Q}_2(\mathbb{R})$ to be the space of all probability measures μ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ with finite second moment (*i.e.*, $\int_{\mathbb{X}} |x|^2 \mu(dx) < \infty$). In particular, we endow the space $\mathbb{Q}_2(\mathbb{R}^d)$ with the following 2-Wasserstein metric: for $\mu,\ \nu \in \mathbb{Q}_2(\mathbb{R}^d)$,

$$W_2(\mu,\nu) \triangleq \inf\left\{\left[\int_0^T |x-y|^2 \rho(dx,dy)\right]^{\frac{1}{2}} : \rho \in \mathbb{Q}_2\left(\mathbb{R}^{2d}\right), \rho\left(.,\mathbb{R}^d\right) = \mu,\ \rho\left(\mathbb{R}^d,.\right) = \nu\right\}.$$

Furthermore, for an $\mathbb{X}$-valued random variable ξ defined on $(\Omega,\mathcal{F},\mathbb{P})$, we denote $\mathbb{P}_\xi \triangleq \mathbb{P}\circ\xi^{-1}$, the law introduced by ξ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$.

We now recall the notion of the differentiability with respect to probability measures. We shall follow the approach introduced in [4]. The main idea is to identify a distribution $\mu \in \mathbb{Q}_2(\mathbb{R}^d)$ with a random variables $\vartheta \in L^2(\mathcal{F};\mathbb{R}^d)$ so that $\mu = \mathbb{P}_\vartheta$. To be more presice, let us assume that the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is *rich enough* in the sense that for every $\mu \in \mathbb{Q}_2(\mathbb{R}^d)$, there is a random variable $\vartheta \in L^2(\mathcal{F};\mathbb{R}^d)$ such that $\mathbb{P}_\vartheta = \mu$. For any function $f : \mathbb{Q}_2(\mathbb{R}^d) \to \mathbb{R}$, we induce a function $\widetilde{f} : (\mathcal{F};\mathbb{R}^d) \to \mathbb{R}$, such that $\widetilde{f}(\vartheta) = f(\mathbb{P}_\vartheta)$, $\vartheta \in L^2(\mathcal{F};\mathbb{R}^d)$.

Definition 2.1. *A function $f : \mathbb{Q}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be differentiable at $\mu_0 \in \mathbb{Q}_2(\mathbb{R}^d)$ if there exists $\upsilon_0 \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ with $\mu_0 = \mathbb{P}_{\upsilon_0}$ such that its lift $\widetilde{f}$ is Fré chet differentiable at υ_0. More precisely, there exists a continuous linear functional $D\widetilde{f}(\upsilon_0) : \mathbb{L}^2(\mathcal{F};\mathbb{R}^d) \to \mathbb{R}$ such that*

$$\widetilde{f}(\upsilon_0 + \xi) - \widetilde{f}(\upsilon_0) = \left\langle D\widetilde{f}(\upsilon_0), \xi\right\rangle + o(\|\xi\|_2) = D_\xi f(\mu_0) + o(\|\xi\|_2) \tag{1}$$

where $\langle .,. \rangle$ *is the dual product on* $\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, *and we will refer to* $D_\xi f\left(\mu_0\right)$ *as the Fréchet derivative of at* μ_0 *in the direction* ξ. *In this case, we have*

$$D_\xi \widetilde{f}\left(\mu_0\right)=\left\langle D\widetilde{f}\left(\upsilon_0\right),\xi\right\rangle=\left.\frac{d}{dt}\widetilde{f}\left(\upsilon_0+t\xi\right)\right|_{t=0},\ \text{with } \mu_0=\mathbb{P}_{\upsilon_0}.$$

From Riesz' representation theorem, there is a unique random variable $\Theta_0\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ *such that* $\left\langle D_\xi\widetilde{f}\left(\upsilon_0\right),\xi\right\rangle=\left(\Theta_0,\xi\right)_2=\mathbb{E}\left[\left(\Theta_0,\xi\right)_2\right]$, *where* $\xi\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$. *It was shown (see the works of* [4]*) that there exists a Boral function* $h\left[\mu_0\right]:\mathbb{R}^d\rightarrow\mathbb{R}^d$, *depending only on the law* $\mu_0=\mathbb{P}_{\upsilon_0}$ *but not on the particular choice of the representative* υ_0 *such that* $\Theta_0=h\left[\mu_0\right]\left(\upsilon_0\right)$. *Thus, we can write* (1) *as*

$$f\left(\mathbb{P}_\upsilon\right)-f\left(\mathbb{P}_{\upsilon_0}\right)=\left\langle h\left[\mu_0\right]\left(\upsilon_0\right),\upsilon-\upsilon_0\right\rangle_2+o\left(\left\|\upsilon-\upsilon_0\right\|_2\right),\forall\upsilon\in\ \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right). \tag{2}$$

we shall denote

$$\partial_\mu f\left(\mathbb{P}_{\upsilon_0},x\right)=h\left[\mu_0\right](x),x\in\mathbb{R}^d.$$

Moreover, we have the following identities:

$$D\widetilde{f}\left(\upsilon_0\right)=\Theta_0=h\left[\mu_0\right]\left(\upsilon_0\right)=\partial_\mu f\left(\mathbb{P}_{\upsilon_0},\upsilon_0\right),$$

and

$$D_\xi f\left(\mathbb{P}_{\upsilon_0}\right)=\left\langle\partial_\mu f\left(\mathbb{P}_{\upsilon_0},\upsilon_0\right),\xi\right\rangle,\ \text{where } \xi=\upsilon-\upsilon_0.$$

Definition 2.2. *We say that the function* $f\in C_b^{1,1}\left(\mathbb{Q}_2\left(\mathbb{R}^d\right)\right)$ *if for all* $\upsilon\in\mathbb{L}^2\left(\mathcal{F},\mathbb{R}^d\right)$, *there exists a* $\mathbb{P}_\upsilon-$*modification of* $\partial_\mu f\left(\mathbb{P}_\upsilon,.\right)$ *such that* $\partial_\mu f:\mathbb{Q}_2\left(\mathbb{R}^d\right)\times\mathbb{R}^d\rightarrow\mathbb{R}^d$ *is bounded and Lipchitz continuous. That is for some* $C>0$, *it holds that*

(1) $\left|\partial_\mu f(\mu,x)\right|\leq C$, $\forall\mu\in\mathbb{Q}_2\left(\mathbb{R}^d\right)$, $\forall x\in\mathbb{R}^d$;
(2) $\left|\partial_\mu f\left(\mu_1,x_1\right)-\partial_\mu f\left(\mu_2,x_2\right)\right|\leq C\left(\mathbb{W}_2\left(\mu_1,\mu_2\right)+\left|x_1-x_2\right|\right)$, $\forall\mu_1,\mu_2\in\mathbb{Q}_2\left(\mathbb{R}^d\right)$ and $\forall x_1,x_2\in\mathbb{R}^d$.

Let $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{\mathbb{P}}\right)$ be a copy of the probability space $\left(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\right)$. For any pair of random variable $(\varkappa,\xi)\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)\times\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, we let $\left(\widehat{\varkappa},\widehat{\xi}\right)$ be an independent copy of $(\varkappa,\xi)$ defined on $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{\mathbb{P}}\right)$.

We consider the product probability space $\left(\Omega\times\widehat{\Omega},\ \mathcal{F}\otimes\widehat{\mathcal{F}},\ \mathbb{F}\otimes\widehat{\mathbb{F}},\ \ \mathbb{P}\otimes\widehat{\mathbb{P}}\right)$ and setting $\left(\widehat{\varkappa},\widehat{\xi}\right)(\omega,\widehat{\omega})=\left(\varkappa\left(\widehat{\omega}\right),\xi\left(\widehat{\omega}\right)\right)$ for any $(\omega,\widehat{\omega})\in\Omega\times\widehat{\Omega}$. Let $\left(\widehat{u}\left(t\right),\widehat{x}\left(t\right)\right)$ be an independent copy of $\left(u\left(t\right),x\left(t\right)\right)$ so that $\mathbb{P}_{x(t)}=\widehat{\mathbb{P}}_{\widehat{x}(t)}$.We denote by $\widehat{\mathbb{E}}$ the expectation under probability measure $\widehat{\mathbb{P}}$. Let

$$\mathbb{U}\left[0,T\right]\triangleq\left\{\Omega\times\left[0,T\right]\rightarrow\mathbb{U}/\ u\text{ is }\mathcal{F}_t-\text{ adapted. }\mathbb{E}\int_0^T\left|u\left(t\right)\right|^2dt<+\infty\right\}.$$

We consider the controlled BDSDE with McKean-Vlasov dynamics

$$\begin{cases}-dy\left(t\right) & =f\left(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y(t)},u\left(t\right)\right)dt-g(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y(t)},\\ & \quad u\left(t\right))dB\left(t\right)-z\left(t\right)dW\left(t\right)\\ y\left(T\right) & =\eta,\end{cases} \tag{3}$$

with the cost functional

$$J(u(.)) = \mathbb{E}\left\{\int_0^T l\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)\right) dt + \Phi\left(y(0),\ \mathbb{P}_{y(0)}\right)\right\}. \quad (4)$$

On the probability space $(\Omega, \mathcal{F}, F, P)$, we introduce the following spaces of processes which will be used later:

$$L^2(\mathcal{F}_T; \mathbb{R}) = \left\{\begin{array}{c} f : \mathbb{R} - \text{valued}, \mathcal{F}_T - \text{measurable random variables}, s.t: \\ E\left[|f|^2\right] < \infty, \end{array}\right\}$$

$$S^2([0,T], \mathbb{R}) = \left\{\begin{array}{c} f : \mathbb{R}\text{-valued } \mathcal{F}_t - \text{measurable stochastic processes}: \\ \mathbb{E}\left(\sup_{0\le t\le T} |f(t)|^2\right) < +\infty, \end{array}\right\}$$

$$M^2([0,T]; \mathbb{R}) = \left\{\begin{array}{c} f : \mathbb{R}\text{-valued } \mathcal{F}_t - \text{ measurable stochastic process:} \\ E\left[\int_0^T |f(t)|^2 dt\right] < \infty. \end{array}\right\}$$

We assume that the following conditions hold

(**H.1**) f, g, l, Φ are continuous and continuously differentiable with respect to y, z, μ, u.

(**H**.2) $f(t, \omega, 0, 0, \delta_0, 0), g(t, \omega, 0, 0, \delta_0, 0) \in M^2([0,T]; \mathbb{R})$,where δ_0 is the Dirac mesure at $0 \in \mathbb{R}$.

(**H**.3) There exist constants $c > 0$ and $0 < \alpha < 1$ such that for any $(y_1, z_1, , \mu_1, u_1)$ and (y_2, z_2, μ_2, u_2) :

$$\begin{aligned} &|f(t, y_1, z_1, \mu_1, u_1) - f(t, y_2, z_2, \mu_2, u_2)|^2 \\ &\le c(|y_1 - y_2|^2 + |z_1 - z_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2) + |u_1 - u_2|^2), \end{aligned}$$

$$\begin{aligned} &|g(t, y_1, z_1, \mu_1, u_1) - g(t, y_2, z_2, \mu_2, u_2)|^2 \\ &\le c(|y_1 - y_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2) + |u_1 - u_2|^2) + \alpha |z_1 - z_2|^2, \end{aligned}$$

(**H**.4) All the derivatives of f , g and l with respect to y, z, μ, u are bounded.

Given $u \in \mathbb{U}[0,T]$, by Juan and Xing [10], there exists a unique pair

$$(y(.), z(.)) \in S^2([0,T]; \mathbb{R}) \times M^2([0,T]; \mathbb{R}),$$

which solves equation(3). We shall need the following extension of the well-know Itô's formula.

Lemma 2.1. *(Pardoux and Peng 1994)*

Let $\alpha \in S^2\left([0,T]; \mathbb{R}^k\right)$, $\beta \in M^2\left([0,T]; \mathbb{R}^k\right)$, $\gamma \in \left([0,T]; \mathbb{R}^{k\times d}\right)$, $\delta \in M^2\left([0,T]; \mathbb{R}^{k\times m}\right)$ be such that (in this lemma $\{W(t) : 0 \le t \le T\}$ and $\{B(t) : 0 \le t \le T\}$ value, respectively, in $\mathbb{R}^m$ and in $\mathbb{R}^d$)

$$\alpha(t) = \alpha(0) + \int_0^t \beta(s) ds + \int_0^t \gamma(s) ds + \int_0^t \delta(s) ds,\ 0 \le t \le T.$$

Then

$$\begin{aligned} |\alpha(t)|^2 &= |\alpha(0)|^2 + 2\int_0^t \langle \alpha(s), \beta(s)\rangle ds + 2\int_0^t \langle \alpha(s), \gamma(s) dB_s\rangle \\ &+2\int_0^t \langle \alpha(s), \delta(s) dW_s\rangle - \int_0^t \|\gamma(s)\|^2 ds + \int_0^t \|\delta(s)\|^2 ds. \end{aligned}$$

$$\mathbb{E}\left|\alpha(t)\right|^2 = \mathbb{E}\left|\alpha(0)\right|^2 + 2\mathbb{E}\int_0^t \langle \alpha(s), \beta(s)\rangle ds - \mathbb{E}\int_0^t \left\|\gamma(s)\right\|^2 ds + \mathbb{E}\int_0^t \left\|\delta(s)\right\|^2 ds.$$

More generally, if $\Phi \in C^2\left(\mathbb{R}^k\right)$.

$$\begin{aligned}\Phi(\alpha(t)) &= \Phi(\alpha(0)) + \int_0^t \left\langle \Phi'(\alpha(s)), \beta(s)\right\rangle + \int_0^t \left\langle \Phi'(\alpha(s)), \gamma(s)\, dB(s)\right\rangle \\ &+ \int_0^t \left\langle \Phi'(\alpha(s)), \delta(s)\, dW(s)\right\rangle - \frac{1}{2}\int_0^t Tr\left[\Phi''(\alpha_s)\gamma(s)\gamma(s)^T\right] ds \\ &+ \frac{1}{2}\int_0^t Tr\left[\Phi''(\alpha(s))\delta(s)\delta(s)^T\right] ds.\end{aligned}$$

Proof. See Pardoux and Peng [13]. □

3. Stochastic Maximum Principle

In this section we study the stochastic maximum principle where the system is defined in 3. Our goal is to give a necessary conditions of optimality. An optimal control u is said to be optimal if

$$J(u(.)) = \inf_{v(.)\in\mathbb{U}[0,T]} J(v(.)). \tag{5}$$

Let u be an optimal control and let $(y(.), z(.))$ be the corresponding trajectory and let v be such that $u + v \in \mathbb{U}$. Since $\mathbb{U}[0,T]$ is convex, then for any $0 \leq \varepsilon \leq 1$

$$u_t^\varepsilon \equiv u_t + \varepsilon v_t \text{ is also in } \mathbb{U}[0,T].$$

We denote by $(y^\varepsilon(.), z^\varepsilon(.))$ the trajectory corresponding to u^ε, we have the following lemma.

Lemma 3.1. *We assume* $(\mathbf{H.1}) - (\mathbf{H.4})$ *hold. Then, we have*

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\left|y^\varepsilon(t) - y(t)\right|^2\right] &= 0, \\ \lim_{\varepsilon\to 0}\mathbb{E}\int_t^T \left|z^\varepsilon(t) - z(t)\right|^2 ds &= 0.\end{aligned}$$

Proof. Note that $y^\varepsilon(t) - y(t)$ satisfies the following BDSDE:

$$\begin{aligned}y^\varepsilon(t) - y(t) &= \int_t^T \left(f\left(s, y^\varepsilon(s), z^\varepsilon(s), \mathbb{P}_{y^\varepsilon(s)}, u^\varepsilon(s)\right) - f\left(s, y(s), z(s), \mathbb{P}_{y(s)}, u(s)\right)\right) ds \\ &+ \int_t^T \left(g\left(s, y^\varepsilon(s), z^\varepsilon(s), \mathbb{P}_{y^\varepsilon(s)}, u^\varepsilon(s)\right) - g\left(s, y(s), z(s), \mathbb{P}_{y(s)}, u(s)\right)\right) dB(s) \\ &- \int_t^T (z^\varepsilon(s) - z(s))\, dW(s).\end{aligned}$$

Applying the generalized Itô formula to $\left|y^\varepsilon(t) - y(t)\right|$, we have

$$\begin{aligned}
&\mathbb{E}\left|y^{\varepsilon}(t)-y(t)\right|^{2}+\mathbb{E}\int_{t}^{T}\left|z^{\varepsilon}(s)-z(s)\right|^{2}ds \\
&\leq 2\mathbb{E}\int_{t}^{T}\left(y^{\varepsilon}(s)-y(s)\right)\left(f\left(s,y^{\varepsilon}(s),z^{\varepsilon}(s),\mathbb{P}_{y^{\varepsilon}(s)},u^{\varepsilon}(s)\right)-f\left(s,y(s),z(s),\mathbb{P}_{y(s)},u(s)\right)\right)ds \\
&+\mathbb{E}\int_{t}^{T}\left|g\left(s,y^{\varepsilon}(s),z^{\varepsilon}(s),\mathbb{P}_{y^{\varepsilon}(s)},u^{\varepsilon}(s)\right)-g\left(s,y(s),z(s),\mathbb{P}_{y(s)},u(s)\right)\right|^{2}ds.
\end{aligned}$$

Hence, from assumption $(\mathbf{H.1})-(\mathbf{H.4})$, we have

$$\begin{aligned}
&\mathbb{E}\left(\left|y^{\varepsilon}(s)-y(s)\right|^{2}\right)+C_{1}\mathbb{E}\int_{t}^{T}\left|z^{\varepsilon}(s)-z(s)\right|^{2}ds \\
&\leq C_{2}\mathbb{E}\int_{t}^{T}\left|y^{\varepsilon}(s)-y(s)\right|^{2}ds+C_{3}\varepsilon^{2}\mathbb{E}\int_{t}^{T}\left|v(s)\right|^{2}ds+C_{4}\mathbb{E}\int_{t}^{T}\left|\mathbb{W}\left(\mathbb{P}_{y^{\varepsilon}(s)},\mathbb{P}_{y(s)}\right)\right|^{2}ds.
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
C_{1}&=\left(1-\tfrac{c}{M}-\alpha\right)\\
C_{2}&=\left(c+M+\tfrac{c}{M}\right)\\
C_{3}&=\left(c+\tfrac{c}{M}\right)\\
C_{4}&=\left(c+\tfrac{c}{M}\right).
\end{aligned}$$

Recall that for the $2-$Wasserstein matric $\mathbb{W}_{2}(.,.)$, we have

$$\begin{aligned}
\mathbb{W}_{2}\left(\mathbb{P}_{y^{\varepsilon}(s)},\mathbb{P}_{y(s)}\right)&=\inf\left\{\left[\mathbb{E}\left|\widetilde{y}^{\varepsilon}(s)-\widetilde{y}(s)\right|^{2}\right]^{\frac{1}{2}},\text{for all }\widetilde{y}^{\varepsilon}(.),\widetilde{y}(.)\in\mathbb{L}^{2}(\mathcal{F};\mathbb{R}),\right.\\
&\qquad\left.\text{with }\mathbb{P}_{y^{\varepsilon}(s)}=\mathbb{P}_{\widetilde{y}^{\varepsilon}}\text{ and }\mathbb{P}_{y(s)}=\mathbb{P}_{\widetilde{y}(s)}\right\}\\
&\leq\left[\mathbb{E}\left|y^{\varepsilon}(s)-y(s)\right|^{2}\right]^{\frac{1}{2}}.
\end{aligned} \tag{7}$$

From $(6),(7)$ and Definition(2.2), we have

$$\mathbb{E}\left|y^{\varepsilon}(s)-y(s)\right|^{2}+C_{1}\mathbb{E}\int_{t}^{T}\left|z^{\varepsilon}(s)-z(s)\right|^{2}ds\leq C\mathbb{E}\int_{t}^{T}\left|y^{\varepsilon}(s)-y(s)\right|^{2}ds+k\varepsilon^{2}.$$

We can choose some M such that $\left(1-\alpha-\frac{c}{M}\right)>0$. By Gronwall's lemma and the Burkholder -Davis-Gundy inequality, the result follows immediately by letting ε go to zero. □

We introduce the following variational equation

$$\begin{cases}
-dx(t) &=\left[f_{y}(t)x(t)+f_{z}(t)r(t)+f_{u}(t)v(t)\right.\\
&\quad\left.+\widehat{\mathbb{E}}\left(\partial_{\mu}f\left(t,y(t),z(t),\mathbb{P}_{y(t)},u(t);\widehat{y}\right)\widehat{x}(t)\right)\right]dt\\
&\quad+\left[g_{y}(t)x(t)+g_{z}(t)r(t)+g_{u}(t)v(t)\right.\\
&\quad\left.+\widehat{\mathbb{E}}\left(\partial_{\mu}g\left(t,y(t),z(t),P_{y(t)},u(t);\widehat{y}\right)\widehat{x}(t)\right)\right]dB(t)\\
&\quad-r(t)\,dW(t)\\
x(T)&=0.
\end{cases} \tag{8}$$

Under the assumptions $(\mathbf{H.1})-(\mathbf{H.4})$, there exists a unique adapted solution $(x(t),r(t))$, $0\leq t\leq T$ satisfying the variational equation (8).

Lemma 3.2. *We assume* $(\mathbf{H.1})-(\mathbf{H.4})$ *hold. Then, we have*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\left|\frac{y^{\varepsilon}(t)-y(t)}{\varepsilon}-x(t)\right|^{2}\right]=0,$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_{t}^{T}\left|\frac{z^{\varepsilon}(s)-z(s)}{\varepsilon}-r(s)\right|^{2}dt\right]=0.$$

Proof. Let $\Lambda^{\varepsilon}(t)=\dfrac{y^{\varepsilon}(t)-y(t)}{\varepsilon}-x(t)$, $\Theta^{\varepsilon}(t)=\dfrac{z^{\varepsilon}(t)-z(t)}{\varepsilon}-r(t)$.

For convenience, we will use following notations

$$\begin{array}{ll} y^{\lambda,\varepsilon}(t) & =y(t)+\lambda\left(y^{\varepsilon}(t)-y(t)\right),\\ z^{\lambda,\varepsilon}(t) & =z(t)+\lambda\left(z^{\varepsilon}(t)-z(t)\right),\\ \widehat{y}^{\lambda,\varepsilon}(t) & =\widehat{y}(t)+\lambda\left(y^{\varepsilon}(t)-\widehat{y}(t)\right),\\ v^{\varepsilon}(t) & =u(t)+\varepsilon v(t). \end{array}$$

By Definition (2.1) and (3), we have the following simple from of the Taylor expansion

$$f\left(\mathbb{P}_{u_0+\xi}\right)-f\left(\mathbb{P}_{u_0}\right)=D_{\xi}f\left(\mathbb{P}_{u_0}\right)+\mathcal{R}(\xi),$$

where $\mathcal{R}(\xi)$ is of order $o\left(\|\xi\|_2\right)$ with $o\left(\|\xi\|_2\right)\to 0$ for $\xi\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$.

From the state Equation (3) and the variational Equation (8), it is easy to get

$$\begin{cases} -d\Lambda^{\varepsilon}(t) & =\dfrac{1}{\varepsilon}\left\{f^{\epsilon}(t)-f(t)-\varepsilon f_y(t)x(t)-\varepsilon f_z(t)r(t)-\varepsilon f_u(t)v(t)\right.\\ & \left.-\widehat{\mathbb{E}}\left(\partial_{\mu}f\left(t,y(t),z(t),\mathbb{P}_{y(t)},u(t);\widehat{y}(t)\right)\widehat{x}(t)\right)\right\}dt\\ & +\dfrac{1}{\varepsilon}\left\{g^{\varepsilon}(t)-g(t)-\varepsilon g_y(t)x(t)-\varepsilon g_z(t)r(t)-\varepsilon g_u(t)v(t)\right.\\ & \left.-\widehat{\mathbb{E}}\left(\partial_{\mu}f\left(t,y(t),z(t),\mathbb{P}_{y(t)},u(t);\widehat{y}(t)\right)\widehat{x}(t)\right)\right\}dB(t)\\ & -\Theta^{\varepsilon}(t)\,dW(t),\\ \Lambda^{\varepsilon}(T) & =0. \end{cases}$$

Now, we decompose $\frac{1}{\varepsilon}\left(f\left(t,y^{\varepsilon}(t),z^{\varepsilon}(t),\mathbb{P}_{y^{\varepsilon}(t)},u^{\varepsilon}(t)\right)-f\left(t,y(t),z(t),\mathbb{P}_{y(t)},u(t)\right)\right)$ into the following parts

$$\frac{1}{\varepsilon}\left[f\left(t,y^{\varepsilon}(t),z^{\varepsilon}(t),\mathbb{P}_{y^{\varepsilon}(t)},u^{\varepsilon}(t)\right)-f\left(t,y(t),z(t),\mathbb{P}_{y(t)},u(t)\right)\right]$$

$$\begin{aligned} &=\frac{1}{\varepsilon}\left[f\left(t,y^{\varepsilon}(t),z^{\varepsilon}(t),\mathbb{P}_{y^{\varepsilon}(t)},u^{\varepsilon}(t)\right)-f\left(t,y(t),z^{\varepsilon}(t),\mathbb{P}_{y^{\varepsilon}(t)},u^{\varepsilon}(t)\right)\right]\\ &+\frac{1}{\varepsilon}\left[f\left(t,y(t),z^{\varepsilon}(t),\mathbb{P}_{y^{\varepsilon}(t)},u^{\varepsilon}(t)\right)-f\left(t,y(t),z(t),\mathbb{P}_{y^{\varepsilon}(t)},u^{\varepsilon}(t)\right)\right]\\ &+\frac{1}{\varepsilon}\left[f\left(t,y(t),z(t),\mathbb{P}_{y^{\varepsilon}(t)},u^{\varepsilon}(t)\right)-f\left(t,y(t),z(t),\mathbb{P}_{y(t)},u^{\varepsilon}(t)\right)\right]\\ &+\frac{1}{\varepsilon}\left[f\left(t,y(t),z(t),\mathbb{P}_{y(t)},u^{\varepsilon}(t)\right)-f\left(t,y(t),z(t),\mathbb{P}_{y(t)},u(t)\right)\right]. \end{aligned}$$

Noting that

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[f\left(t, y^{\varepsilon}(t), z^{\varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)-f\left(t, y(t), z^{\varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)\right] \\
&=\int_0^1\left[f_y\left(t, y^{\lambda, \varepsilon}(t), z^{\varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)\left(\Lambda^{\varepsilon}(t)+x(t)\right)\right] d\lambda dt, \\
&\tfrac{1}{\varepsilon}\left[f\left(t, y(t), z^{\varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)-f\left(t, y(t), z(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)\right] \\
&=\int_0^1\left[f_z\left(t, y^{\varepsilon}(t), z^{\lambda, \varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)\left(\Theta^{\varepsilon}(t)+r(t)\right)\right] d\lambda dt, \\
&\tfrac{1}{\varepsilon}\left[f\left(t, y(t), z(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)-f\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u^{\varepsilon}(t)\right)\right] \\
&=\int_0^1\left[\widehat{E}\left(\partial_\mu f\left(t, y(t), z(t), \mathbb{P}_{\widehat{y}^{\lambda, \varepsilon}(t)}, u^{\varepsilon}(t) ; \widehat{y}(t)\right)\right)\left(\widehat{\Lambda^{\varepsilon}}+\widehat{x}(t)\right)\right] d\lambda dt \quad .
\end{aligned}$$

and

$$\begin{aligned}
&\tfrac{1}{\varepsilon}\left[f\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u^{\varepsilon}(t)\right)-f\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)\right)\right] \\
&=\int_0^1\left[f_u\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u^{\lambda, \varepsilon}(t)\right) v(t)\right] d\lambda dt .
\end{aligned}$$

Analogously, we can have a similar decomposition for g. Therefore, we obtain

$$\begin{aligned}
& d\Lambda^{\varepsilon}(t) \\
& =\int_0^1\left[f_y\left(t, y^{\lambda, \epsilon}(t), z^{\varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)\right] \Lambda^{\varepsilon}(t) d\lambda dt \\
& +\int_0^1\left[f_y\left(t, y^{\lambda, \varepsilon}, z^{\varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)-f_y(t)\right] x(t) d\lambda dt \\
& +\int_0^1\left[f_z\left(t, y(t), z^{\lambda, \varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)\right] \Theta^{\varepsilon}(t) d\lambda dt \\
& +\int_0^1\left[f_z\left(t, y(t), z^{\lambda, \varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)-f_z(t)\right] r(t) d\lambda dt \\
& +\int_0^1\left[\widehat{\mathbb{E}}\left(\partial_\mu f\left(t, y(t), z(t), \mathbb{P}_{\widehat{y}^{\lambda, \varepsilon}(t)}, u^{\varepsilon}(t) ; \widehat{y}(t)\right)\right)\right] \widehat{\Lambda^{\varepsilon}}(t) d\lambda dt \\
& +\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu f\left(t, y(t), z(t), \mathbb{P}_{\widehat{y}^{\lambda, \varepsilon}(t)}, u^{\varepsilon}(t) ; \widehat{y}(t)\right)\right. \\
& \qquad\left.-\partial_\mu f\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t) ; \widehat{y}(t)\right)\right] \widehat{x}(t) d\lambda dt \\
& +\int_0^1\left[f_u\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)+\varepsilon v(t)\right)-f_u(t)\right] v(t) d\lambda dt \\
& +\int_0^1\left[g_y\left(t, y^{\lambda, \varepsilon}(t), z^{\varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)\right] \Lambda^{\varepsilon}(t) d\lambda dB(t) \\
& +\int_0^1\left[g_y\left(t, y^{\lambda, \varepsilon}(t), z^{\varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)-g_y(t)\right] x(t) d\lambda dB(t) \\
& +\int_0^1\left[g_z\left(t, y(t), z^{\lambda, \varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t),} u^{\varepsilon}(t)\right)\right] \Theta^{\varepsilon}(t) d\lambda dB(t) \\
& +\int_0^1\left[g_z\left(t, y(t), z^{\lambda, \varepsilon}(t), \mathbb{P}_{y^{\varepsilon}(t)}, u^{\varepsilon}(t)\right)-g_z(t)\right] r(t) d\lambda dB(t) \\
& +\int_0^1\left[\widehat{\mathbb{E}}\left(\partial_\mu g\left(t, y(t), z(t), \mathbb{P}_{\widehat{y}^{\lambda, \varepsilon}(t)}, u^{\varepsilon}(t) ; \widehat{y}(t)\right)\right)\right] \widehat{\Lambda^{\varepsilon}}(t) d\lambda dt \\
& +\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu g\left(t, y(t), z(t), \mathbb{P}_{\widehat{y}^{\lambda, \varepsilon}(t)}, u^{\varepsilon}(t) ; \widehat{y}(t)\right)\right. \\
& \qquad\left.-\partial_\mu g\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t) ; \widehat{y}(t)\right)\right] \widehat{x}(t) d\lambda dB(t) \\
& +\int_0^1\left[g_u\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)+\varepsilon v(t)\right)-g_u(t)\right] v(t) d\lambda-\Theta^{\varepsilon}(t) dW(t) \\
& \Lambda^{\varepsilon}(T)=0 .
\end{aligned}$$

From $(\mathbf{H.1})-(\mathbf{H.4})$ and Lemma 2.1 , we have

$$\mathbb{E}\left[\left|\Lambda^{\varepsilon}(s)\right|^2\right]+\mathbb{E}\left[\int_t^T\left|\Theta^{\varepsilon}(s)\right|^2 ds\right] \quad \leq C(t) \mathbb{E} \int_0^t\left|\Lambda^{\varepsilon}(s)\right|^2 ds+C_{\varepsilon},$$

where $C_{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$. By Grownwall's inequality, we obtain the desired result. □

Now we deduce from Lemma 3.2 a first expression of the cost derivative.

Proposition 3.1. *The following equality holds:*

$$\frac{d}{d\varepsilon}J\left(u\left(.\right)+\varepsilon v\left(.\right)\right)\Big|_{\varepsilon=0}=$$
$$\mathbb{E}\left\{\int_0^T\left(l_y\left(t\right)x\left(t\right)+l_z\left(t\right)r\left(t\right)+l_u\left(t\right)v\left(t\right)\right)+\widehat{\mathbb{E}}\left[\partial_\mu l\left(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y(t)},u^\varepsilon\left(t\right);\widehat{y}\right)\widehat{x}\left(t\right)\right]\right\}dt$$
$$+\mathbb{E}\left[\Phi_y\left(y\left(0\right),\mathbb{P}_{y(0)}\right).x\left(0\right)\right]+\widehat{\mathbb{E}}\left[\partial_\mu\Phi\left(y\left(0\right),\mathbb{P}_{y(0)};\widehat{y}\left(0\right)\right)\widehat{x}\left(0\right)\right].$$

Proof. By Lemma 3.2 and first order development, we decompose $\frac{1}{\varepsilon}\mathbb{E}\int_0^T\left(l^\varepsilon\left(t\right)-l\left(t\right)\right)dt$ into the following parts

$$\frac{1}{\varepsilon}\mathbb{E}\int_0^T\left(l^\varepsilon\left(t\right)-l\left(t\right)\right)dt$$
$$=\frac{1}{\varepsilon}\mathbb{E}\int_0^T\left[l^\varepsilon\left(t\right)-l\left(t,y\left(t\right),z^\varepsilon\left(t\right),\mathbb{P}_{y^\varepsilon(t)},u^\varepsilon\left(t\right)\right)\right]dt$$
$$+\frac{1}{\varepsilon}\mathbb{E}\int_0^T\left[l\left(t,y\left(t\right),z^\varepsilon\left(t\right),\mathbb{P}_{y^\varepsilon(t)},u^\varepsilon\left(t\right)\right)-l\left(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y^\varepsilon(t)},u^\varepsilon\left(t\right)\right)\right]dt$$
$$+\frac{1}{\varepsilon}\mathbb{E}\int_0^T\left[l\left(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y^\varepsilon(t)},u^\varepsilon\left(t\right)\right)-l\left(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y(t)},u^\varepsilon\left(t\right)\right)\right]dt$$
$$+\frac{1}{\varepsilon}\mathbb{E}\int_0^T\left[l\left(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y(t)},u^\varepsilon\left(t\right)\right)-l\left(t\right)\right]dt.$$

$$\to\mathbb{E}\int_0^T\left[l_y\left(t\right)x\left(t\right)+l_z\left(t\right)r\left(t\right)+l_u\left(t\right)v\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu l\left(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y(t)},u^\varepsilon\left(t\right);\widehat{y}\right)\widehat{x}\left(t\right)\right]\right]dt,$$

as ε tends to 0.

And

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\left[\Phi\left(y^\varepsilon\left(0\right),\mathbb{P}_{y^\varepsilon(0)}\right)-\Phi\left(y\left(0\right),\mathbb{P}_{y(0)}\right)\right]$$
$$=\lim_{\varepsilon\to0}\mathbb{E}\left[\int_0^1\Phi_y\left(y^{\lambda,\varepsilon}\left(0\right),\mathbb{P}_{y^\varepsilon(0)}\right)\left(\frac{y^\varepsilon\left(0\right)+y\left(0\right)}{\varepsilon}\right)d\lambda\right]$$
$$+\lim_{\varepsilon\to0}\mathbb{E}\left[\int_0^1\widehat{\mathbb{E}}\left(\partial_\mu\Phi\left(y\left(0\right),\mathbb{P}_{\widehat{y}^{\lambda,\varepsilon}(0)};\widehat{y}\left(0\right)\right)\right)\left(\frac{\widehat{y^\varepsilon}\left(0\right)+\widehat{y}\left(0\right)}{\varepsilon}\right)d\lambda\right]$$
$$=\mathbb{E}\left[\Phi_y\left(y\left(0\right),\mathbb{P}_{y(0)}\right)x\left(0\right)\right]+\widehat{\mathbb{E}}\left[\partial_\mu\Phi\left(y\left(0\right),\mathbb{P}_{y(0)};\widehat{y}\left(0\right)\right)\widehat{x}\left(0\right)\right].$$

We have

$$\frac{d}{d\varepsilon}J\left(u\left(.\right)+\varepsilon v\left(.\right)\right)$$
$$=\frac{J\left(u\left(.\right)+\varepsilon v\left(.\right)\right)-J\left(u\left(.\right)\right)}{\varepsilon}$$
$$=\mathbb{E}\int_0^T\left[l_y\left(t\right)x\left(t\right)+l_z\left(t\right)r\left(t\right)+l_u\left(t\right)v\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu l\left(t,y\left(t\right),z\left(t\right),\mathbb{P}_{y(t)},v^\varepsilon\left(t\right);\widehat{y}\right)\widehat{x}\left(t\right)\right]\right]dt$$
$$+\mathbb{E}\left[\Phi_y\left(y\left(0\right),\mathbb{P}_{y(0)}\right).x\left(0\right)\right]+\widehat{\mathbb{E}}\left[\partial_\mu\Phi\left(y\left(0\right),\mathbb{P}_{y(0)};\widehat{y}\left(0\right)\right)\widehat{x}\left(0\right)\right]$$

The proof is completed. □

Now, we introduce the adjoint equation involved in the stochastic maximum principle:

$$\begin{cases} dp(t) & = F\left(t, y(t), z(t), \mathbb{P}_{y(t)}, v(t), p(t), q(t)\right) dt \\ & +G\left(t, y(t), z(t), \mathbb{P}_{y(t)}, v(t), p(t), q(t)\right) dW(t) \\ & -q(t)\, dB(t), \\ p(0) & = \Phi_y\left(y(0), \mathbb{P}_{y(0)}\right) + \widehat{\mathbb{E}}\left[\partial_\mu \Phi\left(y(0), \mathbb{P}_{y(0)}; \widehat{y}(0)\right)\right]. \end{cases} \tag{9}$$

where

$$\begin{aligned} & F\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t), p(t), q(t)\right) \\ = \ & f_y\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)\right) p(t) + \widehat{\mathbb{E}}\left[\partial_\mu f\left(t, y(t), z(t), \ \mathbb{P}_{y(t)}, u; \widehat{y}(t)\right) \widehat{p}(t)\right] \\ + \ & g_y\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)\right) q(t) + \widehat{\mathbb{E}}\left[\partial_\mu g\left(t, y(t), z(t), \ \mathbb{P}_{y(t)}, u; \widehat{y}(t)\right) \widehat{q}(t)\right] \\ + \ & l_y\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)\right) + \widehat{\mathbb{E}}\left[\partial_\mu l\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u; \widehat{y}(t)\right)\right], \end{aligned}$$

and

$$\begin{aligned} & G\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t), p(t), q(t)\right) \\ = \ & f_z\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)\right) p(t) + \widehat{\mathbb{E}}\left[\partial_\mu f\left(t, \widehat{y}(t), z(t), \mathbb{P}_{y(t)}, \widehat{u}; y(t)\right) \widehat{p}(t)\right] \\ + \ & g_z\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)\right) q(t) + \widehat{\mathbb{E}}\left[\partial_\mu g\left(t, \widehat{y}(t), z(t), \mathbb{P}_{y(t)}, \widehat{u}; y(t)\right) \widehat{q}(t)\right] \\ + \ & l_z\left(t, y(t), z(t), \mathbb{P}_{y(t)}, u(t)\right) + \widehat{\mathbb{E}}\left[\partial_\mu l\left(t, \widehat{y}(t), z(t), \mathbb{P}_{y(t)}, \widehat{u}; y(t)\right)\right]. \end{aligned}$$

Lemma 3.3. *Let* $(p(.), q(.))$ *be the adapted solution of* (9). *Then*

$$\begin{aligned} & \mathbb{E}^u \langle p(0), x(0) \rangle \\ & = \mathbb{E}^u \int_0^T \Big[\Big\langle p(t), f_y x(t) + f_z r(t) + f_u v(t) + \widehat{\mathbb{E}}\left[\partial_\mu f\left(t, \widehat{y}(t), z(t), \mathbb{P}_{y(t)}, \widehat{u}; y(t)\right) \widehat{p}(t)\right] \Big\rangle \\ & + \Big\langle q(t), g_y x(t) + g_z r(t) + g_u v(t) + \widehat{\mathbb{E}}\left[\partial_\mu g\left(t, \widehat{y}(t), z(t), \mathbb{P}_{y(t)}, \widehat{u}; y(t)\right) \widehat{q}(t)\right] \Big\rangle \Big] dt \\ & - \mathbb{E}^u \int_0^T \left[\langle F, x(t) \rangle + \langle G, r(t) \rangle\right] dt. \end{aligned}$$

Proof. The proof follows imediately from the generalized Itô's formula □

Now by Lemma 3.3, Proposition 3.1, and adjoint equations (9),we have

$$\frac{d}{d\varepsilon} J\left(u(.) + \varepsilon v(.)\right)\Big|_{\varepsilon=0} = \mathbb{E}^u \left\{ \int_0^T \langle l_u + f_u p(t) + g_u q(t), v(t) \rangle \, dt \right\};$$

Defining the generalized Hamiltonian by

$$\begin{aligned} & H(t, y(t), z(t), \mu, v, p(t), q(t)) \\ & \triangleq f(t, y(t), z(t), \ \mu, v(t)) p(t) + g(t, y(t), z(t), \mu, v(t)) q(t) + l(t, y(t), z(t), \mu, v(t)), \\ & \qquad (t, y, z, \mu, v, p, q) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{Q}_2(\mathbb{R}) \times \mathbb{U} \times \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{10}$$

Theorem 3.1. *Let* $u(.)$ *be optimal. then, the maximum principle*

$$\mathbb{E}\left[H_v\left(t, y(t), z(t), \ \mu, u(t), p(t), q(t)\right)\left(v(t) - u(t)\right)\right] \geq 0, \forall v \in \mathbb{U}, a.e., a.s.,$$

holds, where the Hamiltonian function H *is defined by* (10).

4. Application in Backward Doubly Stochastic LQ Control Problem

In this section, we apply our maximum principle to backward doubly stochastic linear-quadratic control problem of Mckean-Vlasov, type the results obtained in section 3, writes as follows:

$$\begin{cases} -dy(t) &= \Big(A(t)\,y(t) + \widetilde{A}(t)\,\mathbb{E}[y(t)] + B(t)\,z(t) + C(t)\,v(t)\Big)\,dt \\ & +D(t)\,y(t)\,dB(t) - z(t)\,dW(t) \\ y(T) &= \eta. \end{cases} \tag{11}$$

The cost functional is a quadratic one, and it has the form

$$J(v(.)) = \frac{1}{2}\mathbb{E}\left[\int_0^T N(.)\,v^2(t)\,dt + M\left[y^2(0)\right]\right], \tag{12}$$

Here we assume that all the coefficients in (11) and (12) are bounded and deterministic functions of t; $N(.) > 0$ and $N^{-1}(.)$ are also bounded. The optimal control problem is to find $u \in \mathbb{U}$ such that

$$J(u(.)) = \inf_{v\in\mathbb{U}} J(v(.)).$$

To solve this problem, we write down the Hamiltonian function

$$\begin{aligned} &H(t, y(t), z(t), v(t), p(t), q(t)) \\ &= \Big(A(t)\,y(t) + \widetilde{A}(t)\,\mathbb{E}[y(t)] + B(t)\,z(t) + C(t)\,v(t)\Big)\,p(t) \\ &+D(t)\,y(t)\,q(t) + \tfrac{1}{2}N(t)\,v^2(t) \end{aligned} \tag{13}$$

where $p(.), q(.)$ satisfies the equation

$$\begin{aligned} &dp(t) = \Big[A(t)\,p(t) + \widetilde{A}(t)\,\mathbb{E}[p(t)] + D(t)\,q(t)\Big]\,dt + B(t)\,p(t)\,dW(t) - q(t)\,dB(t) \\ &p(0) = -My(0). \end{aligned} \tag{14}$$

According to Theorem 3.1, if $u(.)$ is an optimal control process, we have

$$u(t) = N^{-1}(t)\,C(t)\,p(t). \tag{15}$$

Then the state equations and adjoint equations become:

$$\begin{cases} -dy^u(t) &= \Big[A(t)\,y^u(t) + \widetilde{A}(t)\,\mathbb{E}[y^u(t)] + B(t)\,z(t) - N^{-1}(t)\,C^2(t)\,p(t)\Big]\,dt \\ & +D(t)\,y^u(t)\,dB_t - z(t)\,dW(t) \\ y(T) &= \eta \\ dp(t) &= \Big[A(t)\,p(t) + \widetilde{A}(t)\,\mathbb{E}[p(t)] + D(t)\,q(t)\Big]\,dt + B(t)\,p(t)\,dW(t) - q(t)\,dB(t) \\ p(0) &= My^u(0). \end{cases}$$

In order to solve this system we set

$$p(t) = \Phi(t)\,y^u(t) + \Psi(t)\,\mathbb{E}[y^u(t)],$$

where $\Phi(t), \Psi(t)$ are deterministic differential functions which will be specified below. Then, from (14) we get

$$\begin{cases} A(t)\,p(t) + \widetilde{A}(t)\,\mathbb{E}[p(t)] + D(t)\,q(t) \\ = -\Phi(t)\,A(t)\,y^u(t) - \Phi(t)\,\widetilde{A}(t)\,\mathbb{E}[y^u(t)] - \Phi(t)\,B(t)\,z(t) \\ -\Phi(t)\,C(t)\,u(t) - \Psi(t)\left(A(t) + \widetilde{A}(t)\right)\mathbb{E}[y^u(t)] \\ -\Psi(t)\,C(t)\,\mathbb{E}[u(t)] - \frac{d}{dt}\Phi(t)\,y^u(t) - \frac{d}{dt}\Psi(t)\,\mathbb{E}[y^u(t)], \\ B(t)\,p(t) = \Phi(t)\,z(t), \\ -q(t) = -\Phi(t)\,D(t)\,y^u(t). \end{cases} \tag{16}$$

From(15) and by compring the coefficients of $y^u(t)$ and $E[y^u(t)]$, respectively, in the first equation of (16) ,we get

$$\begin{cases} \frac{d}{dt}\Phi(t) - \left(2A(t) + B^2(t) + D^2(t)\right)\Phi(t) + N^{-1}(t)\,C^2(t)\,\Phi^2(t) = 0, \\ t \in [0,T] \\ \Phi(T) = M\,(\geq 0). \end{cases}$$

But this is just a Riccati equation, and it has an unique solution.
Moreover,

$$\begin{cases} \frac{d}{dt}\Psi(t) - 2\left(A(t) + \widetilde{A}(t) + B^2(t) - N^{-1}(t)\,C^2(t)\,\Phi(t)\right)\Psi(t) \\ \quad - \left(B^2(t)\,\Phi^{-1}(t) - C^2(t)\,N^{-1}(t)\right)\Psi^2(t) - 2\widetilde{A}(t)\,\Phi(t) = 0, \\ \Psi(t) = 0. \end{cases}$$

Theorem 4.1. *The optimal control $u \in \mathbb{U}$ for the linear quadratic control problem is given by $u(t) = N^{-1}(t)\,C(t)\,p(t)$.*

5. Conclusions

In conclusion, SMP of mean filed backward doubly optimal control in the convexe case is obtained via the differentiability with respect to probability law, and as an application we study the linear quadratic case.

Recommendations, it is highly recommended that the coming studies should cover the following cases.

- SMP of mean filed backward doubly optimal control in the non convexe case.
- SMP of mean filed forward-backward doubly optimal control in both case.
- Extend the result of RAKIA AHMED et al [20] to the backward doubly system driven by brownian motion and rosenblatt process.

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References

[1] Agram, N., Bachouch, A., Øksendal, B., Proske, F., (2019), Singular control optimal stopping of memory mean field processes, SIAM J. Math. Anal., 51(1), 450–468.

[2] Agram, N., & Øksendal, B., (2015), Malliavin calculus and optimal control of stochastic Volterra equations, Journal of Optimization Theory and Applications, 167(3), 1070-1094.

[3] Agram, N., Øksendal, B., & Yakhlef, S., (2019), New approach to optimal control of stochastic Volterra integral equations, Stochastics, 91(6), 873-894.

[4] Buckdahn , R., Li, J., Ma, J., (2016), A stochastic maximum principle for general mean-field systems, Appl. Math. Optim., 74(3), 507-534.

[5] Buckdahn, R., Djehiche, B., Li, J., Peng, S., (2009), Mean-field backward stochastic differential equations: a limit approach., Ann Probab, 37(4), 1524–1565.

[6] Bensoussan, A., (1982), Lecture on Stochastic Control, in Nonlinear Filtering and Stochastic Control, ser., Lecture Notes in Mathematics. New York: Springer Verlag.
[7] Carmona, R., Delarue, F., (2013), Mean field forward-backward stochastic differential equations, Electronic Communications in Probability, 18, p.1-15.
[8] Han, Y.C., Peng, S.G., Wu, Z., (2010), Maximum principle for backward doubly stochastic control systems with applications, SIAM J. Control Optim., 48 (7), pp 4224–4241.
[9] Kushner, H., (1972), Necessary conditions for continuous parameter stochastic optimisation problems, SIAM Journal on Control and Optimization, 10(3) , 550-565.
[10] Li, J., Xing, C., (2022), General mean-field BDSDEs with continuous coefficients, Journal of Mathematical Analysis and Applications, 506(2), p.125699.
[11] Lasry, J. M., and Lions, P. L., (2006), Jeux ‘a champ moyen. I. Le cas stationnaire., C. R. Math. Acad., Sci. Paris 343(9),P. 619-625.
[12] Pardoux, E., Peng, S., (1990), Adapted solution of a backward stochastic differential equation., Systems & control letters, 14 (1-2), 61-74.
[13] Pardoux , E., and Peng, S., (1994), Backward doubly stochastic differential equations and systems of quasilinear parabolic SPDE's, Probability Theory and Related Fields, 98 (2), pp.209-227.
[14] Peng, S., and Shi, Y., (2003), A type of time-symmetric forward-backward stochastic differential equations, Comptes Rendus Mathematique, 336(9), 773-778.
[15] Peng, S., (1990), A general stochastic maximum principle for optimal control problems, SIAM Journal on Control and Optimization, 28(4), 966-979.
[16] Ren, Y., Lin, A., & Hu, L., (2009), Stochastic PDIEs and backward doubly stochastic differential equations driven by Lévy processes, Journal of Computational and Applied Mathematics, 223(2), 901–907.
[17] Xu, J., and Han, Y. C., (2017), Stochastic maximum principle for delayed backward doubly stochastic control systems, Journal of Nonlinear Sciences & Applications, 10(1), 215–226.
[18] Wu, Z., & Zhang, F., (2011), BDSDEs with locally monotone coefficients and sobolev solutions for SPDEs, Journal of Differential Equations, 251, 759–784.
[19] Wu, Z., (1998), Maximum principle for optimal control problem of fully coupled forward-backward stochastic systems, Systems Sci. Math. Sci., 11, pp. 249-259.
[20] Yahia, R, A., Benchaabane, A., Zeghdoudi, H., (2021), Existence results for second-order neutral stochastic equations driven by Rosenblatt process, Methods of Functional Analysis and Topology, 27(4), 384-400.
[21] Wu, J. B., Liu, Z. M., (2020), Optimal control of mean-field backward doubly stochastic systems driven by Itô-Lévy processes, Int. J. Control, 93(4), pp. 953–970.
[22] Zhu, Q. F., Shi, Y. F., (2015), ‘Optimal control of backward doubly stochastic systems with partial information’, IEEE Trans. Autom. Control, 60 (1), pp.173–178.
[23] Zhang, Q., & Zhao, H., (2007), Stationary solutions of SPDEs and infinite horizon BDSDEs, Journal of Functional Analysis, 252(1), 171–219.
[24] Zhang, L.Q., Shi, Y. F., (2011), Maximum principle for forward-backward doubly stochastic control systems and applications, ESAIM, Control, Optimisatio Calc. Var., 17 (4), pp. 1174–1197.
[25] Zhang, L., Zhou, Q., & Yang, J., (2020), Necessary condition for optimal control of doubly stochastic systems, Mathematical Control & Related Fields, 10(2), 379.

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